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# Conservation laws of nonlinear PDEs arising in elasticity and acoustics in Cartesian, cylindrical, and spherical geometries



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**Abstract**

Conservation laws are computed for various nonlinear partial differential equations that arise in elasticity and acoustics. Using a scaling-homogeneity approach, conservation laws are established for two models describing shear wave propagation in a circular cylinder and a cylindrical annulus. Next, using the multiplier method, conservation laws are derived for a parameterized system of constitutive equations in cylindrical coordinates involving a general expression for the Cauchy stress. Conservation laws for the Khokhlov-Zabolotskaya-Kuznetsov equation and Westervelt-type equations in various coordinate systems are also presented.

# 1 Introduction

One of the beautiful applications of symmetry methods to partial differential equations (PDEs) is the computation of conservation laws which has been a central theme in the work of Prof. Bluman and his collaborators [7, 8, 18]. Conservation laws consist of conserved densities and associated fluxes. The integral of a conserved density often has a physical meaning. Depending on the circumstances, it might express conservation of energy, momentum, angular momentum, and the like. Conservation laws have many uses in the study of PDEs such as establishing the existence and uniqueness of solutions and investigating their stability. They also play a key role in the development of accurate numerical methods to solve nonlinear PDEs.

Of the many techniques [2, 8, 16, 39, 54] to compute conservation laws, we focus on two approaches: (i) a scaling-homogeneity method [21, 24, 25, 43] which uses linear algebra and does not require solving PDEs, and (ii) the multiplier method [5, 39, 40, 53] which is analogous to the integrating factor method [7] for ordinary differential equations.

Applying these methods which are implemented in the codes `ConservationLawsMD.m` [28] and `GeM` [17], respectively, we compute conservation laws of various nonlinear PDEs that arise in elasticity and acoustics. In [38] we derived conservation laws based on a constitutive equation modeling stress in elastic materials with a geometry suitable for Cartesian coordinates. Here we compute conservation laws for PDEs that model the propagation of elastic waves in a circular cylinder and cylindrical annulus. More precisely, we focus on models studied by Kambapalli et al. [31] and Magan et al. [36]. These models belong to a general class of implicit constitutive equations where the strain $\epsilon$ is expressed as a non-invertible function $F(\sigma)$ of the stress $\sigma$. These so-called *implicit theories* originated in work of Rajagopal [44, 45] and have since been advocated by many others (see, e.g., [10, 13, 14] and the references therein).

We start with computing conservation laws for the two models reported in [31] and [36]. Both models are formulated in cylindrical coordinates and assume a power-law constitutive relation. Next, to capture other constitutive relations reported in the literature (see Table 2), we replace the power-law by an arbitrary function, $\epsilon = F(\sigma)$, relating the scalar linearized strain to Cauchy stress. The conserved densities we obtain are independent of $F$ but the fluxes depend on the integral of $F$. For completeness, we also compute conservation laws of the single wave equation for the stress that arises upon elimination of the displacement from the equations of motion.

A couple of equations from nonlinear acoustics are also considered. The first one is the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation [22, 23, 49] (sometimes referred to as the two-dimensional Burgers equation) which models the propagation of sound beams in various nonlinear media, in particular, sound waves generated by parametric acoustic areas. Other applications include the propagation of ultrasound in dissipative media, long waves in ferromagnetic media, and pulsed sound beams in thermo-viscous media. Conservation laws are computed for the KZK equation in Cartesian coordinates as well as cylindrical and spherical coordinates. It turns out that for the KZK equation in Cartesian coordinates, the conservation laws have a couple of space-dependent coefficients; one must be a harmonic function while the other must be a solution of a Poisson equation. Since the Laplace equation has infinitely many solutions there are infinitely many conservation laws. In cylindrical and spherical coordinates the equations for these coefficients are slightly more complicated but the conclusion remains the same: The KZK equation in plane, cylindrical, and spherical geometries has infinite conservation laws.

The second equation is the Westervelt equation [23, 30] named after Peter Westervelt who received the Nobel Prize in Physics in 1964 for his contributions to quantum mechanics. His equation is used to describe the nonlinear propagation of pressure waves in nonlinear media. It has many engineering applications in underwater acoustics, sonar antennas, infrasound in the atmosphere, among others. The equation also models the propagation of high-intensity ultrasound in tissue with numerous medical applications.

The dissipative version of Westervelt's equation in one spatial dimension has been the subject of recent studies by Anco et al. [6] and Márquez et al. [37]. Anco and co-workers not only report new local and non-local conservation laws but also provide an in-depth analysis of various types of symmetries including hidden symmetries for the potential equation, hidden variational structures, recursion and Noether operators, etc. A similar study is carried out in [37] for a slightly different version of the one-dimensional Westervelt equation with an arbitrary nonlinearity.

In this paper we compute conservation laws of the Westervelt equation in more than one space variable. Our computational results confirm recent findings by Sergyeyev [52] who gives a complete description of local conservation laws of the multi-dimensional dissipative Westervelt equation by using the direct method [4, 40] and a theorem from Igonin [29]. In particular, Sergyeyev has shown that an infinite number of conservation laws exists for the case of two or more space variables. This is in contrast with the one-dimensional dissipative Westervelt equation which has only a finite number of local conservation laws [6].

Using the adjoint-symmetry approach, Anco [3] computed conservation laws of Westervelt's equation in spherical coordinates with the standard quadratic nonlinearity. In this paper we get similar results for the Westervelt's equation (in both spherical and cylindrical coordinates) where the quadratic nonlinear term has been replaced with an arbitrary nonlinear function.

This paper is organized as follows. In Section 2, conservation laws are given for the two forementioned models for shear wave propagation in a circular cylinder and cylindrical annulus. The conservation laws for these two models are computed in Section 3 using the scaling-homogeneity approach. In Section 4, conservation laws are computed with the multiplier method for a parameterized system involving an arbitrary function of stress covering a broad class of models for wave propagation in elastic materials formulated in terms of cylindrical coordinates. For specific values of the parameters, the general system includes the two previously mentioned models. Conservation laws for the KZK equation in Cartesian, cylindrically, and spherical coordinates are derived in Section 5. In Section 6, conservation laws are computed with the multiplier method for Westervelt-type equations in multi-space (Cartesian) coordinates as well as cylindrical and spherical coordinates. Finally, a brief discussion of the results and some conclusions are given in Section 7.

## 2 Model equations and some of their conservation laws

In this section we consider two models for shear wave propagation in a circular cylinder and cylindrical annulus. The constitutive equations for these geometries are appropriately expressed in cylindrical coordinates. The models under consideration are based on *implicit* constitutive equations where the strain is expressed as a non-invertible function of the stress. They were introduced by Rajagopal [44, 45] and have numerous applications (see, e.g., [10, 13]).

## 2.1 Model due to Kambapalli et al. [31] with conservation laws

The first model [31] in non-dimensional variables,

$$\sigma_r + \frac{2\sigma}{r} = \delta u_{tt}, \tag{1}$$

$$u_r - \frac{u}{r} = \tfrac{1}{\delta}\sigma(\beta+\sigma^2)^n, \tag{2}$$

describes shear waves in a cylindrical annular region modeled in terms of cylindrical coordinates $(r,\theta,z)$ and time $t$. As usual, subscripts denote partial derivatives, e.g., $\sigma_r = \frac{\partial\sigma}{\partial r}$ and $u_{tt} = \frac{\partial^2 u}{\partial t^2}$.

