

Letter to the Editors

Nonhomogeneous Dispersive Water Waves and Painlevé equations

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A. Mickiewicz University, Uniwersytetu Poznańskiego 2, 61-614 Poznań, Poland**Received Date June 18, 2021; Accepted Date July 21, 2021***Abstract**

In this letter we consider three nonhomogeneous deformations of Dispersive Water Wave (DWW) soliton equation and prove that their stationary flows are equivalent to three famous Painlevé equations, i.e. P_{II} , P_{III} and P_{IV} , respectively.

The six classical Painlevé equations ($P_I - P_{VI}$) are integrable nonlinear second-order ordinary differential equations (ODE's) that defined new transcendental functions globally in the complex plane. Although first discovered from strictly mathematical considerations, nowadays the Painlevé equations play an important role in a variety of physical applications. In particular includes such topics as: the transport of particles across boundaries (Nernst–Planck equations) [1], Hele–Shaw problems in viscous fluids [2], dilute Bose–Einstein condensates in an external one dimensional field (Gross–Pitaevskii equation) [3] as well as various approaches to quantum field theory, statistical mechanics, plasma physics, nonlinear waves, quantum gravity, and nonlinear fiber optics. Besides, in last decades there has been considerable interest in the Painlevé equations also due to the fact that they arise as various reductions of the nonlinear soliton PDE's, which are solvable by inverse scattering method. In the literature they mainly appeared as similarity reductions and scaling reductions of particular soliton equations (see for example [4]–[11] and references therein).

In this letter we announce the existence of another relationship between soliton systems and Painlevé equations. Actually, Painlevé equations are just equivalent to stationary flows of particular, nonhomogeneous deformations of soliton systems. Here we prove the equivalence between three Painlevé equations, i.e.

$$P_{II} : \quad q_{\tau\tau} = 2q^3 + \tau q + \alpha, \quad (1)$$

$$P_{III} : \quad \tau q q_{\tau\tau} = \tau q_{\tau}^2 - q q_{\tau} + \gamma \tau q^4 + \alpha q^3 + \beta q + \delta \tau \quad (2)$$

and

$$P_{IV} : \quad qq_{\tau\tau} = \frac{1}{2}q_{\tau}^2 + \frac{3}{2}q^4 + 4\tau q^3 + 2(\tau^2 - \alpha)q^2 + \beta, \quad (3)$$

where $\alpha, \beta, \gamma, \delta$ are constants, and stationary flows of the Dispersive Water Wave (DWW) PDE, deformed by its own local master symmetries. Nevertheless, our preliminary research shows that such relation exists on the level of whole soliton hierarchies and multi component Painlevé-type equations.

Let us consider the DWW soliton equation in Antonowicz-Fordy representation [12]

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} \frac{1}{4}v_{xxx} + uv_x + \frac{1}{2}vu_x \\ u_x + \frac{3}{2}vv_x \end{pmatrix} \equiv \mathcal{K}_2. \quad (4)$$

It is nonlinear PDE related to a linear spectral problem

$$(\partial_x^2 + u + v\lambda)\psi = \lambda^2\psi.$$

Equation (4) belongs to tri-Hamiltonian soliton hierarchy [12]

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_n} = \mathcal{K}_n = \pi_0\gamma_n = \pi_1\gamma_{n-1} = \pi_2\gamma_{n-2}, \quad n = 1, 2, \dots \quad (5)$$

where γ_r are exact one-forms and three Poisson operators are

$$\begin{aligned} \pi_0 &= \begin{pmatrix} -\frac{1}{2}v\partial_x - \frac{1}{2}\partial_x v & \partial_x \\ \partial_x & 0 \end{pmatrix}, & \pi_1 &= \begin{pmatrix} \frac{1}{4}\partial_x^3 + \frac{1}{2}u\partial_x + \frac{1}{2}\partial_x u & 0 \\ 0 & \partial_x \end{pmatrix}, \\ \pi_2 &= \begin{pmatrix} 0 & \frac{1}{4}\partial_x^3 + \frac{1}{2}u\partial_x + \frac{1}{2}\partial_x u \\ \frac{1}{4}\partial_x^3 + \frac{1}{2}u\partial_x + \frac{1}{2}\partial_x u & \frac{1}{2}v\partial_x + \frac{1}{2}\partial_x v \end{pmatrix}. \end{aligned}$$

The hierarchy (5) can be generated by recursion operator and its adjoint

$$N = \pi_1\pi_0^{-1} = \begin{pmatrix} 0 & \frac{1}{4}\partial_x^2 + u + \frac{1}{2}u_x\partial_x^{-1} \\ 1 & v + \frac{1}{2}v_x\partial_x^{-1} \end{pmatrix}, \quad N^\dagger = \begin{pmatrix} 0 & 1 \\ \frac{1}{4}\partial_x^2 + u - \frac{1}{2}\partial_x^{-1}u_x & v - \frac{1}{2}\partial_x^{-1}v_x \end{pmatrix},$$

$$\mathcal{K}_{n+1} = N^n\mathcal{K}_1, \quad \gamma_n = d\mathcal{H}_n = (N^\dagger)^n\gamma_0, \quad n = 0, 1, 2, \dots$$

