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Search for integrable two-component versions of the lattice equations in the ABS-list

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Abstract

We search and classify two-component versions of the quad equations in the ABS list, under certain assumptions. The independent variables will be called y, z and in addition to multilinearity and irreducibility the equation pair is required to have the following specific properties: (1) The two equations forming the pair are related by $y \leftrightarrow z$ exchange. (2) When $z = y$ both equations reduce to one of the equations in the ABS list. (3) Evolution in any corner direction is by a multilinear equation pair. One straightforward way to construct such two-component pairs is by taking some particular equation in the ABS list (in terms of y), using replacement $y \leftrightarrow z$ for some particular shifts, after which the other equation of the pair is obtained by property (1). This way we can get 8 pairs for each starting equation. One of our main results is that due to condition (3) this is in fact complete for H1, H3, Q1, Q3. (For H2 we have a further case, Q2, Q4 we did not check.) As for the CAC integrability test, for each choice of the bottom equations we could in principle have 8^2 possible side-equations. However, we find that only equations constructed with an *even* number of $y \leftrightarrow z$ replacements are possible, and for each such equation there are two sets of "side" equation pairs that produce (the same) genuine Bäcklund transformation and Lax pair.

1 Introduction

Within the topic of integrable discrete systems [14], equations that can be defined on a single quadrilateral of the Cartesian $\mathbb{Z} \times \mathbb{Z}$ lattice have been studied in great detail. One common equation type is defined by the following:

Definition. 1. [Acceptable one-component quad equations]

1.1 The equation depends on all corner variables of the elementary quadrilateral.

1.2 The equation is affine linear in each corner variable.

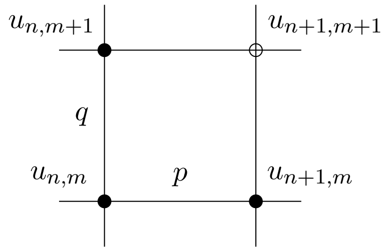

Figure 1. Corner variables on an elementary quadrilateral.

1.3 The equation is irreducible.

1.4 Uniformity: Every quadrilateral in the plane carries the same equation (depending on corresponding corner variables)

The geometric description is in Figure 1: subscript m labels the points in the vertical direction and n in the horizontal direction. The lattice parameters p, q are associated with horizontal and vertical directions, respectively. In practice we use shorthand notation in which a shift in the n-direction is indicated by a tilde, and in the m-direction by a hat

$$u_{n,m} = u, \quad u_{n+1,m} = \widetilde{u}, \quad u_{n,m+1} = \widehat{u}, \quad u_{n+1,m+1} = \widehat{\widetilde{u}}.$$

If the conditions in Definition **1** are satisfied one can define evolution starting from staircase- or corner-like initial conditions.

For lattice equations a necessary property for integrability is "Multidimensional Consistency". It means that the equations can be *consistently* extended into higher dimensions, which is related to the existence of a *hierarchy* of integrable continuous equations ([14], Sec 3.2). For 2D quad equations it means in practice Consistency-Around-a-Cube (CAC), that is, the original quad equation can be put on a 3D cube in a consistent way. Consider Figure 2 and assume that the original 2D lattice equation is on the bottom of the cube. Certain modifications of that equation are then placed on the back and left sides. Typically these equations are obtained by cyclic permutation:

$$\widetilde{\ } \to \widehat{\ } \to \overline{\ } \to \widetilde{\ } \qquad p \to q \to r \to p \qquad n \to m \to k \to n \tag{1}$$

where we have also introduced a bar to denote shift in the vertical direction, where steps are counted by k : $u_{n,m,k+1} = \overline{u}$. The equations on the opposing sides are obtained by the perpendicular shift. We then have 6 equations

$$\text{bottom:} \quad Q_{12}(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; p, q) = 0. \qquad \text{top:} \quad Q_{12}(\overline{u}, \overline{\widetilde{u}}, \overline{\widehat{u}}, \overline{\widehat{\widetilde{u}}}; p, q) = 0, \tag{2a}$$

$$\text{back:} \quad Q_{23}(u, \widehat{u}, \overline{u}, \overline{\widehat{u}}; q, r) = 0, \qquad \text{front:} \quad Q_{23}(\widetilde{u}, \widehat{\widetilde{u}}, \overline{\widetilde{u}}, \overline{\widehat{\widetilde{u}}}; q, r) = 0, \tag{2b}$$

$$\text{left:} \quad Q_{31}(u, \overline{u}, \widetilde{u}, \overline{\widetilde{u}}; r, p) = 0, \qquad \text{right:} \quad Q_{31}(\widehat{u}, \overline{\widehat{u}}, \widehat{\widetilde{u}}, \overline{\widehat{\widetilde{u}}}; r, p) = 0. \tag{2c}$$

The consistency problem arrives as follows: Take u, $\widetilde{u}$, $\widehat{u}$, $\overline{u}$ as initial values, then from bottom, back and left equations we can compute the values of $\widehat{\widetilde{u}}$, $\overline{\widehat{u}}$ and $\overline{\widetilde{u}}$, respectively. After these are substituted into the top, front and right equations we get independently 3 values for $\overline{\widehat{\widetilde{u}}}$ and these values must be the same. This introduces severe conditions.