The first equation expresses the balance of linear momentum. The second equation is a consequence of the constitutive relation. These equations of motion for stress $\sigma(r,t)$ and displacement $u(r,t)$ are for a specific (but rather simple) model where the linear strain is a nonlinear function of stress expressed as a power law where the arbitrary exponent $n \geq 0$ is a natural or rational number (see also [32, 35]).

The reciprocal of constant parameter $\delta$, i.e., $\frac{1}{\delta} = \frac{\alpha}{\sqrt{\gamma}}$, is the displacement gradient which involves two material parameters $\alpha \geq 0$ and $\gamma \geq 0$. An auxiliary constant parameter $\beta$ has been introduced[1] to assure that the system is scaling homogeneous (see Section 3).

Equations (1)-(2) are a system of evolution equations in $r$. Noting that $\theta$ and $z$ do not explicitly appear, a *conservation law* for (1)-(2) simply reads

$$\mathrm{D}_r\, T^r + \mathrm{D}_t\, T^t \doteq 0, \tag{3}$$

where the total derivatives $\mathrm{D}_r$ and $\mathrm{D}_t$ are defined as

$$\mathrm{D}_r = \frac{\partial}{\partial r} + \sigma_r\frac{\partial}{\partial \sigma} + u_r\frac{\partial}{\partial u} + \sigma_{rr}\frac{\partial}{\partial \sigma_r} + u_{rr}\frac{\partial}{\partial u_r} + \sigma_{rt}\frac{\partial}{\partial \sigma_t} + u_{rt}\frac{\partial}{\partial u_t} + \ldots, \tag{4}$$

and

$$\mathrm{D}_t = \frac{\partial}{\partial t} + \sigma_t\frac{\partial}{\partial \sigma} + u_t\frac{\partial}{\partial u} + \sigma_{tt}\frac{\partial}{\partial \sigma_t} + u_{tt}\frac{\partial}{\partial u_t} + \sigma_{rt}\frac{\partial}{\partial \sigma_r} + u_{rt}\frac{\partial}{\partial u_r} + \ldots. \tag{5}$$

In (3), $T^r$ is a *conserved density* and $T^t$ is the corresponding *flux* which are functions of $r$, $t$, $\sigma$, and $u$, and their partial derivatives with respect to $t$. Note that all partial derivatives of $\sigma$ and $u$ with respect to $r$ can be eliminated by using the PDEs. The $\doteq$ means that equality should only hold on solutions $\sigma(r,t)$ and $u(r,t)$ of PDE system.

System (1)-(2) has the following conservation laws

$$\mathrm{D}_r(r^2\sigma) + \mathrm{D}_t(-\delta r^2 u_t) \doteq 0, \tag{6}$$

$$\mathrm{D}_r(r^2 t\sigma) + \mathrm{D}_t(\delta r^2(u - t u_t)) \doteq 0, \tag{7}$$

$$\mathrm{D}_r(r u \sigma_t) + \mathrm{D}_t\Big( -\tfrac{r}{2(n+1)\delta}\left((\beta+\sigma^2)^{n+1} - (n+1)\delta^2(u_t^2 - 2u u_{tt})\right)\Big) \doteq 0. \tag{8}$$

[1]Once the computations are done one can set $\beta = 1$ in the conservation laws.

Conservation law (6) is (1) itself (after multiplication by $r^2$). Similarly, multiplying (1) by $r^2 t$ allows one to straightforwardly recast the equation in the form (7). Both conservation laws are easy to spot and do not require (2) to hold. Conservation (8) can be readily verified but getting it by inspection would require ingenuity. It will be shown in Sections 3 and 4 how these conservation laws can be computed algorithmically.

Differentiating (1) with respect to $r$ once and (2) with respect to $t$ twice and equating $u_{ttr} = u_{rtt}$ yields

$$\sigma_{rr} + \frac{\sigma_r}{r} - \frac{4\sigma}{r^2} = \big(\sigma(\beta + \sigma^2)^n\big)_{tt}. \tag{9}$$

Conservation laws for this single wave equation for the stress will be addressed in Section 4.2.

### 2.2 Model due to Magan et al. [36] with conservation laws

The second model [36], again in non-dimensional form, reads

$$\sigma_r + \frac{\sigma}{r} = \delta u_{tt}, \tag{10}$$

$$u_r = \tfrac{1}{\delta}\sigma(\beta + \sigma^2)^n, \tag{11}$$

with the meaning of the symbols as in Section 2.1. The above system describes the propagation of axial displacement waves and shear stress waves in a circular cylinder and a cylindrical annulus. Based on that geometry the use of cylindrical coordinates is recommended.

Some of the conservation laws of (10)-(11) are

$$\mathrm{D}_r(r\sigma) + \mathrm{D}_t(-\delta r u_t) \doteq 0, \tag{12}$$

$$\mathrm{D}_r(rt\sigma) + \mathrm{D}_t(\delta r(u - tu_t)) \doteq 0, \tag{13}$$

$$\mathrm{D}_r(ru\sigma_t) + \mathrm{D}_t\Big( - \tfrac{r}{2(n+1)\delta}\left((\beta + \sigma^2)^{n+1} - (n+1)\delta^2(u_t^2 - 2uu_{tt})\right)\Big) \doteq 0. \tag{14}$$

Note that last conservation law coincides with (8).

Here again, by eliminating the displacement $u$, the first-order system (10)-(11) can be replaced by a wave equation for the stress

$$\sigma_{rr} + \frac{\sigma_r}{r} - \frac{\sigma}{r^2} = \big(\sigma(\beta + \sigma^2)^n\big)_{tt}, \tag{15}$$

for which conservation laws will be computed in Section 4.2.

## 3 Computation of conservation laws using scaling homogeneity

In this section we discuss the scaling-homogeneity approach [21, 24, 25, 38, 43] for the computation of conservation laws. To keep matters transparent we will show how to compute conservation law (8) of (1)-(2). Since that conservation law has an arbitrary $n$, we start with $n = 1$ and repeat the computations for $n = 2$ and 3. Once these densities and fluxes are computed, conservation law (8) for arbitrary $n$ is obtained by pattern matching or some straightforward computation as shown at the end of this section.

### 3.1 Scaling homogeneity

One can readily verify that (1)-(2) has a two-parameter family of scaling symmetries,

$$(r, t, \sigma, u, \beta) \to (\kappa^{-(2n+1)p+q}\, r, \kappa^{-(n+1)p+q}\, t, \kappa^{p}\, \sigma, \kappa^{q}\, u, \kappa^{2p}\, \beta), \tag{16}$$

parameterized by the arbitrary real numbers $p$ and $q$. The constant $\kappa \neq 0$ is an arbitrary scaling parameter. If we had not introduced an auxiliary parameter $\beta$ with an appropriate scale, then $p = 0$ and (1)-(2) would only have a one-parameter family of scaling symmetries.

The scaling homogeneity of (1)-(2) can be expressed in terms of weights, e.g., $W(r) = -(2n+1)p + q = -W(D_r)$ and $W(t) = -(n+1)p + q = -W(D_t)$, where *weight* ($W$) of a variable (or differential operator) is the exponent of $\kappa$ associated with that variable (operator). Thus, (16) is equivalent with

$$W(D_r) = (2n+1)p - q, W(D_t) = (n+1)p - q, W(\sigma) = p, W(u) = q, W(\beta) = 2p. \tag{17}$$

The *rank* of a monomial is its total weight. For example, $(\beta + \sigma^2)^n$ has rank $2np$. An expression is called *uniform in rank* if all its monomials have the same ranks.