where \mathcal{H}_r are Hamiltonian densities. In particular

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} 2 \\ v \end{pmatrix}, & \gamma_1 &= \begin{pmatrix} v \\ u + \frac{3}{4}v^2 \end{pmatrix}, & \gamma_2 &= \begin{pmatrix} u + \frac{3}{4}v^2 \\ \frac{1}{4}v_{xx} + \frac{3}{2}uv + \frac{5}{8}v^3 \end{pmatrix}, \quad \dots \\ \mathcal{H}_0 &= 2u + \frac{1}{2}v^2, & \mathcal{H}_1 &= uv + \frac{1}{4}v^3, & \mathcal{H}_2 &= -\frac{1}{8}v_x^2 + \frac{1}{2}u^2 + \frac{3}{4}uv^2 + \frac{5}{32}v^4, \quad \dots \\ \mathcal{K}_1 &= \begin{pmatrix} u_x \\ v_x \end{pmatrix}, & \mathcal{K}_2 &= \begin{pmatrix} \frac{1}{4}v_{xxx} + uv_x + \frac{1}{2}vu_x \\ u_x + \frac{3}{2}vv_x \end{pmatrix}, \quad \dots \end{aligned}$$

In addition, with the DWW hierarchy of symmetries \mathcal{K}_n is related a hierarchy of master symmetries $\sigma_m = N^{m+1}\sigma_{-1}$, non-local in general, except the first three

$$\sigma_{-1} = \begin{pmatrix} -v \\ 2 \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} 2u + xu_x \\ v + xv_x \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} \frac{3}{4}v_{xx} + \frac{1}{4}xv_{xxx} + uv + xuv_x + \frac{1}{2}xvu_x \\ xu_x + \frac{3}{2}xvv_x + v^2 + 2u \end{pmatrix}, \quad \dots$$

Notice that

$$\sigma_{-1} = \pi_0 \zeta, \quad \sigma_0 = \pi_1 \zeta, \quad \sigma_1 = \pi_2 \zeta, \quad \zeta = \begin{pmatrix} 2x \\ xv \end{pmatrix}.$$

Both, symmetries \mathcal{K}_n and master symmetries σ_m constitute so called Virasoro algebra (hereditary algebra)

$$[\mathcal{K}_m, \mathcal{K}_n] = 0, \quad [\sigma_m, \mathcal{K}_n] = n\mathcal{K}_{n+m}, \quad [\sigma_m, \sigma_n] = (n - m)\sigma_{n+m}.$$

Let us consider three Hamiltonian nonhomogeneous deformations of the DWW equation (4) by its local master symmetries

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \mathcal{K}_2 + \sigma_{-1} = \pi_0(\gamma_2 + \zeta) = \begin{pmatrix} \frac{1}{4}v_{xxx} + uv_x + \frac{1}{2}vu_x - v \\ u_x + \frac{3}{2}vv_x + 2 \end{pmatrix}, \quad (6)$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \mathcal{K}_2 + \sigma_0 = \pi_1(\gamma_1 + \zeta) = \begin{pmatrix} \frac{1}{4}v_{xxx} + uv_x + \frac{1}{2}vu_x + 2u + xu_x \\ u_x + \frac{3}{2}vv_x + v + xv_x \end{pmatrix}, \quad (7)$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \mathcal{K}_2 + \sigma_1 = \pi_2(\gamma_0 + \zeta) = \begin{pmatrix} \frac{1}{4}v_{xxx} + uv_x + \frac{1}{2}vu_x + \frac{3}{4}v_{xx} + \frac{1}{4}xv_{xxx} + uv + xuv_x + \frac{1}{2}xvu_x \\ u_x + \frac{3}{2}vv_x + xu_x + \frac{3}{2}xvv_x + v^2 + 2u \end{pmatrix}. \quad (8)$$

In what follows, we will show the equivalence between stationary flows, i.e. $t = 0$, of equations (6)-(8) and P_{II} , P_{IV} , P_{III} , respectively.

First, let us consider the stationary flow of equation (6)

$$0 = \begin{pmatrix} -\frac{1}{2}v\partial_x - \frac{1}{2}\partial_x v & \partial_x \\ \partial_x & 0 \end{pmatrix} \begin{pmatrix} u + \frac{3}{4}v^2 + 2x \\ \frac{1}{4}v_{xx} + \frac{3}{2}uv + \frac{5}{8}v^3 + xv \end{pmatrix}.$$

It is equivalent to the pair of equations

$$0 = -\left(\frac{1}{2}v\partial_x + \frac{1}{2}\partial_x v\right) \left(u + \frac{3}{4}v^2 + 2x\right) + \left(\frac{1}{4}v_{xx} + \frac{3}{2}uv + \frac{5}{8}v^3 + xv\right)_x,$$

$$0 = \left(u + \frac{3}{4}v^2 + 2x\right)_x.$$

Integrating them once we find

$$0 = u + \frac{3}{4}v^2 + 2x + c,$$

$$0 = \frac{1}{2}c_1 v_x + \frac{1}{4}v_{xx} + \frac{3}{2}uv + \frac{5}{8}v^3 + xv - 2\alpha.$$

Eliminating u we get a single equation

$$\frac{1}{4}v_{xx} = \frac{1}{2}v^3 + cv + 2xv + 2\alpha,$$

which after rescaling

$$v = 2q, \quad x = \frac{1}{2}\tau$$

turns into

$$q_{\tau\tau} = 2q^3 + (c + \tau)q + \alpha,$$

which is P_{II} (1) with the choice $c = 0$.

