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On the Propagation of Equatorial Waves Interacting With a Non-Uniform Current

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Abstract

We consider the propagation of equatorial waves of small amplitude, in a flow with an underlying non-uniform current. Without making the too restrictive rigid-lid approximation, by exploiting the available Hamiltonian structure of the problem, we derive the dispersion relation for the propagation of coupled long-waves: a surface wave and an internal wave. Also, we investigate the above-mentioned model of wave-current interactions in the general case with arbitrary vorticities.

Keywords : Wave-current interactions; Hamiltonian formulation;
Mathematics Subject Classification (2010): 35Q35 · 37N10 · 37K05

1 Introduction

The propagation of ocean waves in equatorial regions presents great physical relevance and offers many mathematical challenges. For example, the equatorial region of the Pacific extends over 13000 km (about one third of its total length) and it is responsible for one of the most important climate phenomena: the El Nino events, starting every few years and having a global impact several months afterwards (see the discussion in [12]). Moreover, the ocean flow in this region is characterized by underlying depth-varying currents (westward at the surface and eastward at about 100-200 m depth, while at great depths the water is almost still – see [1]). A further complication is the fact that equatorial ocean regions present the strongest stratification, with a thermocline that is quite well-defined, separating two layers of practically constant density. The study of wave-current interactions in equatorial regions is of great current interest (see [3], [4], [5], [11] and references therein). As we noted above the El Nino events is one of the most important climate phenomena. In the present paper we consider the model of wave-current interactions (see [3]) with parameters which correspond to El Nino case. Also, we investigate the general case with arbitrary vorticities (the case of only non-zero constant vorticity in the near-surface homogeneous layer have been considered in [15]).

2 Preliminaries

Near the Equator the Coriolis force acts like a waveguide, inducing the azimuthal propagation of waves and currents, and we can therefore investigate two-dimensional flows in the f-plane approximation. Following the setting of the recent paper [3], we consider equations of motion of the fluid in the domain bounded by flat bed $z = -h$ from below and with the upper boundary $z = h_1 + \eta_1(x,t)$ which is a free surface of elevation (where $h \approx 8$ and $h_1 \approx 0.24$ are some non-dimensional constants, which reflect the fact that the near-surface layer is about 100-200 m deep, the average depth of the Pacific ocean near the Equator being about 4km). The thermocline $z = \eta(x,t)$ separates the two layers of different constant densities in the domain under consideration. Thus, the thermocline divides the domain into two parts: the deep (colder) layer

$$D(t) = \{(x,z) : -h < z < \eta(x,t)\}$$

and the relatively shallow near-surface layer

$$D_1(t) = \{(x,z) : \eta(x,t) < z < h_1 + \eta_1(x,t)\}.$$

We consider an inviscid setting, in which the wind effects are captured by the wind-drift near-surface current of constant vorticity (see the data in [10] for the reasonable assumption that wind-generated currents are appropiately described by the assumption of constant vorticity, and note that non-zero vorticity means non-uniform currents – see the considerations in [7]). Therefore, above the thermocline, in the region $D_1(t)$ the equations of motion are Euler's equations

$$u_{1,t} + u_1 u_{1,x} + \varpi_1 u_{1,z} + \omega\varpi_1 = -p_x$$

$$\varpi_{1,t} + u_1 \varpi_{1,x} + \varpi_1 \varpi_{1,z} - \omega u_1 = -p_z - g,$$

where (u_1, ϖ_1) is the velocity field in upper region, p is the pressure, $\omega = 0,15$ is the non-dimensional constant that captures the Coriolis effect due to the Earth's rotation and $g \approx 2 \times 10^4$ is the non-dimensional gravity. Taking into account the equations of mass conservation (for constant density) and the vorticity distribution typical of mixing in the near-surface layer, we have

$$u_{1,x} + \varpi_{1,z} = 0$$

$$u_{1,z} - \varpi_{1,x} = \gamma_1,$$

where $\gamma_1 < 0$ is the vorticity (constant in the near-surface layer, the sign corresponding to the westward trade winds in the equatorial Pacific, for an eastward orientation of the horizontal axis, as in the paper [3]).