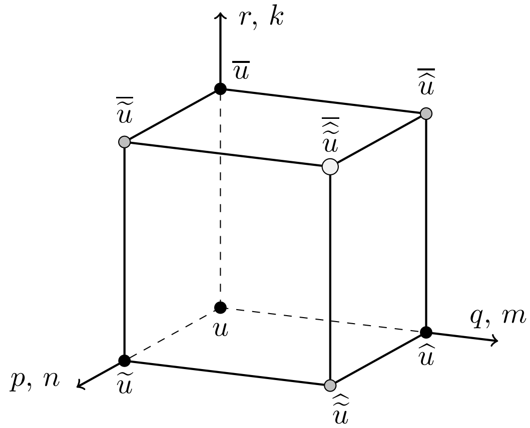

Figure 2. The consistency cube.

Several isolated examples of integrable quad-equations were found already in the 1980s by considering continuous equations and the permutability property of their Bäcklund transformations ([14], Sec. 2.4-5). A major development in this field was the classification of integrable quad-equations by Adler, Bobenko and Suris [1], under the assumptions of D4 symmetry and the "tetrahedron property". (The tetrahedron property was essential in the classification work. It states that the triply shifted quantity computed in three ways from (2) does not depend on u.) The result of this classification is the so-called "ABS-list", its main components being the H and Q lists:

H-list

$$\mathrm{H}_1: \quad (u-\widehat{\widetilde{u}})(\widehat{u}-\widetilde{u}) = p^2-q^2 \tag{3a}$$

$$\mathrm{H}_2: \quad (u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u}) = (p-q)(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}})+p^2-q^2 \tag{3b}$$

$$\mathrm{H}_3: \quad p(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})-q(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}}) = \delta^2(p^2-q^2) \tag{3c}$$

Q-list:

$$\mathrm{Q}_1: \quad p(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-q(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}}) = \delta^2 pq(q-p) \tag{4a}$$

$$\begin{aligned}\mathrm{Q}_2: \quad & p(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-q(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})+pq(p-q)(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}})\\ &= pq(p-q)(p^2-pq+q^2)\end{aligned} \tag{4b}$$

$$\begin{aligned}\mathrm{Q}_3: \quad & p(1-q^2)(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})-q(1-p^2)(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})\\ &= (p^2-q^2)\left((\widehat{u}\widetilde{u}+u\widehat{\widetilde{u}})+\delta^2\frac{(1-p^2)(1-q^2)}{4pq}\right)\end{aligned} \tag{4c}$$

$$\begin{aligned}\mathrm{Q}_4 \text{ from [11]}: \quad & sn(\alpha)(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}})-sn(\beta)(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}})-sn(\alpha-\beta)(\widetilde{u}\widehat{u}+u\widehat{\widetilde{u}})\\ &+k\, sn(\alpha)sn(\beta)sn(\alpha-\beta)(1+u\widetilde{u}\widehat{u}\widehat{\widetilde{u}}) = 0.\end{aligned} \tag{4d}$$

However, it is well known that there are other CAC-compatible equations if some conditions used by ABS are relaxed, for example $\widehat{\widetilde{u}}\,u-\widetilde{u}\,\widehat{u}=0$, which breaks the tetrahedron condition.

One of the assumptions used to generate the ABS list was that equations on opposing sides are related by the corresponding shift, as can be seen in (2). This assumption was relaxed in the work of Boll [3, 4], while still keeping the tetrahedron property. On the other hand, in the classification of Hietarinta [13] the tetrahedron assumption was not made but the search was restricted to equations that were quadratic homogeneous.

When we have a set of consistent equations on the sides of the cube one can use the “side” equations to construct a Lax pair or a Bäcklund transformation, which should generate the “bottom” equation (see e.g., [14], Sec. 3.3). But in [13] it was found that many equations can pass the CAC test without being integrable, in other words, sometimes the Lax pair generated from the side equations is trivial. This means that CAC is only a necessary test and must be verified by the existence of a genuine Lax or Bäcklund pair.

2 Previous work on two-component equations

Multi-component quad equations have also been studied and various types of equations have been proposed. For example discrete versions of the Boussinesq equations have been proposed, often in three-component form [12], but after eliminating one variable one obtains in some cases a two component form still on the elementary quadrilateral (e.g., [18], (4.8)). Several two-component equations were also proposed in [8]. However, none of these equations satisfy the exchange conditions **2.2** in Definition **2** below.

Furthermore in this paper we restrict our attention to equations that can be considered as multi-component generalizations of the equations in the ABS-list. One such equation was given in [5] (table 5, with name change $x \to y$, $y \to z$)

$$\begin{cases} (y-\hat{\tilde{y}})(\tilde{z}-\hat{z})-p^2+q^2=0, \\ (z-\hat{\tilde{z}})(\tilde{y}-\hat{y})-p^2+q^2=0. \end{cases} \tag{5}$$

Clearly the limit $z \to y$ (for any shift: none, tilde, hat, tilde-hat) takes both equations to H1. Furthermore the equations are related by $y \leftrightarrow z$ exchange for all shifts. Also note that in the first equation the *once shifted* variables have been changed by $y \to z$.