If (16) was not known it could be computed with linear algebra as follows. Require that (1)-(2) is uniform in rank, that is,

$$W(\sigma) + W(D_r) = W(\sigma) - W(r) = W(u) + 2W(D_t), \tag{18}$$

$$W(u) + W(D_r) = (2n+1)W(\sigma), \quad W(\beta) = 2W(\sigma). \tag{19}$$

Solve these equations for $W(D_r)$ and $W(D_t)$, to get

$$W(D_r) = (2n+1)W(\sigma) - W(u), \;\; W(D_t) = (n+1)W(\sigma) - W(u), \tag{20}$$

where the arbitrary $W(\sigma) > 0$ and $W(u) > 0$ should be selected so that $W(D_r)$ and $W(D_t)$ are strictly positive and preferably as small as possible. To get the lowest possible weights, we take $W(D_r) = 1$ and $W(D_t) = n$ for which (17) simplifies into

$$W(D_r) = n+1, \; W(D_t) = 1, \; W(\sigma) = 1, \; W(u) = n, \; W(\beta) = 2. \tag{21}$$

This simple set of weights will be used in the computations below.

### 3.2 Computing conservation law (8)

Conservation law (3) is *linear* in the density (and flux). Consequently, a linear combination of densities with constant coefficients is also a density. Vice versa, if a density has coefficients that are, for example, powers of the parameter $\beta$, it can be split into independent densities according to those powers. The aim is to produce linearly *independent* densities that are as short as possible. In particular, they should be free of constant terms and any terms that could be moved into the flux.

The key idea of the scaling-homogeneity method is that all the terms in the density must have the same rank. That homogeneity in rank is because (3) should only hold on solutions of (1)-(2). Consequently, densities, fluxes, and conservation laws themselves inherit (or adopt) the scaling homogeneity of that system (and all its other continuous and discrete symmetries).

For example, in (8) the density has rank 1, the flux has rank $n+1$ and the entire conservation law has rank $n+2$. Note that (8) is of first degree in $r$.

Using the scaling-homogeneity method and `ConservationLawsMD.m` [28], the user should select a value of $n$ (for example, $n = 1$), specify the rank of the density to be computed (e.g., rank one), and the desired highest degree in $r$ and $t$ (e.g., degree 1).

**Step 1**: Computation of a candidate density ($T^r$) for $n = 1$. The code uses (21) with $n = 1$ to build a $T^r$ as a linear combination with constant coefficients of scaling homogenous monomials involving $\beta, \sigma$, and $u$ (and their $t$-derivatives) and the independent variables $r$ and $t$ (up to first degree) so that each monomial has rank 1. Table 1 shows the terms needed together with their respective ranks.

| Coefficient $r^{p_1}t^{p_2}$ | Rank | Differential term | Rank | Rank of product |
|---|---|---|---|---|
| 1 | 1 | $\sigma$, $u$ | 1 | 1 |
| $r$ | $-2$ | $\beta\sigma$, $\sigma^3$, $\beta u$, $\sigma^2 u$, $\sigma u^2$, $u^3$, $u\sigma_t$ | 3 | 1 |
| $t$ | $-1$ | $\sigma^2$, $\sigma u$, $u^2$ | 1 | 1 |

**Table 1.** Coefficients of type $r^{p_1}t^{p_2}$ (with $p_1, p_2$ natural numbers) of at most degree 1 (i.e., $0 \le p_1 + p_2 \le 1$) are paired with differential terms so that the respective products all have rank 1.

The lists of differential terms are free of trivial (constant) terms, total derivatives with respect to $t$, and divergence-equivalent terms, meaning terms that only differ by a $t$-derivative. For example, $\sigma u_t$ and $u\sigma_t$ of rank 3 are divergence-equivalent since $\sigma u_t = \mathrm{D}_t(\sigma u) - u\sigma_t$. Hence, $\sigma u_t$ is removed but $u\sigma_t$ is kept. Doing so, produces a candidate for $T^r$ which is free of terms that could have been moved into the flux $T^t$.

Using the twelve terms in Table 1,

$$T^r = c_1\sigma + c_2 u + c_3\beta r\sigma + c_4 t\sigma^2 + c_5 r\sigma^3 + c_6\beta r u + c_7 t\sigma u + c_8 r\sigma^2 u + c_9 t u^2 + c_{10} r\sigma u^2 + c_{11} r u^3 + c_{12} r u\sigma_t, \tag{22}$$

where $c_1$ through $c_{12}$ are undetermined coefficients. See [21, 27, 38, 43] for additional details on how candidate densities are constructed algorithmically.

**Step 2**: Computation of the undetermined coefficients. To find the constants $c_i$, one computes

$$\mathrm{D}_r T^r = c_1\sigma_r + c_2 u_r + c_3\beta(\sigma + r\sigma_r) + 2c_4 t\sigma\sigma_r + \cdots + c_{11}u^2(u + 3ru_r) + c_{12}(u\sigma_t + ru_r\sigma_t + ru\sigma_{tr}), \tag{23}$$

and, using (1)-(2), replaces $\sigma_r$, $u_r$, $\sigma_{tr} = \sigma_{rt}$, to obtain

$$P = -\tfrac{c_1}{r}(2\sigma - \delta r u_{tt}) + \tfrac{c_2}{r\delta}\left(\delta u + r\sigma(\beta + \sigma^2)\right) + \cdots + \tfrac{c_{12} r}{\delta}\left(\sigma\sigma_t(\beta + \sigma^2) + \delta^2 u u_{ttt}\right). \tag{24}$$

Since $P = \mathrm{D}_r T^r$ must match $-\mathrm{D}_t T^t$ for some flux $T^t$ (computed below in Step 3), $P$ must be *exact*. Therefore, the variational derivative (a.k.a. Euler operator) [24, 25] for each of the dependent variables applied to $P$ must vanish. The code applies the Euler operator for $\sigma$

$$\mathcal{E}_\sigma = \sum_{k=0}^{K} (-\mathrm{D}_t)^k \frac{\partial}{\partial \sigma_{kt}} = \frac{\partial}{\partial \sigma} - \mathrm{D}_t \frac{\partial}{\partial \sigma_t} + \mathrm{D}_t^2 \frac{\partial}{\partial \sigma_{tt}} - \mathrm{D}_t^3 \frac{\partial}{\partial \sigma_{ttt}} + \dots, \tag{25}$$

to $P$ wherein the highest derivative of $\sigma$ is $\sigma_t$. Hence, $K = 1$ and

$$\mathcal{E}_\sigma P = \frac{\partial P}{\partial \sigma} - \mathrm{D}_t \frac{\partial P}{\partial \sigma_t} = -\frac{2c_1}{r} + \frac{\beta c_2}{\delta} - \beta c_3 + \frac{r\beta^2 c_6}{\delta} + \frac{2t}{\delta r}(\beta r c_7 - 4\delta c_4)\sigma + \dots + 2\delta r c_8 u u_{tt}. \tag{26}$$