Denoting with (u, ϖ) the velocity field in lower region $D(t)$ the equations of motion are again Euler's equations

$$u_t + uu_x + \varpi u_z + \omega\varpi = -\frac{1}{1+r}p_x,$$

$$\varpi_t + u\varpi_x + \varpi\varpi_z - \omega u = -\frac{1}{1+r}p_z - g,$$

with $r \ll 1$ a constant that takes into account the stable density stratification (the less dense, warmer, fluid overlying the abyssal colder fluid). Below the thermocline, in the region $D(t)$, the equations of mass conservation and vorticity distribution can be written as follows:

$$u_x + \varpi_z = 0,$$

$$u_z - \varpi_x = \gamma,$$

where $\gamma >$ is the vorticity, constant in the region below the thermocline (thus capturing the eastward orientation of the Equatorial Undercurrent, which occurs practically from about 100-200 m depth to 500m depth: see the data provided in [4] and [5].

We now present the relevant boundary conditions. Firstly, at the free surface $z = h_1 + \eta_1(x,t)$ the pressure is the constant atmospheric pressure, so that we complement the above-metioned equations with the following the dynamic and kinematic boundary conditions

$$\varpi_1 = \eta_{1,t} + u_1\eta_{1,x} \text{ on } z = h_1 + \eta_1(x,t),$$

$$\varpi_1 = \eta_t + u_1\eta_x \text{ on } z = \eta(x,t),$$

$$\varpi = \eta_t + u\eta_x \text{ on } z = \eta(x,t),$$

$$\varpi = 0 \text{ on } z = -h.$$

Besides, as noted in [3], the available field data for equatorial flows suggests a continuous transition between the two layers (velocity discontinuities across the thermocline are not detected in measurements), so that that it is required that a tangential velocity balance holds:

$$\varpi_1\eta_x + u_1 = \varpi\eta_x + u \text{ on } z = \eta(x,t)\,,$$

in addition to the condition that the pressure is continuous across the thermocline $z = \eta(x,t)$.

The complicated nature of the governing equations described above is to some degree compensated by the fact that they have a Hamiltonian structure (see the discussion in [3]). We recall that Zakharov's discovery [17] of the Hamiltonian formulation for irrotational deep-water gravity waves represented an important advance, enabling in-depth studies (see the discussion in [8]). Subsequently Hamiltonian formulations for irrotational internal waves were provided in [9], for waves with constant vorticity [16] and for wave-current interactions in stratified rotational flows with piecewise constant vorticity (see [6] and [3]). In all these results, the key idea is that for harmonic functions one reduce the two-dimensions to one (on the boundary) by means of Dirichlet-Neumann operators (scalar with no stratification, but if there is a thermocline, one needs a coupling of a scalar and a matrix Dirichlet-Neumann operator – see the discussion in [9], [6] and [3]). For this to be analytically tractable, one needs the assumption of irrotational flow (as in [9]), or of constant vorticity (as in [6] and [3], where a nonlinear separation result was proved, showing that, at the level of the nonlinear governing equations, wave-current interactions for flows with constant vorticity correspond to a harmonic wave on a non-uniform pure-current background state). To take advantage of this feature, we define

$$\Phi(x,t) = \varphi(x,\eta(x,t),t), \qquad \Phi_1(x,t) = \varphi_1(x,\eta(x,t),t)\,,$$

and

$$\Phi_2(x,t) = \varphi_1(x,h_1+\eta_1(x,t),t)\,,$$

where φ in $D(t)$ and φ_1 in $D_1(t)$ are harmonic perturbed velocity potentials which are defined as follows:

$$\begin{cases} u = \varphi_x + \gamma\,(z+h) \quad \text{and} \quad \varpi = \varphi_z \quad \text{in} \quad D(t), \\ u_1 = \varphi_{1,x} + \gamma_1 z + \gamma h \quad \text{and} \quad \varpi_1 = \varphi_{1,z} \quad \text{in} \quad D(t). \end{cases}$$

Then, let

$$\mathfrak{u} = (\eta,\eta_1,\xi,\xi_1)^T\,,$$

where the superscript denotes the transpose and

$$\xi = (1+r)\,\Phi - \Phi_1\,, \quad \xi_1 = \Phi_2\,,$$

with a positive constant r (defined above; typically $r = 10^{-3}$ – see [12]).