Is this equation integrable? At least it should have the CAC property. With (5) as the bottom equation we have to choose the side equations. It would be natural to try the cyclic rule (1), which produces

$$\text{bottom:} \quad \begin{cases} (y-\hat{\tilde{y}})(\tilde{z}-\hat{z})+q-p=0, \\ (z-\hat{\tilde{z}})(\tilde{y}-\hat{y})+q-p=0, \end{cases} \tag{6a}$$

$$\text{back:} \quad \begin{cases} (y-\bar{\hat{y}})(\hat{z}-\bar{z})+r-q=0, \\ (z-\bar{\hat{z}})(\hat{y}-\bar{y})+r-q=0, \end{cases} \tag{6b}$$

$$\text{left:} \quad \begin{cases} (y-\bar{\tilde{y}})(\bar{z}-\tilde{z})+p-r=0, \\ (z-\bar{\tilde{z}})(\bar{y}-\tilde{y})+p-r=0. \end{cases} \tag{6c}$$

This indeed passes the CAC test with the triply shifted variables being

$$\bar{\hat{\tilde{y}}} = \frac{p\tilde{y}(\bar{y}-\hat{y})+q\hat{y}(-\bar{y}+\tilde{y})+r\bar{y}(\hat{y}-\tilde{y})}{p(\bar{y}-\hat{y})+q(-\bar{y}+\tilde{y})+r(\hat{y}-\tilde{y})}, \tag{7a}$$

$$\bar{\hat{\tilde{z}}} = \frac{p\tilde{z}(\bar{z}-\hat{z})+q\hat{z}(-\bar{z}+\tilde{z})+r\bar{z}(\hat{z}-\tilde{z})}{p(\bar{z}-\hat{z})+q(-\bar{z}+\tilde{z})+r(\hat{z}-\tilde{z})}. \tag{7b}$$

Note that this has the tetrahedron property (no unshifted y, z) and that the y and z variables are separated in the final formulae.

The above construct in which variables with an odd number of shifts are exchanged, was described already in [2] as Toeplitz extension. It was generalized to all the equations in the ABS list in [9] by the same rule: exchanging the singly shifted variables, see also [16]. This approach was developed further by including other replacements by symmetry arguments [19]. Another approach in deriving multi-component versions of the ABS list was given in [17] where such equations were derived by from the star-triangle relations.

One result in [19] was the following two-component version of H1 (Eqs. (2.25), (2.28),(2.29)) consisting of

$$\text{bottom:} \qquad \begin{cases} (z-\widehat{\widetilde{y}})(\widetilde{z}-\widehat{y})+q-p=0, \\ (y-\widehat{\widetilde{z}})(\widetilde{y}-\widehat{z})+q-p=0, \end{cases} \tag{8a}$$

$$\text{back:} \qquad \begin{cases} (z-\overline{\widehat{y}})(\widehat{y}-\overline{z})+r-q=0, \\ (y-\overline{\widehat{z}})(\widehat{z}-\overline{y})+r-q=0, \end{cases} \tag{8b}$$

$$\text{left:} \qquad \begin{cases} (y-\overline{\widetilde{y}})(\overline{y}-\widetilde{y})+p-r=0, \\ (z-\overline{\widetilde{z}})(\overline{z}-\widetilde{z})+p-r=0, \end{cases} \tag{8c}$$

Here the bottom and back equations are related by cyclic permutation, but the left equation is of entirely different type, in fact separating the y and z variables. This peculiar combination passes the CAC test, and the triply shifted variables are

$$\overline{\widehat{\widetilde{y}}} = \frac{p\widetilde{z}(\widehat{y}-\overline{z})+q\widehat{y}(\overline{z}-\widetilde{z})+r\overline{z}(-\widehat{y}+\widetilde{z})}{p(\widehat{y}-\overline{z})+q(\overline{z}-\widetilde{z})+r(-\widehat{y}+\widetilde{z})}, \tag{9a}$$

$$\overline{\widehat{\widetilde{z}}} = \frac{p\widetilde{y}(\overline{y}-\widehat{z})+q\widehat{z}(-\overline{y}+\widetilde{y})+r\overline{y}(-\widetilde{y}+\widehat{z})}{p(\overline{y}-\widehat{z})+q(-\overline{y}+\widetilde{y})+r(-\widetilde{y}+\widehat{z})}, \tag{9b}$$

and they have the tetrahedron property.

3 Classification of two-component generalizations of the ABS list

The puzzling triplet (8) suggests that there may be interesting phenomena specific for two-component equations. The purpose of this paper is to search and classify such equations.

3.1 The domain of the search

Since the fully generic case of the problem is too hard to tackle we restrict our attention to equation pairs with the following properties:

Definition. 2 [Acceptable two-component quad equations]

2.1 Both equations of the pair are affine multilinear and irreducible.

2.2 Exchange rule: The two equations that form the pair are related by the exchange rule $y \leftrightarrow z$, $\widetilde{y} \leftrightarrow \widetilde{z}$, $\widehat{y} \leftrightarrow \widehat{z}$, $\widehat{\widetilde{y}} \leftrightarrow \widehat{\widetilde{z}}$.

2.3 Evolution: From the pair of equations one can solve for any of the corner variable pairs $\{y,z\},\{\widetilde{y},\widetilde{z}\}$, $\{\widehat{y},\widehat{z}\}$, $\{\widehat{\widetilde{y}},\widehat{\widetilde{z}}\}$

2.4 Strong multilinearity: When any resolved variable pair is written as a pair of polynomial equations, the polynomials are again multilinear and irreducible.