Next, the Euler operator for $u$ is applied to $P$ which has a third-order term $u_{ttt}$. Hence, $K = 3$ and

$$\mathcal{E}_u P = \frac{\partial P}{\partial u} - \mathrm{D}_t \frac{\partial P}{\partial u_t} + \mathrm{D}_t^2 \frac{\partial P}{\partial u_{tt}} - \mathrm{D}_t^3 \frac{\partial P}{\partial u_{ttt}} = \frac{c_1}{r} + 2\beta c_6 + \frac{t}{\delta r}(2\beta r c_9 - \delta c_7)\sigma + \dots + 4\delta c_8 \sigma u_{tt} + 4\delta r c_{10} u u_{tt}. \tag{27}$$

Requiring that $\mathcal{E}_\sigma P \equiv 0$ and $\mathcal{E}_u P \equiv 0$ on the yet space (where monomials in $r, t, \sigma$, $u$, $\sigma_t$, $u_t$, $\sigma_{tt}$, $u_{tt}$, etc., are treated as independent) gives a *linear* system for the coefficients $c_i$ (not shown). Solving reveals that all $c_i$ must be zero except $c_{12}$ which is arbitrary. Substitution of the solution and setting $c_{12} = 1$ into (22), yields $T^r = r u \sigma_t$ which agrees with the density in (8). Although computed for $n = 1$, in this example the density turns out to be independent of $n$.

**Step 3**: Computation of the flux $T^t$ for $n = 1$. Upon substitution of the constants $c_i$ into (24),

$$P = \tfrac{r}{\delta}\left(\sigma\sigma_t(\beta + \sigma^2) + \delta^2 u u_{ttt}\right), \tag{28}$$

which must be integrated with respect to $t$ to get the flux $T^t$. Actually, since $P = \mathrm{D}_r T^r = -\mathrm{D}_t T^t$, after integration by parts by hand or with *Mathematica*, one has

$$T^t = -\int P \, dt = -\tfrac{r}{4\delta}\left((\beta + \sigma^2)^2 - 2\delta^2(u_t^2 - 2u u_{tt})\right) \tag{29}$$

which matches the flux in (8) for $n = 1$.

In other examples, the expressions for $P$ are long and complicated. Since repeated integration by parts with *Mathematica* might fail, `ConservationLawsMD.m` uses the *homotopy operator* [40, p. 372] to reduce the integration with respect to $t$ to a standard integral with respect to a scaling parameter $\lambda$. The homotopy operator turns out to be a very useful tool for the computation of conservation laws (see, e.g., [5, 24, 25, 26, 27, 38, 42, 43]).

Application of the homotopy operator requires a few steps: (i) computation of an integrand for each of the dependent variables ($\sigma$ and $u$), (ii) scaling of the dependent variables (and their derivatives) with $\lambda$, and (iii) evaluation of one-dimensional integral with respect to $\lambda$.

In terms of the homotopy operator $\mathcal{H}_{\mathbf{u}}$,

$$T^t = -\mathcal{H}_{\mathbf{u}} P = -\int_0^1 (I_\sigma P + I_u P)[\lambda \mathbf{u}]\, \frac{d\lambda}{\lambda}, \tag{30}$$

where $\mathbf{u} = (\sigma, u)$ and $[\lambda\mathbf{u}]$ denotes that (in the integrands $I_\sigma P$ and $I_u P$) $\sigma$ is replaced by $\lambda\sigma$, $u$ by $\lambda u$, $\sigma_t$ by $\lambda\sigma_t$, $u_t$ by $\lambda u_t$, etc. The integrand for $\sigma$ [24, 26, 43] reads

$$\begin{aligned} I_\sigma P &= \sum_{k=1}^{K} \left( \sum_{i=0}^{k-1} \sigma_{it} (-\mathrm{D}_t)^{k-(i+1)} \right) \frac{\partial P}{\partial \sigma_{kt}} \\ &= (\sigma \mathrm{I})(\frac{\partial P}{\partial \sigma_t}) + (\sigma_t \mathrm{I} - \sigma \mathrm{D}_t)(\frac{\partial P}{\partial \sigma_{tt}}) + (\sigma_{tt}\mathrm{I} - \sigma_t \mathrm{D}_t + \sigma \mathrm{D}_t^2)(\frac{\partial P}{\partial \sigma_{ttt}}) + \ldots, \end{aligned} \tag{31}$$

where I denotes the identity operator. Applied to (28) where $K = 1$, one gets

$$I_\sigma P = (\sigma \mathrm{I})(\frac{\partial P}{\partial \sigma_t}) = \tfrac{r}{\delta} \sigma^2 (\beta + \sigma^2). \tag{32}$$

Using (31) for $u$ (instead of $\sigma$) and $K = 3$,

$$I_u P = -r\delta (u_t^2 - 2 u u_{tt}). \tag{33}$$

Then, from (30),

$$\begin{aligned} T^t &= -\frac{r}{\delta} \int_0^1 \left( \sigma^2(\beta + \sigma^2) - \delta^2 (u_t^2 - 2uu_{tt}) \right) [\lambda \mathbf{u}]\, \frac{d\lambda}{\lambda} \\ &= -\frac{r}{\delta} \int_0^1 \lambda \left( \sigma^2 (\beta + \lambda^2 \sigma^2) - \delta^2 (u_t^2 - 2uu_{tt}) \right) d\lambda \\ &= -\tfrac{r}{4\delta} \left( 2\beta\sigma^2 + \sigma^4 - 2\delta^2 (u_t^2 - 2uu_{tt}) \right). \end{aligned} \tag{34}$$

After adding $-\frac{r\beta^2}{4\delta}$ (which does not depend on $t$) to $T^t$, one gets (29). Once the density $T^r$ and flux $T^t$ are computed the code verifies that they indeed satisfy (3).

**Step 4**: Computation of the flux $T^t$ for arbitrary $n$. Recall that in this example $T^r$ is valid for any value of $n$. To determine the flux $T^t$ for arbitrary $n$, it suffices to compute the conservation laws for $n = 2$ and $n = 3$ for (1)-(2) and do some pattern matching. Or, one can provide the form of $T^r$ (e.g., $c_1 r u \sigma_t$) and let `ConservationLawsMD.m` compute $T^t$ automatically.

Alternatively, one can evaluate

$$\mathrm{D}_r T^r = \mathrm{D}_r (r u \sigma_t) = u\sigma_t + r u_r \sigma_t + r u \sigma_{tr} \tag{35}$$

on solutions of (1)-(2) yielding

$$P = \tfrac{r}{\delta} \left( \sigma\sigma_t (\beta + \sigma^2)^n + \delta^2 u u_{ttt} \right). \tag{36}$$

Thus,

$$T^t = -\int P\, dt = -\tfrac{r}{2(n+1)\delta} \left( (\beta + \sigma^2)^{n+1} - 2(n+1)\delta^2 (u_t^2 - 2uu_{tt}) \right) \tag{37}$$

which matches the flux in (8).

Conducting a search with `ConservationLawsMD.m` for conservation laws of (1)-(2) and (10)-(11), respectively, where the densities have coefficients up to *third degree* in $r$ and $t$, yields (6)-(8) and (12)-(14), respectively. No additional conservation laws for either model could be found. This is in stark contrast with the successful computation of seven conservation laws for a similar model in Cartesian coordinates [38] which is conjectured to have infinitely many conservation laws.