Using the Fourier transform

$$\hat{f} = \int_{\mathbb{R}} f\,(x)\,e^{-ikx}dx$$

for f in the Schwartz class $S(\mathbb{R})$ in each component of $\mathfrak{u}$ above-mentioned linearised equations of motion are transformed for any fixed $k \in R$ into the linear autonomous system of ordinary differential equations (see [3]):

$$\partial_t \overset{\wedge}{\mathfrak{u}}\,(k,t) = M\,(k)\,\overset{\wedge}{\mathfrak{u}}\,(k,t)\,, \tag{1}$$

where

$$M(k) = \begin{pmatrix} -i\gamma hk + i\mu k\Theta\,(k) & i\mu_1 k\,\sec h\,(h_1 k)\,\Theta\,(k) & k^2\Theta\,(k) & k^2 \sec h\,(h_1 k)\,\Theta\,(k) \\ i\mu k \sec h\,(h_1 k)\,\Theta\,(k) & -i\Gamma_1 k + i\mu_1 k\,\Theta_1\,(k) & k^2 \sec h\,(h_1 k)\,\Theta\,(k) & k^2\Theta_1\,(k) \\ \Gamma - \mu^2\Theta\,(k) & -\mu\mu_1 \sec h\,(h_1 k)\,\Theta\,(k) & -i\gamma hk + i\mu k\Theta\,(k) & i\mu k \sec h\,(h_1 k)\,\Theta\,(k) \\ -\mu\mu_1 \sec h\,(h_1 k)\,\Theta\,(k) & \omega\Gamma_1 - g - \mu_1^2\Theta_1\,(k) & i\mu_1 k \sec h\,(h_1 k)\,\Theta\,(k) & -i\Gamma_1 k + i\mu_1 k\Theta_1\,(k) \end{pmatrix}$$

and

$$\begin{aligned} \mu &= \tfrac{(1+r)\gamma - \gamma_1 + r\omega}{2}\,, \qquad \mu_1 = \tfrac{\gamma_1+\omega}{2}\,, \\ \omega &= 0,15\,, \qquad \Gamma = -r\hat{g}\,, \qquad \hat{g} = g - \omega h\gamma, \\ \Theta\,(k) &= \tfrac{\tanh(hk)}{k[1+r+\tanh(hk)\tanh(h_1 k)]}\,, \\ \Theta_1\,(k) &= \tfrac{\tanh(hk)+(1+r)\tanh(h_1 k)}{k[1+r+\tanh(hk)\tanh(h_1 k)]}\,. \end{aligned}$$

Here a fixed nondimensional value of k corresponds to a harmonic oscillation of the free surface and/or of the thermocline of wavelength $\frac{1000\pi}{|k|}$ m. Therefore we have that the unique solution to (1) with initial data

$$\overset{\wedge}{\mathfrak{u}}_0\,(k) = \int_{\mathbb{R}} \mathfrak{u}_0\,(x)\,e^{ikx}dx,$$

is

$$\overset{\wedge}{\mathfrak{u}}(k,t) = e^{M(k)t}\overset{\wedge}{\mathfrak{u}}_0(k)$$

for $t \geq 0$. This solution corresponds, by means of the inverse Fourier transform, to the solution

$$\mathfrak{u}(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{M(k)t}\overset{\wedge}{\mathfrak{u}}_0(k)\, e^{ikx} dk$$

of the linearized problem under consideration, with initial data $\mathfrak{u}_0 \in S(\mathbb{R})$. Note that the Schwartz class is good to describe localized waves, which arise as perturbations of a pure-current background state with a flat free surface and a flat thermocline.

An eigenvalue with non-zero real part leads to instability due to growth in time (see the discussion in [3]). On the other hand, to a purely imaginary eigenvalue

$$\Lambda(k) = -ikc \quad \text{with} \quad c \in \mathbb{R}\backslash\{0\}$$

of the 4×4 matrix $M(k)$, with corresponding eigenvector $b(k) \neq 0$, we can associate the oscillatory mode solution

$$e^{M(k)t}b(k) = e^{-ikx}b(k)\,.$$

Also, a purely imaginary eigenvalue $\Lambda(k) = -ikc$ of $M(k)$ corresponds to the fundamental oscillation mode

$$e^{ik(x-ct)}b(k)$$

with frequency $\frac{|k|}{2\pi}$, propagating at the constant speed c and the general solution of (1) is a linear combination of solutions of the form $t^n e^{\Lambda(k)t}U$, where n is a nonnegative integer, U is a constant vector and $\Lambda(k)$ is an eigenvalue of $M(k)$ (see the discussion in [3]). This way, the linear wave propagation is reduced to the study of the eigenvalues of the matrix $M(k)$.