Remarks:

- We use $\bullet$ to indicate when the exchange rule **2.2** has been applied, That is, if B is obtained from A by the exchange rule we write $B=\overset{\bullet}{A}$. Obviously $(\overset{\bullet}{A})^{\bullet}=A$.

- Multilinearity does not imply unique evolution. Consider the pair

$$\begin{aligned} z\widehat{\widetilde{y}}+\widehat{y}\widehat{z}+2\widehat{y}\widetilde{z}+\widetilde{z}\widetilde{y} &= 0,\\ y\widehat{\widetilde{z}}+\widehat{y}\widehat{z}+2\widehat{z}\widetilde{y}+\widetilde{z}\widetilde{y} &= 0. \end{aligned}$$

  As given it is resolved for $\{\widehat{\widetilde{y}},\widehat{\widetilde{z}}\}$ and for $\{y,z\}$. However, if one tries solve for $\{\widetilde{y},\widetilde{z}\}$ or $\{\widehat{y},\widehat{z}\}$ there will be square roots and therefore evolution in the NW or SE direction is not uniquely determined. Note also that the one-component reduction of this pair is not multilinear.

- Multilinearity does not imply strong multilinearity. Consider the pair of equations

$$\begin{aligned} (2y-z-\widehat{\widetilde{y}})(2\widehat{z}-\widehat{y}-2\widetilde{z}+\widetilde{y}) &= p^2-q^2,\\ (2z-y-\widehat{\widetilde{z}})(2\widehat{y}-\widehat{z}-2\widetilde{y}+\widetilde{z}) &= p^2-q^2, \end{aligned}$$

  which is resolved for $\widehat{\widetilde{y}},\widehat{\widetilde{z}}$. It is obviously multilinear and reduces to H1. When this pair is solved for $\{\widetilde{y},\widetilde{z}\}$ one obtains the pair

$$\begin{aligned} 3(\widehat{y}-\widetilde{y})(2y-\widehat{\widetilde{y}}-z)(y-2z+\widehat{\widetilde{z}}) &= (p^2-q^2)(2\widehat{\widetilde{y}}+\widehat{\widetilde{z}}-3y),\\ 3(\widehat{z}-\widetilde{z})(2z-\widehat{\widetilde{z}}-y)(z-2y+\widehat{\widetilde{y}}) &= (p^2-q^2)(2\widehat{\widetilde{z}}+\widehat{\widetilde{y}}-3z), \end{aligned}$$

  which is resolved for both $\{\widetilde{y},\widetilde{z}\}$ and $\{\widehat{y},\widehat{z}\}$. However, this is not multilinear because y and z appear quadratically. (As a consequence, if we attempt to resolve for y,z from this pair there is a superfluous solution.)

3.2 Results for acceptable pairs

Proposition 3.1. *1. For any given equation in the ABS list one can get a two component version satisfying the conditions in Definition* **2** *by applying to the original equation any* *one of the following eight replacements*

$$0: \quad \textit{none}, \tag{10a}$$
$$1: \quad y \to z, \tag{10b}$$
$$2: \quad \widetilde{y} \to \widetilde{z}, \tag{10c}$$
$$3: \quad \widehat{y} \to \widehat{z}, \tag{10d}$$
$$4: \quad y \to z,\ \widetilde{y} \to \widetilde{z}, \tag{10e}$$
$$5: \quad y \to z,\ \widehat{y} \to \widehat{z}, \tag{10f}$$
$$6: \quad \widetilde{y} \to \widetilde{z},\ \widehat{y} \to \widehat{z}, \tag{10g}$$
$$7: \quad y \to z,\ \widetilde{y} \to \widetilde{z},\ \widehat{y} \to \widehat{z}, \tag{10h}$$

after which the other member of the pair is obtained by the exchange rule **2.2**

2. For H1, H3, Q1 and Q3 this result is complete.

For H2 we have a counterexample on completeness, given below, while for Q2 and Q4 uniqueness is open.

Proof. 1. It is easy to verify that from an equation in the ABS list, any of the substitutions (10) results in a pair satisfying all properties of Definition **2**.

2. It is a bit more laborious to show that there are no others. For this purpose we generate multilinear equations (with arbitrary coefficients) for all four resolutions, i.e. equation pairs of the type

$$\begin{cases} \widehat{\widetilde{y}}\, L_1(y,z,\widetilde{y},\widetilde{z},\widehat{y},\widehat{z}) + P_1(y,z,\widetilde{y},\widetilde{z},\widehat{y},\widehat{z}) + C = 0, \\ \widehat{\widetilde{z}}\, L_1(z,y,\widetilde{z},\widetilde{y},\widehat{z},\widehat{y}) + P_1(z,y,\widetilde{z},\widetilde{y},\widehat{z},\widehat{y}) + C = 0, \end{cases} \tag{11a}$$

$$\begin{cases} \widetilde{y}\, L_2(y,z,\widehat{y},\widehat{z},\widehat{\widetilde{y}},\widehat{\widetilde{z}}) + P_2(y,z,\widehat{y},\widehat{z},\widehat{\widetilde{y}},\widehat{\widetilde{z}}) + C = 0, \\ \widetilde{z}\, L_2(z,y,\widehat{z},\widehat{y},\widehat{\widetilde{z}},\widehat{\widetilde{y}}) + P_2(z,y,\widehat{z},\widehat{y},\widehat{\widetilde{z}},\widehat{\widetilde{y}}) + C = 0, \end{cases} \tag{11b}$$