# 4 Computation of conservation laws with a multiplier method

Several methods are available to compute conservation laws (see, e.g., [7, 8, 39]). We refer the reader to [2, 16, 39, 54] for reviews and comparisons of commonly-used approaches.

In this section, we discuss the multiplier method [7, 39, 40, 53] for the computation of conservation laws. Using the multiplier approach, we derive the conservation laws (6)-(8) corresponding to the system (1)-(2). Similarly, the conservation laws (12)-(14) for the system (10)-(11) are also obtained. Instead of treating these systems separately, we combine them in a single system of PDEs by introducing suitable constants. Furthermore, by considering a general function $F(\sigma)$, this set-up covers a broad class of models for wave propagation in elastic materials formulated in cylindrical coordinates.

## 4.1 Conservation laws for (1)-(2) and (10)-(11)

Model (1)-(2) by Kambapalli et al. [31] and system (10)-(11) by Magan et al. [36] can be combined and generalized as follows:

$$\sigma_r + \frac{\kappa_1 \sigma}{r} = \delta u_{tt}, \tag{38}$$

$$u_r + \frac{\kappa_2 u}{r} = \tfrac{1}{\delta} F(\sigma). \tag{39}$$

Eliminating $u$ from (38)-(39) yields

$$\sigma_{rr} + (\kappa_1 + \kappa_2)\frac{\sigma_r}{r} - \kappa_1(1-\kappa_2)\frac{\sigma}{r^2} = F(\sigma)_{tt}. \tag{40}$$

For $F(\sigma) = \sigma(\beta + \sigma^2)^n$ and $\kappa_1 = 2$ and $\kappa_2 = -1$, system (38)-(39) matches (1)-(2) and (40) becomes (9). Likewise, for $\kappa_1 = 1$ and $\kappa_2 = 0$, one gets (10)-(11) and (15). Note that $\kappa_1 + \kappa_2 = 1$ for either set of parameters.

Several constitutive relations, $\epsilon = F(\sigma)$, relating scalar linearized strain ($\epsilon$) to Cauchy stress ($\sigma$, also scalar) have been reported in the literature. A dozen of those are summarized in Table 2 together with references where additional information can be obtained.

The examples are meant to illustrate the form $F(\sigma)$ can take. Not all these functions $F(\sigma)$ might be relevant for (38)-(39). One should check the cited references to derive the appropriate one-dimensional version of the implicit constitutive relation, $\boldsymbol{\epsilon} = \mathbf{F}(\boldsymbol{\sigma})$, relating the linearized strain ($\boldsymbol{\epsilon}$) and Cauchy stress ($\boldsymbol{\sigma}$) tensors for purely elastic deformations. It is also important to consider the most suitable coordinate system for the geometrical configuration and derive the correct equations of motion for the problem at hand. The reader is referred to [10, 13] for a discussion about the appropriateness and limitations of the constitutive relations within their physical contexts.

| $F(\sigma)$ | Remarks | References |
|---|---|---|
| $\sigma\big(1+\sigma^2\big)^n$ | $n \geq 0$ rational | [31, [Eq. (2.17)], [32, Eq. (28)], [34, Eqs. (4.2.7) & (5.2.6)], [35, Eq. (2.20)], [38, Eq. (1)] |
| $\frac{1}{\delta}\Big(\beta\sigma + \alpha\sigma(1+\frac{\gamma}{2}\sigma^2)^n\Big)$ | $n$ rational | [20, Eq. (2.4)], [32, Eq. (28)] |
| $\frac{\alpha\delta}{2}\sigma\big(1+\frac{1}{1+\sigma^2}\big)^n$ | $n \geq 0$ rational | [34, Eqs. (4.7.4) & (5.6.2)], [35, Eq. (2.21)], [36, Eq. (2.9)], [47, Eq. (2.17)] |
| $\frac{\alpha\delta}{2}\sigma(\frac{1}{1+\sigma^2})^n$ | $n \geq \frac{1}{2}$, rational | [36, Eq. (2.10)] |
| $\frac{\alpha}{\delta}\sigma\left(\frac{1}{(1+\lvert\sigma\rvert^n)^{\frac{1}{n}}}\right)$ | $n > 0$, integer | [20, Eq. (2.5)] [47, Eq. (3.10), $n=1$] |
| $\frac{\alpha}{\delta}\Big(-1+\frac{1}{1+\beta\sigma}+\frac{\gamma\sigma}{\sqrt{1+\iota\sigma^2}}\Big)$ | | [12, Eq. (25)], [14, Eq. (29)], [41, Eq. (15)], [51, Eq. (3.2)] |
| $\frac{\alpha}{\delta}\Big(-\frac{\beta\sigma}{1+\beta\sigma}+\frac{\mu\sigma}{\sqrt{1+\gamma^2\sigma^2}}\Big)$ | | [9, Eq. (47)] |
| $\frac{\alpha}{\delta}\Big(1-\frac{1}{1+\frac{\sigma}{1+\tilde{\delta}\lvert\sigma\rvert}}\Big) + \frac{\beta}{\delta}\sigma\big(1+\frac{1}{1+\gamma\sigma^2}\big)^n$ | $n$ rational | [20, Eq. (2.7)], [46, Eq. (3.13)] |
| $\frac{1}{\delta}\mathcal{N}\arctan(\theta\sigma)$ | | [51, Eq. (3.1)] |
| $\frac{1}{\delta}\big(-\alpha\tanh(\beta\sigma)+\gamma\sigma\frac{1}{\sqrt{1+\iota\sigma^2}}\big)$ | | [11, Eq. (13)] |
| $\frac{\alpha}{\delta}\Big(1-\mathrm{e}^{-\beta\sigma}+\frac{\gamma\sigma}{1+\iota\lvert\sigma\rvert}\Big)$ | | [12, Eq. (28)] |
| $\frac{1}{\delta}\Big(\beta_2\sigma+\beta_3\sigma\mathrm{e}^{(1+\beta_4\sigma^2)^{\frac{n}{2}}}\Big)$ | $n \geq 0$, integer | [19, Eq. (28)] |
| $\frac{\alpha\gamma}{\delta}\big(\frac{\sigma}{1+\lvert\sigma\rvert}\big) + \frac{\alpha}{\delta}\Big(1-\mathrm{e}^{-\frac{\beta\sigma}{1+\tilde{\delta}\lvert\sigma\rvert}}\Big)$ | | [20, Eq. (2.6)], [46, Eq. (3.12)] |
| $\frac{\beta}{\delta}\sigma\big((1+\gamma\sigma^n)^{-\frac{1}{n}}\big) + \frac{\alpha}{\delta}\Big(1-\mathrm{e}^{-\frac{\lambda\sigma}{1+\tilde{\delta}\sigma}}\Big)$ | $n$ rational | [20, Eq. (2.6), n = 1], [33, Eq. (28)], [46, Eq. (3.14)], [48, Eq. (34), n = 1] |

**Table 2.** Examples of scalar (1D) versions of constitutive relations from the literature. The meaning of the material parameters ($\alpha, \beta, \gamma, \iota$, etc.) can be found in the cited references.