Finding accurate estimate for the eigenvalues of $M(k)$ turns out to be quite a challenge (see [3] and [14]), but this is at the core of understanding the evolution of wave packets at the surface and along the thermocline. In this context, it is worth to note that, $\Lambda(k) \in \mathbb{C}$ is an eigenvalue of $M(k)$ with eigenvector $(v_1, v_2, v_3, v_4)^T$ if and only if

$$\lambda(k) = \frac{i\Lambda(k)}{k}$$

is an eigenvalue with corresponding eigenvector $(v_1, v_2, iv_3, iv_4)^T$ of the real matrix

$$R(k) = \begin{pmatrix} \gamma h - \mu\Theta(k) & -\mu_1 \sec h(h_1 k)\Theta(k) & \Theta(k) & \sec h(h_1 k)\Theta(k) \\ -\mu \sec h(h_1 k)\Theta(k) & \Gamma_1 - \mu_1 \Theta_1(k) & \sec h(h_1 k)\Theta(k) & \Theta_1(k) \\ -\Gamma + \mu^2\Theta(k) & \mu\mu_1 \sec h(h_1 k)\Theta(k) & \gamma h - \mu\Theta(k) & -\mu \sec h(h_1 k)\Theta(k) \\ \mu\mu_1 \sec h(h_1 k)\Theta(k) & -\omega\Gamma_1 + g + \mu_1^2\Theta_1(k) & -\mu_1 \sec h(h_1 k)\Theta(k) & \Gamma_1 - \mu_1\Theta_1(k) \end{pmatrix}.$$

3 The propagation of linear waves in the El Nino setting

A careful consideration of the field data reveals that $h_1/h \ll 1$ typically. Moreover, during an El Nino event the trade winds in the equatorial mid-Pacific loose considerably in intensity and, as a consequence, the depth of the thermocline diminishes a great deal throughout the Pacific. These physical consideration motivate us to investigate the linear problem in the limiting case $h_1 \to 0$.

The following lemma holds.

Lemma 1. In the limiting case $h_1 \to 0$ we have that the matrix $R(k)$ has only real eigenvalues, given explicitly by:

$$\lambda_{1,2} = \gamma h, \qquad \lambda_{3,4} = \gamma h - (\mu_1\Theta + \mu\Theta) \pm \sqrt{\Theta^2\left(\mu_1+\mu\right)^2 + \Theta\hat{g}\left(1+r\right)}.$$

Proof. To investigate roots of the characteristic polynomial

$$p(\lambda) = det(R - \lambda I_4)\,,$$

we first perform two sets of operations to simplify its structure. Add the first row multiplied by μ to the third row, and the second row multiplied by μ_1 to the fourth row, and in the outcome add the third column multiplied by μ to the first column and the fourth column multiplied by μ_1 to the second column to obtain a determinant expressed in terms of λ, which corresponds to the wave speed relative to the maximum speed of the current, given by $\overset{\Lambda}{p}(\lambda) = \begin{vmatrix} -X & 0 & \Theta & \Theta \\ 0 & \gamma_1 h_1 - X & \Theta & \Theta \\ -\Gamma - 2\mu X & 0 & -X & 0 \\ 0 & \hat{g} - 2\mu_1 X & 0 & \gamma_1 h_1 - X \end{vmatrix}$,

where

$$h_1 = 0, \quad X = \lambda - \gamma h, \quad \hat{g} = g - \omega h\gamma, \quad \Gamma = -r\hat{g}.$$

Therefore, we have

$$\overset{\Lambda}{p}(\lambda) = \begin{vmatrix} -X & 0 & \Theta & \Theta \\ 0 & -X & \Theta & \Theta \\ r\hat{g} - 2\mu X & 0 & -X & 0 \\ 0 & \hat{g} - 2\mu_1 X & 0 & -X \end{vmatrix} =$$

$$X^4 + (2\mu_1\Theta + 2\mu\Theta)\,X^3 + \left(-\Theta\hat{g} - r\hat{g}\Theta\right)X^2.$$