$$\begin{cases} \widehat{y}\, L_3(y,z,\widetilde{y},\widetilde{z},\widehat{\widetilde{y}},\widehat{\widetilde{z}}) + P_3(y,z,\widetilde{y},\widetilde{z},\widehat{\widetilde{y}},\widehat{\widetilde{z}}) + C = 0, \\ \widehat{z}\, L_3(z,y,\widetilde{z},\widetilde{y},\widehat{\widetilde{z}},\widehat{\widetilde{y}}) + P_3(z,y,\widetilde{z},\widetilde{y},\widehat{\widetilde{z}},\widehat{\widetilde{y}}) + C = 0, \end{cases} \tag{11c}$$

$$\begin{cases} y\, L_4(\widetilde{y},\widetilde{z},\widehat{y},\widehat{z},\widehat{\widetilde{y}},\widehat{\widetilde{z}}) + P_4(\widetilde{y},\widetilde{z},\widehat{y},\widehat{z},\widehat{\widetilde{y}},\widehat{\widetilde{z}}) + C = 0, \\ z\, L_4(\widetilde{z},\widetilde{y},\widehat{z},\widehat{y},\widehat{\widetilde{z}},\widehat{\widetilde{y}}) + P_4(\widetilde{z},\widetilde{y},\widehat{z},\widehat{y},\widehat{\widetilde{z}},\widehat{\widetilde{y}}) + C = 0, \end{cases} \tag{11d}$$

where L_j are linear and P_j quadratic multilinear polynomials in the indicated variables. Next some coefficients in L_j, P_j are fixed by the condition that the $z \mapsto y$ reduction leads to one of the equations H1, H3, Q1, Q3. This still leaves 3 free coefficients in each L_j and 9 in P_j. The pairs in (11) describe the same evolution and therefore if we solve $\{\widehat{\widetilde{y}}, \widehat{\widetilde{z}}\}$ from (11a), say, and substitute to the other equations they should all vanish. This leads to 384 smallish equations, which can be solved by starting with the simplest ones and proceeding step by step. This is not difficult, only tedious. For each of the equations H1, H3, Q1, Q3 the solution process eventually splits into eight branches as listed in (10). ∎

3.2.1 H2

For H2 we found an equation that does not fit into the result of Proposition 3.1:

$$\begin{aligned}
&(\widehat{\widetilde{y}} - (y+z)/2)(\widehat{y} - \widetilde{y} + \widehat{z} - \widetilde{z}) \\
&+ \nu_1\,(y + \widetilde{y} + z + \widetilde{z} + 2\,\epsilon\,p)(\widehat{y} - \widehat{z}) \\
&+ \nu_2\,(y + \widehat{y} + z + \widehat{z} + 2\,\epsilon\,q)(\widetilde{y} - \widetilde{z}) \\
&+ \nu_3\,(\widehat{y} - \widetilde{y} + \widehat{z} - \widetilde{z} - 2\,\epsilon\,(p-q))(y-z) \\
&- \epsilon\,(p-q)(2\,\widehat{\widetilde{y}} + y + z + \widehat{y} + \widetilde{y} + \widehat{z} + \widetilde{z}) \\
&- 2\epsilon^2(p^2 - q^2) = 0,
\end{aligned} \tag{12}$$

together with its $z \leftrightarrow y$ reflection. This pair satisfies the strong multilinearity condition if all parameters ν_i are nonzero. It reduces to H2 when all $z = y$, and clearly the parameters ν_j disappear in this reduction. We do not know whether (12) is integrable or linearizable.

4 Integrability

4.1 Integrability by CAC

The example (6) shows that if one uses the replacement rule 6 and its cyclic variants for the bottom back and left equations (which we denote as $(6,6,6)$) the system has CAC property. On the other hand in example (8) the replacements are given by $(4,5,0)$. The question then arises as to which combinations among the 8^3 possibilities have the CAC property. The result is as follows:

Proposition 4.1. *For each equation in the ABS list the following eight replacement rules have the CAC property: (0,0,0), (0,4,5), (4,5,0), (4,6,5), (5,0,4), (5,4,6), (6,5,4), (6,6,6), where the numbers in the triplet are the replacement rules used for bottom, back, and left equations on the cube. The rules given in* (10) *must be modified cyclically to fit the corresponding side. The top, front and right equations are obtained by a perpendicular shift.*

Proof. By direct computation. Since there are no free parameters the computations for the 512 cases are easy to automatize. ■

Remarks:

- The only replacements appearing in the list are 0,4,5,6, which correspond to replacements of even number of variables.
- The cases (0,4,5), (4,5,0), (5,0,4), are related by rotation around the $(y,z) - (\overline{\widehat{\widetilde{y}}}, \overline{\widehat{\widetilde{z}}})$ axis, the same holds for (4,6,5), (6,5,4), (5,4,6). There are therefore only four essentially different triplets.
- The fact that there are two kinds of side equation pairs for each bottom equation pair follows from the $y \leftrightarrow z$ symmetry. For if we do this exchange only on the variables with a bar-shift, the top pair does not change (and neither does the bottom pair), but the side equations will change.

- Our end result agrees with the result of [19], which was derived by an entirely different approach.