Therefore we investigate system (38)-(39) for a class of constitutive relations with an unspecified $F(\sigma)$ and arbitrary $\kappa_1$ and $\kappa_2$. System (38)-(39) can be expressed as

$$E^1(r,t,\sigma,u,\sigma_r,u_{tt}) = 0, \quad E^2(r,t,\sigma,u,u_r) = 0, \tag{41}$$

where

$$E^1 = \sigma_r + \frac{\kappa_1 \sigma}{r} - \delta u_{tt}, \quad E^2 = u_r + \frac{\kappa_2 u}{r} - \tfrac{1}{\delta} F(\sigma). \tag{42}$$

The conservation law is written in characteristic form [39, 40, 53] as

$$\mathrm{D}_r T^r + \mathrm{D}_t T^t = \Lambda^1 \left(\sigma_r + \frac{\kappa_1 \sigma}{r} - \delta u_{tt}\right) + \Lambda^2 \left(u_r + \frac{\kappa_2 u}{r} - \tfrac{1}{\delta} F(\sigma)\right), \tag{43}$$

with $\mathrm{D}_r$ and $\mathrm{D}_t$ given in (4) and (5). The differential functions $\Lambda^1$ and $\Lambda^2$ are the multipliers (a.k.a. characteristics). We assume $\Lambda^1(r,t,\sigma,u,\sigma_t,u_t)$ and the same dependencies for $\Lambda^2$. Once restricted to *solutions* of (41), the characteristic equation (43) yields a local conservation law (3). As outlined in Olver [40], the determining equations for the multipliers $\Lambda^1$ and $\Lambda^2$ are established by applying the Euler operators to the characteristic equation (43) with respect to the dependent variables $\sigma$ and $u$. Hence,

$$\mathcal{E}_\sigma \left[\Lambda^1 \left(\sigma_r + \frac{\kappa_1 \sigma}{r} - \delta u_{tt}\right) + \Lambda^2 \left(u_r + \frac{\kappa_2 u}{r} - \tfrac{1}{\delta} F(\sigma)\right)\right] = 0, \tag{44}$$

$$\mathcal{E}_u \left[\Lambda^1 \left(\sigma_r + \frac{\kappa_1 \sigma}{r} - \delta u_{tt}\right) + \Lambda^2 \left(u_r + \frac{\kappa_2 u}{r} - \tfrac{1}{\delta} F(\sigma)\right)\right] = 0, \tag{45}$$

where $\mathcal{E}_\sigma$ and $\mathcal{E}_u$ are the standard Euler operators (see (25) and (27), respectively). Equations (44) and (45) result in an overdetermined system of PDEs for the multipliers $\Lambda^1$ and $\Lambda^2$. The derivation and solution of such determining equations has been implemented in `GeM` [15, 16, 17] and other symbolic packages [54].

After some simplifications, the determining equations are

$$\begin{aligned}
\Lambda^1_{\sigma_t} = 0,\ \Lambda^1_{u_t u_t} = 0,\ \Lambda^1_{\sigma} = 0,\ \Lambda^2_{u_t} = 0,\ \Lambda^2_{\sigma_t \sigma_t} &= 0, \\
\Lambda^1_{u_t} + \Lambda^2_{\sigma_t} = 0,\ \Lambda^1_u - \Lambda^2_\sigma - \Lambda^1_{t u_t} - u_t \Lambda^1_{u u_t} &= 0, \\
\Lambda^1_u - \Lambda^2_\sigma + \Lambda^2_{t\sigma_t} + u_t \Lambda^2_{u\sigma_t} + \sigma_t \Lambda^2_{\sigma\sigma_t} &= 0, \\
2\Lambda^1_u + \Lambda^1_{t u_t} + u_t \Lambda^1_{u u_t} &= 0, \\
\left(\Lambda^2_{t\sigma_t} - \Lambda^2_\sigma + u_t \Lambda^2_{u\sigma_t} + \sigma_t \Lambda^2_{\sigma\sigma_t}\right) F(\sigma) + \left(\sigma_t \Lambda^2_{\sigma_t} - \Lambda^2\right) F'(\sigma) & \\
+ \tfrac{\delta}{r} \left[k_2 u \left(\Lambda^2_\sigma - u_t \Lambda^2_{u\sigma_t} - \sigma_t \Lambda^2_{\sigma\sigma_t} - \Lambda^2_{t\sigma_t}\right) + k_1 \Lambda^1 - r\Lambda^1_r - k_2 u_t \Lambda^2_{\sigma_t}\right] &= 0, \\
\Lambda^2_u F(\sigma) + \delta \Lambda^2_r + \delta^2 (\Lambda^1_{uu} u_t^2 + \Lambda^1_{tt} + 2\Lambda^1_{tu} u_t) & \\
- \tfrac{\delta}{r} \left[k_1 \sigma \left(\Lambda^1_u - u_t \Lambda^1_{u_t u} - \Lambda^1_{u_t t}\right) + k_2 u \Lambda^2_u + k_2 \Lambda^2 - k_1 \sigma_t \Lambda^1_{u_t}\right] &= 0.
\end{aligned} \tag{46}$$

Solving (46) for an arbitrary $F(\sigma)$ yields the following multipliers

$$\Lambda^1 = c_1 r^{\kappa_1} + c_2 t r^{\kappa_1} - c_3 r^{\kappa_1 + \kappa_2} u_t, \quad \Lambda^2 = c_3 r^{\kappa_1 + \kappa_2} \sigma_t. \tag{47}$$

Next, one substitutes (47) into (43) and determines $T^r$ and $T^t$. Direct integration [18] as implemented in `GeM` [15, 16, 17] yields

$$\begin{aligned}
&\mathrm{D}_r \left[c_1 r^{\kappa_1} \sigma + c_2 r^{\kappa_1} t\sigma + c_3 r^{\kappa_1 + \kappa_2} u \sigma_t\right] + \mathrm{D}_t \left[-c_1 \delta r^{\kappa_1} u_t + c_2 \delta r^{\kappa_1} (u - t u_t)\right. \\
&\left. + c_3\, r^{\kappa_1 + \kappa_2} \left(\tfrac{\delta}{2}(u_t^2 - 2 u u_{tt}) - \tfrac{1}{\delta} \int F(\sigma)\, d\sigma\right)\right] \doteq 0.
\end{aligned} \tag{48}$$

Alternatively, one could use a homotopy operator method [5, 43] to compute $T^r$ and $T^t$.

Any density-flux pair for system (38)-(39) with multipliers of the form $\Lambda^1(r,t,\sigma,u,\sigma_t,u_t)$ and $\Lambda^2(r,t,\sigma,u,\sigma_t,u_t)$ is therefore a linear combination of the three conserved densities and associated fluxes presented in Table 3 together with the corresponding multipliers. For the function $F(\sigma)$ one could take examples from Table 2.

| Multipliers $(\Lambda_1,\Lambda_2)$ | Density $T^r$ | Flux $T^t$ |
|---|---|---|
| $(r^{\kappa_1},\,0)$ | $r^{\kappa_1}\sigma$ | $-\delta r^{\kappa_1}u_t$ |
| $(tr^{\kappa_1},\,0)$ | $r^{\kappa_1}t\sigma$ | $\delta r^{\kappa_1}(u-tu_t)$ |
| $(-r^{\kappa_1+\kappa_2}u_t,\,r^{\kappa_1+\kappa_2}\sigma_t)$ | $r^{\kappa_1+\kappa_2}u\sigma_t$ | $r^{\kappa_1+\kappa_2}\left(\frac{\delta}{2}(u_t^2-2uu_{tt})-\frac{1}{\delta}\int F(\sigma)\,d\sigma\right)$ |

**Table 3.** Multipliers, conserved densities, and fluxes for system (38)-(39) where $\kappa_1$ and $\kappa_2$ are arbitrary. The densities for (1)-(2) corresponding to $\kappa_1=2$ and $\kappa_2=-1$ are given in (6)-(8). Those for (10)-(11) corresponding to $\kappa_1=1$ and $\kappa_2=0$ can be found in (12)-(14).