Now, it is easy to see that all the roots of the equation $\overset{\Lambda}{p}(\lambda) = 0$ are real. Indeed, we have

$$\overset{\Lambda}{p}(\lambda) = X^4 + (2\mu_1\Theta + 2\mu\Theta)\,X^3 + \left(-\Theta\hat{g} - r\hat{g}\Theta\right)X^2 = 0$$

or

$$\left(X^2 + (2\mu_1\Theta + 2\mu\Theta)\,X + \left(-\Theta\hat{g} - r\hat{g}\Theta\right)\right)X^2 = 0.$$

Consequently,

$$\left(\lambda - \gamma h - (\mu_1\Theta + \mu\Theta) + \sqrt{\Theta^2\left(\mu_1+\mu\right)^2 + \Theta\hat{g}\left(1+r\right)}\right) \times$$

$$\left(\lambda - \gamma h - (\mu_1\Theta + \mu\Theta) - \sqrt{\Theta^2\left(\mu_1+\mu\right)^2 + \Theta\hat{g}\left(1+r\right)}\right)(\lambda - \gamma h)^2 = 0.$$

Therefore,

$$\lambda_{1,2} = \gamma h - (\mu_1\Theta + \mu\Theta) \pm \sqrt{\Theta^2\left(\mu_1+\mu\right)^2 + \Theta\hat{g}\left(1+r\right)}, \lambda_{3,4} = \gamma h.$$

■

Taking into account that $\Theta = \frac{\tanh(hk)}{k(1+r)}$, $\hat{g} = g - \omega h\gamma$ (where $g >> \gamma h\omega$) and $\mu_1 + \mu = \frac{(1+r)(\gamma+\omega)}{2}$ we get

$$\lambda_{1,2} = \gamma h - \frac{\tanh(hk)}{2k}(\gamma+\omega) \pm \sqrt{\frac{\tanh^2(hk)}{k^2}(\gamma+\omega)^2 + \frac{\tanh(hk)}{k}(g-\omega h\gamma)}, \quad (2)$$

$$\lambda_{3,4} = \gamma h. \quad (3)$$

Therefore, the matrix $M(k)$ has the following purely imaginary eigenvalues

$$\Lambda_{1,2} = -\frac{ik}{2}\left(\gamma h - \frac{\tanh(hk)}{2k}(\gamma+\omega)\right) \pm ik\sqrt{\frac{\tanh^2(hk)}{k^2}(\gamma+\omega)^2 + \frac{\tanh(hk)}{k}(g-\omega h\gamma)}.$$

$$\Lambda_3 = -ik\gamma h.$$

which corresponding to oscillations with speeds given by formulas $(2)-(3)$.

Thus, the previous considerations prove the main result.

Theorem 1. In an El Nino setting, the propagation of linear waves generated by equation (1) is characterized by the fundamental oscillation modes propagating at the constant speeds given by the formulas $(2)-(3)$.

4 The propagation of linear waves for a case of non-zero constant vorticities.

In [14] the model of wave-current interactions (see [3])) in the case of the absence of vorticity has been considered. However, this permits a more detailed analysis, and then the model with non-zero vorticities can be regarded as a perturbation of the the irrotational setting. In this respect, the above-mentioned paper [14] and [15] where $\gamma = 0$ and $\gamma_1 \neq 0$, are closely connected with general case and reasoning from here can be adapted to the case with nonzero γ_1 and γ if we replace λ with $X = \lambda - \gamma h$ (see Lemma 1 in [14]), taking into account some technical details.

Thus, we can investigate the model with non-zero vorticities based on general scheme of proofs from [14, 15] with necessary references to the above-mentioned papers.

As we noted in Intoduction, $\Lambda(k) \in \mathbb{C}$ is an eigenvalue of $M(k)$ with eigenvector $(v_1, v_2, v_3, v_4)^T$ if and only if $\lambda(k) = \frac{i\Lambda(k)}{k}$ is an eigenvalue with corresponding eigenvector $(v_1, v_2, iv_3, iv_4)^T$ of the real matrix $R(k)$. Moreover, the following leema holds (see [3], page 28)

Lemma 2. The matrix $M(k)$ has four distinct purely imaginary eigenvalues for $0 < |k| < 64$.

Thus, we can investigate the model in the physically relevant regime $0 < |k| < 64$, with non-zero vorticities based on general scheme of proofs from [14, 15] with necessary references to the above-mentioned papers.