Second, let us consider the stationary flow of equation (7)

$$0 = \begin{pmatrix} \frac{1}{4}\partial_x^3 + \frac{1}{2}u\partial_x + \frac{1}{2}\partial_x u & 0 \\ 0 & \partial_x \end{pmatrix} \begin{pmatrix} v + 2x \\ u + \frac{3}{4}v^2 + xv \end{pmatrix}.$$

Again, it is equivalent to the pair of equations

$$\begin{aligned} 0 &= \left(\frac{1}{4}\partial_x^3 + \frac{1}{2}u\partial_x + \frac{1}{2}\partial_x u \right) (v + 2x), \\ 0 &= \left(u + \frac{3}{4}v^2 + xv \right)_x. \end{aligned}$$

Denote $\gamma = v + 2x$ and multiply the first equation by γ . Then, we find

$$\begin{aligned} 0 &= \gamma \left(\frac{1}{4}\partial_x^3 + \frac{1}{2}u\partial_x + \frac{1}{2}\partial_x u \right) \gamma = \left(\gamma\gamma_{xx} - \frac{1}{2}\gamma_x^2 + u\gamma^2 \right)_x, \\ 0 &= \left(u + \frac{3}{4}v^2 + xv \right)_x. \end{aligned}$$

Integrating them once we get

$$\begin{aligned} 0 &= \gamma\gamma_{xx} - \frac{1}{2}\gamma_x^2 + u\gamma^2 - \beta, \\ 0 &= u + \frac{3}{4}v^2 + xv - c. \end{aligned}$$

Eliminating u we derive a single equation

$$\gamma\gamma_{xx} = \frac{1}{2}\gamma_x^2 + \frac{3}{4}\gamma^4 - 2x\gamma^3 + (x^2 - c)\gamma^2 + \beta,$$

which after rescaling

$$\gamma = -2^{\frac{1}{4}}q, \quad x = 2^{\frac{1}{4}}\tau, \quad 2^{-\frac{1}{2}}c = \alpha$$

turns into P_{IV} (3).

Finally, let us consider the stationary flow of equation (8)

$$0 = \begin{pmatrix} 0 & \frac{1}{4}\partial_x^3 + \frac{1}{2}u\partial_x + \frac{1}{2}\partial_x u \\ \frac{1}{4}\partial_x^3 + \frac{1}{2}u\partial_x + \frac{1}{2}\partial_x u & \frac{1}{2}v\partial_x + \frac{1}{2}\partial_x v \end{pmatrix} \begin{pmatrix} 2 + 2x \\ v + xv \end{pmatrix}. \quad (9)$$

In order to integrate it, notice that

$$\begin{pmatrix} b & 0 \\ a & b \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{4}\partial_x^3 + \frac{1}{2}u\partial_x + \frac{1}{2}\partial_x u \\ \frac{1}{4}\partial_x^3 + \frac{1}{2}u\partial_x + \frac{1}{2}\partial_x u & \frac{1}{2}v\partial_x + \frac{1}{2}\partial_x v \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ = \begin{pmatrix} (\frac{1}{4}bb_{xx} - \frac{1}{8}b_x^2 + \frac{1}{2}ub^2)_x \\ (\frac{1}{4}ab_{xx} + \frac{1}{4}ba_{xx} - \frac{1}{4}a_xb_x + uab + \frac{1}{2}vb^2)_x \end{pmatrix}.$$

Passing to new independent variable $y = x + 1$ and substituting $a = 2y$ and $b = yv = z$, integration of (9) gives a pair of equations

$$\begin{aligned} 0 &= zz_{yy} - \frac{1}{2}z_y^2 + 2uz^2 + \delta, \\ 0 &= \frac{1}{2}yz_{yy} - \frac{1}{2}z_y + 2yuz + \frac{1}{2}vz^2 + \frac{1}{2}\beta. \end{aligned}$$

Multiplying the first equation by y , the second one by $-z$ and adding them we obtain a single equation

$$yzz_{yy} = yz_y^2 - zz_y + \frac{1}{y}z^4 + \frac{1}{2}\beta z + \alpha y$$

which, after transformation

$$z = \tau q, \quad y = \frac{1}{2}\tau^2,$$

turns into

$$\tau qq_{\tau\tau} = \tau q_\tau^2 - qq_\tau + 4\tau q^4 + \beta q + \delta\tau, \tag{10}$$

i.e. P_{III} (2) with $\alpha = 0$ and $\gamma = 4$.

Let us conclude this letter with an open question, whether one can relate deformed DWW equations (6)-(8) to some non-isospectral problems, according to the idea developed in [5].

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