In summary, once the dependencies of the multipliers are selected, the code `GeM` computes the multipliers (47) as well as the densities and associated fluxes using one of several approaches [16]: direct integration [18], homotopy techniques [5, 25, 26, 43] or a scaling-symmetry based formula for the conserved vector [1]. Instructions for how to use `GeM` can be found on Cheviakov's website [17].

Restricted to the models in Section 2 for which $\kappa_1+\kappa_2=1$, with `ConservationLawsMD.m` we computed conservation laws of (38)-(39) with $F(\sigma)=\sigma(\beta+\sigma^2)^n$ and $\kappa_2=1-\kappa_1$. Doing so for $n=1$ and coefficients up to degree three in $r$ and $t$, the output of `ConservationLawsMD.m` confirms the results in Table 3.

### 4.2 Conservation laws of (40)

For completeness, we compute low-order conservation laws for the wave equation (40) associated with (38)-(39). The results for (9) and (15) associated with (1)-(2) and (10)-(11), respectively, are special cases. The results are summarized in Table 4.

## 5 Application 1: Khokhlov-Zabolotskaya-Kuznetsov equation

In this section we investigate the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation which is primarily used to model the propagation of sound beams in various nonlinear media. We compute conservation laws for the KZK equation in Cartesian coordinates as well as cylindrical and spherical coordinates.

| Multiplier $\Lambda$ | Density $T^r$ | Flux $T^t$ |
|---|---|---|
| $r^{\kappa_2}$ | $r^{\kappa_2-1}(r\sigma_r+\kappa_1\sigma)$ | $-r^{\kappa_2}\sigma_t F'$ |
| $tr^{\kappa_2}$ | $tr^{\kappa_2-1}(r\sigma_r+\kappa_1\sigma)$ | $r^{\kappa_2}(F-t\sigma_t F')$ |
| $r^{1+\kappa_1}$ | $r^{\kappa_1}\big(r\sigma_r-(1-\kappa_2)\sigma\big)$ | $-r^{1+\kappa_1}\sigma_t F'$ |
| $tr^{1+\kappa_1}$ | $tr^{\kappa_1}\big(r\sigma_r-(1-\kappa_2)\sigma\big)$ | $r^{\kappa_1+1}(F-t\sigma_t F')$ |
| $\frac{1}{r}$ | $\frac{1}{r^2}(r\sigma_r+2\sigma)$ | $-\frac{1}{r}\sigma_t(\beta+\sigma^2)^{n-1}\big(\beta+(2n+1)\sigma^2\big)$ |
| $\frac{t}{r}$ | $\frac{t}{r^2}(r\sigma_r+2\sigma)$ | $\frac{1}{r}(\beta+\sigma^2)^{n-1}\big(\sigma(\beta+\sigma^2)-t\sigma_t\big(\beta+(2n+1)\sigma^2\big)\big)$ |
| $r^3$ | $r^2(r\sigma_r-2\sigma)$ | $-r^3\sigma_t(\beta+\sigma^2)^{n-1}\big(\beta+(2n+1)\sigma^2\big)$ |
| $tr^3$ | $tr^2(r\sigma_r-2\sigma)$ | $r^3(\beta+\sigma^2)^{n-1}\big(\sigma(\beta+\sigma^2)-t\sigma_t\big(\beta+(2n+1)\sigma^2\big)\big)$ |
| $1$ | $\frac{1}{r}(r\sigma_r+\sigma)$ | $-\sigma_t(\beta+\sigma^2)^{n-1}\big(\beta+(2n+1)\sigma^2\big)$ |
| $t$ | $\frac{t}{r}(r\sigma_r+\sigma)$ | $(\beta+\sigma^2)^{n-1}\big(\sigma(\beta+\sigma^2)-t\sigma_t\big(\beta+(2n+1)\sigma^2\big)\big)$ |
| $r^2$ | $r(r\sigma_r-\sigma)$ | $-r^2\sigma_t(\beta+\sigma^2)^{n-1}\big(\beta+(2n+1)\sigma^2\big)$ |
| $tr^2$ | $tr(r\sigma_r-\sigma)$ | $r^2(\beta+\sigma^2)^{n-1}\big(\sigma(\beta+\sigma^2)-t\sigma_t\big(\beta+(2n+1)\sigma^2\big)\big)$ |

**Table 4.** The top set are the multipliers, conserved densities, and fluxes for the parameterized wave equation (40) with arbitrary $\kappa_1$ and $\kappa_2$. The middle set is for (9) where $\kappa_1 = 2$ and $\kappa_2 = -1$. The bottom set is for (15) where $\kappa_1 = 1$ and $\kappa_2 = 0$.

## 5.1 Conservation laws of the Khokhlov-Zabolotskaya-Kuznetsov equation in Cartesian coordinates

The KZK equation [23, Ch. 3, Eq. (3.65)],

$$p_{zt} - \tfrac{c_0}{2}\nabla_\perp^2 p - \tfrac{\delta}{2c_0^3}p_{ttt} - \tfrac{\beta}{2\rho_0 c_0^3}(p^2)_{tt} = 0, \tag{49}$$

describes the propagation of a beam in a nonlinear medium. For simplicity of notation we have replaced the retarded time variable ($\tau$) by $t$. The second term in (49) accounts for the diffraction of the beam. The operator $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian acting on the plane perpendicular to the $z-$axis along which the beam propagates. For acoustic waves, $p$ denotes the sound pressure, $\rho_0$ a reference density of the medium, and $c_0$ the speed of sound. The positive parameters $\beta$ and $\delta$ are coefficients of nonlinearity and diffusion (viscosity, absorption), respectively. The reader is referred to [49] for a detailed discussion of various derivations of (49) and historical references.

Clearing denominators yields

$$\delta p_{ttt} + \tilde{\beta}(p^2)_{tt} - 2c_0^3 p_{zt} + c_0^4(p_{xx} + p_{yy}) = 0, \tag{50}$$

where $\tilde{\beta} = \frac{\beta}{\rho_0}$, for which we will compute low-order conservation laws with the multiplier method. The conservation law in characteristic form reads as

$$\mathrm{D}_t\, T^t + \mathrm{D}_x\, T^x + \mathrm{D}_y\, T^y + \mathrm{D}_z\, T^z = \Lambda\left(\delta p_{ttt} + \tilde{\beta}(p^2)_{tt} - 2c_0^3 p_{zt} + c_0^4(p_{xx} + p_{yy})\right), \tag{51}$$

where $\Lambda$ is a characteristic (a.k.a. multiplier). Total derivative $\mathrm{D}_t$ is given in (5) and $\mathrm{D}_x$, $\mathrm{D}_y$, and $\mathrm{D}_z$ are defined analogously. When (51) is evaluated on solutions of (50) one gets a local conservation law. To compute low-order conservation laws we assume $\Lambda(x, y, z, t, p, p_x, p_y, p_z, p_t)$.