Lemma 3. The eigenvalues of $R(k)$ for $0<|k|<64$ satifies the following estimates:

$$|\lambda_i| \le (\frac{16}{k^2}\tanh^2(k(h+h_1)) + 2\gamma_1^2 h_1^2 - 8\gamma_1 h_1 \mu_1 \Theta_1 + 2\Theta_1\gamma_1^2 h_1$$

$$+\frac{4}{k}(g-\omega\gamma h)\tanh k(h+h_1) - 8\mu\mu_1\Theta\frac{\tanh(h_1 k)}{k})^{1/2} + \gamma h. \tag{4}$$

Proof. To investigate roots of the characteristic polynomial $p(\lambda) = det(R-\lambda I_4)$, we first perform two sets of operations to simplify its structure. Add the first row multiplied by μ to the third row, and the second row multiplied by μ_1 to the fourth row, and in the outcome add the third column multiplied by μ to the first column and the fourth column multiplied by μ_1 to the second column to obtain a determinant edxpressed in terms of $X=\lambda-\gamma h$, corresponding to the wave speed relative to the maximum speed of the

$$\hat{p}(\lambda) = \begin{vmatrix} -X & 0 & \Theta & s\Theta \\ 0 & \gamma_1 h_1 - X & s\Theta & \Theta_1 \\ -\Gamma - 2\mu\lambda & 0 & -X & 0 \\ 0 & \hat{g} - 2\mu_1 X & 0 & \gamma_1 h_1 - X \end{vmatrix},$$

where $s = sech(h_1 k)$, $\Gamma = r\omega\gamma h - rg, \hat{g} = g + \gamma_1^2 h_1 - \omega\gamma h$.
We obtain:

$$\begin{aligned}
&X^4 + (-2\gamma_1 h_1 + 2\mu_1\Theta_1 + 2\mu\Theta)X^3 + \\
&(\gamma_1^2 h_1^2 - \Theta_1\gamma_1^2 h_1 - \Theta_1(g-\omega\gamma h) - 4\mu\Theta\gamma_1 h_1 \\
&+4\mu\mu_1\left(\Theta\Theta_1 - s^2\Theta^2\right) - r\Theta(g-\omega\gamma h))X^2 + \\
&(2\gamma_1 h_1 r\Theta(g-\omega\gamma h) + 2\mu\Theta\gamma_1^2 h_1^2 \\
&-2\left(rg\mu_1 + \mu g + \mu\gamma_1^2 h_1 - r\omega\gamma h\mu_1\right)\left(\Theta\Theta_1 - s^2\Theta^2\right))X - \\
&\gamma_1^2 h_1^2 r\Theta(g-\omega\gamma h) + (rg^2 + r\gamma_1^2 h_1(g-\omega\gamma h) \\
&-r\omega\gamma hg)\left(\Theta\Theta_1 - s^2\Theta^2\right) \qquad = 0.
\end{aligned} \tag{5}$$

In view of above-mentioned Lemma 2 (see [3]) all the roots of this equation are real. Taking into account coefficients of equation (5) we have following equalities:

$$X_1 + X_2 + X_3 + X_4 = -2\gamma_1 h_1 + 2\mu_1\Theta_1 + 2\mu\Theta, \tag{6}$$

$$\begin{aligned}
&X_1(X_2+X_3+X_4) + X_2(X_3+X_4) + X_3X_4 = \\
&\gamma_1^2 h_1^2 - \Theta_1\gamma_1^2 h_1 - \Theta_1(g-\omega\gamma h) - 4\mu\Theta\gamma_1 h_1 \\
&+4\mu\mu_1\Theta\frac{\tanh(h_1 k)}{k} - r\Theta(g-\omega\gamma h),
\end{aligned} \tag{7}$$

where X_i are roots of the equation (5). Thus, by squaring the right-hand and left-hand sides of equation (6) and using equation (7) multiplied by 2, carrying same manipulations as in papers [14, 15] we obtain the validity of Lemma 3 and since the further proof of this fact repeats almost word for word the corresponding proof from the above-mentioned papers, we omit the further details. ■