4.2 Decoupling

It is easy to see that in the (0,0,0) case the equations are decoupled, since in each pair one equation depends only on the y variables and the other only on the z variables. We will now look whether the other sets can also be decoupled somehow.

Since the CAC analysis is completely algebraic the variable names do not matter: instead of $y, z, \widetilde{y}, \widetilde{z}, \widehat{y}, \dots$ we could have used $a, b, c, d, \dots$ and the algebra would have been the same.

In the case of (4,5,0) given in (8) we see that (8c) is already decoupled and we can separate variables into two set, $S_a = \{y, \widetilde{y}, \overline{y}, \overline{\widetilde{y}}\}$ and $S_b = \{z, \widetilde{z}, \overline{z}, \overline{\widetilde{z}}\}$. Insisting that any particular equation depends only on variables from one set, we can augment these sets using (8) and its shifted version to

$$(4,5,0): \qquad S_a = \{y, \widetilde{y}, \overline{y}, \overline{\widetilde{y}}, \widehat{z}, \overline{\widehat{z}}, \widehat{\widetilde{z}}, \overline{\widehat{\widetilde{z}}}\}, \quad S_b = \{z, \widetilde{z}, \overline{z}, \overline{\widetilde{z}}, \widehat{y}, \overline{\widehat{y}}, \widehat{\widetilde{y}}, \overline{\widehat{\widetilde{y}}}\}. \tag{13a}$$

Thus there are 6 equations depending on variables from S_a and they satisfy CAC all be themselves, similarly for S_b. It seems that all equations that pass the CAC test do decouple in the described manner. For example while (6,5,4) passes CAC and decouples, (6,4,5) does not decouple nor pass CAC. The sets for the other integrable combinations are as follows:

$$(0,0,0): \qquad S_a = \{y, \widetilde{y}, \overline{y}, \overline{\widetilde{y}}, \widehat{y}, \overline{\widehat{y}}, \widehat{\widetilde{y}}, \overline{\widehat{\widetilde{y}}}\}, \quad S_b = \{z, \widetilde{z}, \overline{z}, \overline{\widetilde{z}}, \widehat{z}, \overline{\widehat{z}}, \widehat{\widetilde{z}}, \overline{\widehat{\widetilde{z}}}\}, \tag{13b}$$

$$(6,5,4): \qquad S_a = \{y, \widetilde{z}, \overline{y}, \overline{\widetilde{z}}, \widehat{z}, \overline{\widehat{z}}, \widehat{\widetilde{y}}, \overline{\widehat{\widetilde{y}}}\}, \quad S_b = \{z, \widetilde{y}, \overline{z}, \overline{\widetilde{y}}, \widehat{y}, \overline{\widehat{y}}, \widehat{\widetilde{z}}, \overline{\widehat{\widetilde{z}}}\}, \tag{13c}$$

$$(6,6,6): \qquad S_a = \{y, \widetilde{z}, \overline{z}, \overline{\widetilde{y}}, \widehat{z}, \overline{\widehat{y}}, \widehat{\widetilde{y}}, \overline{\widehat{\widetilde{z}}}\}, \quad S_b = \{z, \widetilde{y}, \overline{y}, \overline{\widetilde{z}}, \widehat{y}, \overline{\widehat{z}}, \widehat{\widetilde{z}}, \overline{\widehat{\widetilde{y}}}\}. \tag{13d}$$

These can be described as follows: (0,0,0) is decoupled as it stands; (4,5,0) is decoupled if for odd number of hat-shits we exchange $z \leftrightarrow y$; (6,5,4) can be decoupled if for odd total number of tilde and hat-shifts we exchange, while bar shift has no effect; for (6,6,6) exchange is needed when the total number of all kinds of shifts is odd.

If the exchanges needed for decoupling are transferred to a property of the lattice itself, then for (0,0,0) the lattice is uniform; for (4,5,0) we should change the lattice on alternate planes in the hat direction; for (6,5,4) we have checkerboard lattice in tilde and hat direction without change in bar direction; for (6,6,6) we need a lattice that is alternating in every direction.

The possibility of decoupling follows in part from our assumption that the two equations are related by $y \leftrightarrow z$ replacement. But there are other two-component equation pairs for which decoupling is not possible, for example the discrete Boussinesq equation given e.g. as the pair (3.3) of [15] has $\widetilde{w}, \widetilde{z}, \widehat{w}, \widehat{z}$ in both equations of the pair.

The above decoupling approach explains the algebra quite well, but in practice the variables are not featureless symbols but the tilde-hat-bar decorations have a dynamic meaning. That is, if we have a solution $y = y(n, m, k)$, $z = z(n, m, k)$ for the set of equations then $\widehat{y}$, $\widehat{z}$ are obtained by changing $m \to m + 1$ in the concrete formula for y and z. If the y and z solutions are different (for example solitons traveling in different directions) it may be better to keep the dependent variables uncoupled and the equations coupled than vice versa.

4.3 Integrability by BT

It has been observed that CAC is necessary but not always sufficient for integrability [13]. However, the existence of a genuine Bäcklund Transformation (BT) (or equivalently, a nontrivial Lax pair) is a proof of integrability.

Proposition 4.2. *For each equation in the ABS list the following eight replacement rules (0,0,0), (0,4,5), (4,5,0), (5,0,4), (4,6,5), (6,5,4), (5,4,6), (6,6,6), generate a genuine BT. That is, if one assigns any of the listed triplet replacements on the bottom, back, and left pairs of equations (or their cyclic permutations), then one can freely choose two pairs of "side" equations for BT and together with their perpendicular shift equations they generate by variable elimination the third pair.*

For example if the bottom pair is generated by replacement 5, back pair by 4 then they together generate left pair of type 6, after three of the four variables on the right pair are eliminated. The same left pair is also generated if bottom and back equations are both generated by replacement 6.

4.4 The Lax pair

One of the more important proofs of integrability is by construction of a Lax pair. But here one must note that there are "fake" Lax pairs [7, 13] and therefore one must verify that the Lax pair is genuine and able to generate the equation(s) in question. The general formula for constructing Lax pairs from a CAC consistent system is given e.g., in [14] Section 3.3.1, and furthermore there are now even computer programs that can do that [5, 6].

It is perhaps sufficient to consider an example, for which we choose type 5 Q1. As noted before type 5 bottom equation goes together with side equations 4,6 as well as 0,4.

4.4.1 Example: Q1, sides 4,6 generate bottom 5

We construct the Lax pair for Q1;5 from a back equation pair of type 4

$$q(z-\overline{y})(\widehat{z}-\overline{\widehat{y}})-r(z-\widehat{z})(\overline{y}-\overline{\widehat{y}})=\delta^2 qr(r-q), \tag{14a}$$
$$q(y-\overline{z})(\widehat{y}-\overline{\widehat{z}})-r(y-\widehat{y})(\overline{z}-\overline{\widehat{z}})=\delta^2 qr(r-q), \tag{14b}$$

and a left equation of type 6

$$r(y-\widetilde{z})(\overline{z}-\overline{\widetilde{y}})-p(y-\overline{z})(\widetilde{z}-\overline{\widetilde{y}})=\delta^2 rp(p-r), \tag{14c}$$
$$r(z-\widetilde{y})(\overline{y}-\overline{\widetilde{z}})-p(z-\overline{y})(\widetilde{y}-\overline{\widetilde{z}})=\delta^2 rp(p-r), \tag{14d}$$

Together they should generate a bottom equation of type 5, i.e.,

$$p(z-\widehat{z})(\widetilde{y}-\widehat{\widetilde{y}})-q(z-\widetilde{y})(\widehat{z}-\widehat{\widetilde{y}})=\delta^2 pq(q-p), \tag{14e}$$
$$p(y-\widehat{y})(\widetilde{z}-\widehat{\widetilde{z}})-q(y-\widetilde{z})(\widehat{y}-\widehat{\widetilde{z}})=\delta^2 pq(q-p). \tag{14f}$$

In order to construct the Lax pair we solve the first 4 equations of (14) for the double shifted quantities and then replace the barred quantities as follows:

$$\overline{y}=\frac{f}{k},\quad \overline{z}=\frac{g}{l},\quad \widetilde{\overline{y}}=\frac{\widetilde{f}}{\widetilde{\overline{k}}},\quad \widetilde{\overline{z}}=\frac{\widetilde{g}}{\widetilde{\overline{l}}},\quad \widehat{\overline{y}}=\frac{\widehat{f}}{\widehat{\overline{k}}},\quad \widehat{\overline{z}}=\frac{\widehat{g}}{\widehat{\overline{l}}}.$$

For the left equations this leads to

$$\frac{\widetilde{f}}{\widetilde{\overline{k}}} = \frac{g[p\widetilde{z}+r(y-\widetilde{z})]+pl[\delta^2 r(r-p)-y\widetilde{z}]}{pg+l[-py+r(y-\widetilde{z})]}, \tag{15a}$$

$$\frac{\widetilde{g}}{\widetilde{\overline{g}}} = \frac{f[p\widetilde{y}+r(z-\widetilde{y})]+pk[\delta^2 r(r-p)-z\widetilde{y}]}{pf+k[-pz+r(z-\widetilde{y})]}. \tag{15b}$$

These can be written in matrix form

$$\widetilde{\psi}=\mathcal{L}_{Q1;6}\,\psi \quad \text{where} \quad \psi=(f,k,g,l)^T \quad \text{and} \quad \mathcal{L}_{Q1;6}=\begin{pmatrix}\mathbf{0} & \mathbf{L}\\ \dot{\mathbf{L}} & \mathbf{0}\end{pmatrix} \quad \text{with} \tag{16a}$$

$$\mathbf{L}=\lambda\begin{pmatrix}p\widetilde{z}+r(y-\widetilde{z}) & p[\delta^2 r(r-p)-y\widetilde{z}]\\ p & -py+r(y-\widetilde{z})\end{pmatrix}, \tag{16b}$$

where the parameter λ is the splitting factor (and may depend on $y,\widetilde{z}$ and therefore it is possible that $\lambda\neq\dot{\lambda}$).