Remark. From physical point of view γ_1 is a negative constant (see [3], page 11) greater than 1 (approximately equal to -12.5) and since $r < 1$ (and therefore $\mu_1 = \frac{\gamma_1 + r}{2} < 0$) we get that term $8\gamma_1 h_1 \mu_1 \Theta_1$ is a positive. Thus, formula (4) acquires a more simpler form:

$$|\lambda_i| \leq (\frac{16}{k^2}\tanh^2\left(k\left(h+h_1\right)\right) + 2\gamma_1^2 h_1^2 + 2\Theta_1\gamma_1^2 h_1$$
$$+\frac{4}{k}\left(g-\omega\gamma h\right)\tanh k\left(h+h_1\right) - 8\mu\mu_1\Theta\frac{\tanh\left(h_1 k\right)}{k})^{1/2} + \gamma h. \tag{8}$$

Besides, to obtain roots of the equation we also can use the following standard substitution

$$X = \delta - \frac{-\gamma_1 h_1 + \mu_1\Theta_1 + \mu\Theta}{2}.$$

and the problem boils down to the consideration of the following equation

$$\delta^4 + p\delta^2 + q\delta + l = 0, \tag{9}$$

where

$$\begin{gathered}
p = \gamma_1^2 h_1^2 - \Theta_1\gamma_1^2 h_1 - \Theta_1\left(g-\omega\gamma h\right) - 4\mu\Theta\gamma_1 h_1 - r\Theta\left(g-\omega\gamma h\right) + \\
4\mu\mu_1\Theta\tfrac{\tanh(h_1 k)}{k} - \tfrac{3}{2}\left(\gamma_1 h_1 - \mu_1\Theta_1 - \mu\Theta\right)^2; \\
q = 2\gamma_1 h_1 r\Theta\left(g-\omega\gamma h\right) + 2\mu\Theta\gamma_1^2 h_1^2 - 2\left(rg\mu_1 + \mu g + \mu\gamma_1^2 h_1 - r\omega\gamma h\mu_1\right)\times \\
\Theta\tfrac{\tanh(h_1 k)}{k} + \left(-\gamma_1 h_1 + \mu_1\Theta_1 + \mu\Theta\right)^3 + \left(\gamma_1 h_1 - \mu_1\Theta_1 - \mu\Theta\right)\times \\
\Big(\gamma_1^2 h_1^2 - \Theta_1\gamma_1^2 h_1 - \Theta_1\left(g-\omega\gamma h\right) - 4\mu\Theta\gamma_1 h_1 + 4\mu\mu_1\Theta\tfrac{\tanh(h_1 k)}{k} - r\Theta\left(g-\omega\gamma h\right)\Big); \\
l = \tfrac{1}{4}\left(\gamma_1 h_1 - \mu_1\Theta_1 - \mu\Theta\right)^2\left(\gamma_1^2 h_1^2 - \Theta_1\gamma_1^2 h_1 - \Theta_1\left(g-\omega\gamma h\right) - 4\mu\Theta\gamma_1 h_1 + \right. \\
\left. 4\mu\mu_1\Theta\tfrac{\tanh(h_1 k)}{k} - r\Theta\left(g-\omega\gamma h\right)\right) + \left(\gamma_1 h_1 - \mu_1\Theta_1 - \mu\Theta\right)\left(\gamma_1 h_1 r\Theta\left(g-\omega\gamma h\right) + \right. \\
\left. \mu\Theta\gamma_1^2 h_1^2 - \left(rg\mu_1 + \mu g + \mu\gamma_1^2 h_1 - r\omega\gamma h\mu_1\right)\Theta\tfrac{\tanh(h_1 k)}{k}\right) - \tfrac{3}{16}\left(\gamma_1 h_1 - \mu_1\Theta_1 - \mu\Theta\right)^4 - \\
\gamma_1^2 h_1^2 r\Theta\left(g-\omega\gamma h\right) + \left(g^2 + \gamma_1^2 h_1\left(g-\omega\gamma h\right) - \omega\gamma h g\right) r\Theta\tfrac{\tanh(h_1 k)}{k}.
\end{gathered}$$

Following a procedure from the paper [14] under notations

$$p_1 = \frac{p^2 - 4l}{4} - \frac{1}{3}p^2 = -\frac{1}{12}p^2 - l,$$

$$q_1 = \frac{2}{27}p^3 - \frac{1}{3}p\frac{p^2-4l}{4} - \frac{q^2}{8} = -\frac{1}{108}p^3 + \frac{1}{3}pl - \frac{q^2}{8},$$

$$d = -q_1/\left(\sqrt{-\frac{p_1}{3}}\right)^3,$$

$$Q = \left(\frac{1}{3}p_1\right)^3 + \left(\frac{1}{2}q_1\right)^2 = \left(-\frac{1}{12}p^2 - l\right)^3 + \left(-\frac{1}{216}p^3 + \frac{1}{6}pl - \frac{q^2}{16}\right)^2$$

we consider the following three cases generated by equation $y^3 - 3y = d$ which is related to resolvent cubic of the above-mentioned quartic equation (9):

i) $Q < 0$ (which corresponds to the case $|d| < 2$);

ii) $Q = 0$ (which corresponds to the case $|d| = 2$);

iii) $Q > 0$ (which corresponds to the case $|d| > 2$).