Similarly, from the back equation we get

$$\widehat{\psi}=\mathcal{M}_{Q1;4}\,\psi \quad \text{where} \quad \mathcal{M}_{Q1;4}=\begin{pmatrix}\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \dot{\mathbf{M}}\end{pmatrix} \quad \text{with} \tag{17a}$$

$$\mathbf{M}=\kappa\begin{pmatrix}q\widehat{z}+r(z-\widehat{z})] & q[\delta^2 r(r-q)-z\widehat{z}]\\ q & -qz+r(z-\widehat{z})]\end{pmatrix}. \tag{17b}$$

The commutativity condition $\widehat{\mathcal{L}}\mathcal{M}=\widetilde{\mathcal{M}}\mathcal{L}$ now implies

$$\widehat{\mathbf{L}}\,\dot{\mathbf{M}}=\widetilde{\mathbf{M}}\,\mathbf{L},\qquad \text{or equivalently}\quad \widehat{\dot{\mathbf{L}}}\,\mathbf{M}=\widetilde{\dot{\mathbf{M}}}\,\dot{\mathbf{L}}.$$

In order to get the equations from this we have to fix the separations constants λ, κ. One elegant way to do that is to require $\det\mathcal{L}=\det\mathcal{M}=1$. Here it leads to

$$\lambda^2 =1/[(\delta^2p^2-(y-\widetilde{z})^2)(p-r)r],\quad \dot{\lambda}^2=1/[(\delta^2p^2-(z-\widetilde{y})^2)(p-r)r], \tag{18a}$$

$$\kappa^2 =1/[(\delta^2q^2-(y-\widehat{y})^2)(q-r)r],\quad \dot{\kappa}^2=1/[(\delta^2p^2-(z-\widehat{z})^2)(q-r)r]. \tag{18b}$$

From these one can relatively easily derive $(\widetilde{\kappa}\dot{\kappa}\,q)^2=(\lambda\widehat{\lambda}p)^2$, but in practice we need

$$\widetilde{\kappa}\dot{\kappa}\,q=\lambda\widehat{\lambda}p, \tag{19}$$

and its exchanged version, in order to derive the bottom $Q1;5$-equation. (The apparent asymmetry in (19) is due to the block structure of the Lax matrices).

4.4.2 Example: Q1, sides 0,4 generate bottom 5

A back equation pair of type 0 is given by

$$q(y-\overline{y})(\widehat{y}-\widehat{\overline{y}})-r(y-\widehat{y})(\overline{y}-\widehat{\overline{y}})=\delta^2 qr(r-q), \tag{20a}$$

$$q(z-\overline{z})(\widehat{z}-\widehat{\overline{z}})-r(z-\widehat{z})(\overline{z}-\widehat{\overline{z}})=\delta^2 qr(r-q), \tag{20b}$$

and a left equation of type 4

$$r(z-\widetilde{y})(\overline{z}-\widetilde{\overline{y}})-p(z-\overline{z})(\widetilde{y}-\widetilde{\overline{y}})=\delta^2 rp(p-r), \tag{20c}$$

$$r(y-\widetilde{z})(\overline{y}-\widetilde{\overline{z}})-p(y-\overline{y})(\widetilde{z}-\widetilde{\overline{z}})=\delta^2 rp(p-r). \tag{20d}$$

Now comparing equations (14) and (20) we find that the sets are the same if we exchange all barred quantities and only them: $\overline{y}\leftrightarrow\overline{z}$, $\widetilde{\overline{y}}\leftrightarrow\widetilde{\overline{z}}$, $\widehat{\overline{y}}\leftrightarrow\widehat{\overline{z}}$. From the Lax matrix point of view this means permuting the blocks, i.e.,

$$\mathcal{L}_{Q1;0}=\begin{pmatrix}\mathbf{0} & \dot{\mathbf{L}}\\ \mathbf{L} & \mathbf{0}\end{pmatrix},\quad \mathcal{M}_{Q1;5}=\begin{pmatrix}\dot{\mathbf{M}} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{pmatrix},$$

while keeping the previous definitions (16b) and (17b). Thus we end up with the same conditions.

5 Conclusions

In this paper we have searched for two-component versions of the equations in the ABS list. In addition to the standard assumptions placed on quad equations we assumed the following (the dependent variables are named y, z)

1. The two equations forming the pair are related by $y\leftrightarrow z$ exchange (for all shifts).
2. When $z=y$ both equations reduce to one of the equations in the ABS list.
3. Evolution in any corner direction is by a pair of multilinear equations.

Condition 3 in more detail: one must be able to solve for any corner variable pair (e.g., $\widetilde{y}=A/B$, $\widetilde{z}=C/D$) and when this is written as equations (e.g., $B\widetilde{y}-A=0$, $D\widetilde{z}-C=0$) they must be multilinear in all of the dependent variables. We call this *strong multilinearity*.

The above conditions turn out to be quite strong and as a result we found that the only possibility is that the pair of equations is obtained from the original one component equation by a simple replacement (Proposition 3.1), the only caveats are H2, for which we have a counterexample, and Q2 and Q4 which we did not check.

Since one can get several candidate pairs for each original equation there rises a question related to multidimensional consistency, namely how we should populate the sides of the consistency cube. We found eight combinations that satisfy the CAC-condition, as described in Proposition 4.1. Our end result is essentially the same as the one obtained in [19] by symmetry arguments. Note also that if the CAC condition is satisfied for some bottom equation, it is satisfied with two different sets of side equations. Also, both pairs of side equations work as a Bäcklund transformations generating the same bottom equation, and from both pairs one gets the same Lax pair. We gave the details for Q1 of type 5.

With the above equations one can ask some natural questions which include: what are their semi-continuous and fully continuous limits, and what are their soliton solutions, in particular how do the different components of a soliton solution interact.

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