Then, by the similar manipulations as in [14] under condition $p_1 = -\frac{1}{12}p^2 - l < 0$, we obtain that $M(k)$ has the four disjoint purely imaginary eigenvalues:

$$\Lambda_{1,2} = -\frac{ik}{2}\left(\sqrt{2\widetilde{z}} \mp \sqrt{-2\left(p + \widetilde{z} + \frac{q}{\sqrt{2\widetilde{z}}}\right)}\right) + ik\frac{-2\gamma h - \gamma_1 h_1 + \mu_1\Theta_1 + \mu\Theta}{2}, \tag{10}$$

$$\Lambda_{3,4} = -\frac{ik}{2}\left(-\sqrt{2\widetilde{z}} \mp \sqrt{-2\left(p + \widetilde{z} - \frac{q}{\sqrt{2\widetilde{z}}}\right)}\right) + ik\frac{-2\gamma h - \gamma_1 h_1 + \mu_1\Theta_1 + \mu\Theta}{2}, \tag{11}$$

where $\widetilde{z} = \widetilde{y}\sqrt{\frac{1}{36}p^2 + \frac{1}{3}l} - \frac{1}{3}p$ and $\widetilde{y}$ are some constants depending on d as in formulas below (according to cases (i-iii)):

i) $-1 < \widetilde{y} \le -\frac{1}{2}d$ for $d \in (0, 2]$ and $-\frac{1}{2}d \le \widetilde{y} < 1$ for $d \in [-2, 0)$.

ii) $\widetilde{y} = -1$ for $d = 2$ and $\widetilde{y} = 1$ for $d = -2$

iii) $\widetilde{y} = \sqrt[3]{\frac{d+\sqrt{d^2-4}}{2}} + \frac{1}{\sqrt[3]{\frac{d+\sqrt{d^2-4}}{2}}}$.

In other words, by formulas (10) – (11) we get four types of oscillations with speeds given by formulas (12) – (13) below

$$\lambda_{1,2} = \gamma h + \frac{1}{2}\left(\sqrt{2\widetilde{z}} \mp \sqrt{-2\left(p + \widetilde{z} + \frac{q}{\sqrt{2\widetilde{z}}}\right)}\right) - \frac{-\gamma_1 h_1 + \mu_1\Theta_1 + \mu\Theta}{2}, \tag{12}$$

$$\lambda_{3,4} = \gamma h + \frac{1}{2}\left(-\sqrt{2\widetilde{z}} \mp \sqrt{-2\left(p + \widetilde{z} - \frac{q}{\sqrt{2\widetilde{z}}}\right)}\right) - \frac{-\gamma_1 h_1 + \mu_1\Theta_1 + \mu\Theta}{2}, \tag{13}$$

where $\frac{1}{12}p^2 + l > 0$ (note that since g is a dominated term, this condition is not very restrictive. Besides for long-waves, in our case μ_1 and μ are sufficiently close to zero, thus this condition holds directly).

The previous considerations prove the following result.

Theorem 2. The propogation of linear waves generated by equation (1) is characterized by the the fundamental oscillation modes propagating at the constant speeds given by the formulas (12) − (13). Moreover, the following estimates hold:

$$|\lambda_i| \leq (\frac{16}{k^2}\tanh^2\left(k\left(h+h_1\right)\right) + 2\gamma_1^2 h_1^2 - 8\gamma_1 h_1 \mu_1 \Theta_1 + 2\Theta_1\gamma_1^2 h_1$$

$$+\frac{4}{k}\left(g-\omega\gamma h\right)\tanh k\left(h+h_1\right) - 8\mu\mu_1\Theta\frac{\tanh\left(h_1 k\right)}{k})^{1/2} + \gamma h.$$

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