Taking the variational derivative of the characteristic equation (51) yields

$$\mathcal{E}_p\left[\Lambda\left(\delta p_{ttt} + \tilde{\beta}(p^2)_{tt} - 2c_0^3 p_{zt} + c_0^4(p_{xx} + p_{yy})\right)\right] = 0, \tag{52}$$

where $\mathcal{E}_p$ is the Euler operator [43]

$$\begin{aligned}\mathcal{E}_p &= \sum_{k=0}^{K}\sum_{\ell=0}^{L}\sum_{m=0}^{M}\sum_{n=0}^{N}(-\mathrm{D}_x)^k(-\mathrm{D}_y)^\ell(-\mathrm{D}_z)^m(-\mathrm{D}_t)^n\frac{\partial}{\partial p_{kx\,\ell y\,mz\,nt}}\\ &= \frac{\partial}{\partial p} - \mathrm{D}_x\frac{\partial}{\partial p_x} - \mathrm{D}_y\frac{\partial}{\partial p_y} - \mathrm{D}_z\frac{\partial}{\partial p_z} - \mathrm{D}_t\frac{\partial}{\partial p_t} + \mathrm{D}_x^2\frac{\partial}{\partial p_{xx}} + \mathrm{D}_y^2\frac{\partial}{\partial p_{yy}}\\ &\quad + \mathrm{D}_z^2\frac{\partial}{\partial p_{zz}} + \mathrm{D}_t^2\frac{\partial}{\partial p_{tt}} + \cdots + \mathrm{D}_z\,\mathrm{D}_t\frac{\partial}{\partial p_{zt}} + \cdots - \mathrm{D}_t^3\frac{\partial}{\partial p_{ttt}} + \ldots,\end{aligned} \tag{53}$$

where $K$, $L$, $M$ and $N$ are the highest-orders of respective derivatives of $p$ appearing in the expression $\mathcal{E}_p$ acts upon.

Equation (52) results in an overdetermined system of PDEs for the multiplier $\Lambda$ (which can be automatically computed with `GeM`):

$$\Lambda_{tt} = 0,\ \Lambda_p = 0,\ \Lambda_{p_x} = 0,\ \Lambda_{p_y} = 0,\ \Lambda_{p_z} = 0,\ \Lambda_{p_t} = 0,\ \Lambda_{tz} = \tfrac{c_0}{2}\left(\Lambda_{xx} + \Lambda_{yy}\right). \tag{54}$$

Solving (54) yields

$$\Lambda = \phi(x, y, z) + t\,\psi(x, y, z), \tag{55}$$

where $\phi(x, y, z)$ and $\psi(x, y, z)$ must satisfy

$$\phi_{xx} + \phi_{yy} = \tfrac{2}{c_0}\psi_z, \quad \psi_{xx} + \psi_{yy} = 0. \tag{56}$$

The Laplace equation has infinitely many solutions $\psi(x, y; z)$, called harmonic functions, which in this case are parameterized by $z$. Once a solution for $\psi(x, y; z)$ is selected, one has to solve the above Poisson equation for $\phi(x, y, z)$, using, e.g., the Green's function approach. Since there are infinitely many solutions for $\phi$ (with corresponding $\psi$) their exist infinite many conservation laws for (49). From (51), (55), and (56) one obtains

$$\begin{aligned}&\mathrm{D}_t\left[\Lambda(\delta p_{tt} + 2\tilde{\beta}pp_t - 2c_0^3 p_z) - \Lambda_t(\delta p_t + \tilde{\beta}p^2)\right] + \mathrm{D}_x\left[c_0^4(\Lambda p_x - p\Lambda_x)\right]\\ &+\mathrm{D}_y\left[c_0^4(\Lambda p_y - p\Lambda_y)\right] + \mathrm{D}_z\left[2c_0^3 p\Lambda_t\right] = \Lambda\left(\delta p_{ttt} + \tilde{\beta}(p^2)_{tt} - 2c_0^3 p_{zt} + c_0^4(p_{xx} + p_{yy})\right)\\ &-c_0^4 p(\Lambda_{xx} + \Lambda_{yy} - \tfrac{2}{c_0}\Lambda_{tz})\end{aligned} \tag{57}$$

for arbitrary $p(x, y, z, t)$. Note that the last term in (57) vanishing since the last equation in (56) holds. Then, when $p(x, y, z, t)$ is a solution system (50),

$$\begin{aligned} &\mathrm{D}_t \left[\Lambda(\delta p_{tt} + 2\tilde{\beta} p p_t - 2c_0^3 p_z) - \Lambda_t(\delta p_t + \tilde{\beta} p^2)\right] + \mathrm{D}_x \left[c_0^4(\Lambda p_x - p\Lambda_x)\right] \\ &+ \mathrm{D}_y \left[c_0^4(\Lambda p_y - p\Lambda_y)\right] + \mathrm{D}_z \left[2c_0^3 p\Lambda_t\right] \doteq 0, \end{aligned} \tag{58}$$

a local conservation law follows. The conserved densities and fluxes for system (50) with multiplier (55) are

$$\begin{aligned} T^t &= (\phi + t\psi)(\delta p_{tt} + 2\tilde{\beta} p p_t - 2c_0^3 p_z) - \psi(\delta p_t + \tilde{\beta} p^2), \\ T^x &= c_0^4 \left[(\phi + t\psi) p_x - p(\phi_x + t\psi_x)\right], \\ T^y &= c_0^4 \left[(\phi + t\psi) p_y - p(\phi_y + t\psi_y)\right], \\ T^z &= 2c_0^3 p\psi, \end{aligned} \tag{59}$$

where $\phi(x, y, z)$ and $\psi(x, y, z)$ are solutions of (56).

## 5.2 Conservation laws of the Khokhlov-Zabolotskaya-Kuznetsov equation in cylindrical and spherical coordinates

Using the multiplier method, we compute some conservation laws of the KZK equation [22, Ch. 8, Eq. (8.1)] in cylindrical and spherical coordinates,

$$p_{rr} + \frac{p_r}{r} + \frac{\delta}{c_0^4} p_{ttt} + \frac{\beta}{\rho_0 c_0^4}(p^2)_{tt} - \frac{2}{c_0} p_{zt} = 0, \tag{60}$$

where the meaning of the symbols is analogous to (49). Taking a more general form,

$$c_0^4(p_{rr} + \frac{2m p_r}{r}) + \delta p_{ttt} + \tilde{\beta} F(p)_{tt} - 2c_0^3 p_{zt} = 0, \tag{61}$$

with $\tilde{\beta} = \frac{\beta}{\rho_0}$, allows us to treat the cylindrical ($m = \frac{1}{2}$) and spherical ($m = 1$) cases and an arbitrary nonlinear function $F(p)$ at all once. Using `GeM` and the procedure outlined in the previous section then yields the determining equations for the multiplier $\Lambda(r, z, t, p, p_r, p_z, p_t)$:

$$\Lambda_{tt} = 0,\ \Lambda_p = 0,\ \Lambda_{p_r} = 0,\ \Lambda_{p_z} = 0,\ \Lambda_{p_t} = 0,\ \Lambda_{rr} = \frac{2}{c_0}\Lambda_{tz} + \frac{2m}{r^2}(r\Lambda_r - \Lambda), \tag{62}$$

provided $F'' \neq 0$. Solving (62) one obtains

$$\Lambda = \phi(r, z) + t\,\psi(r, z), \tag{63}$$

where $\phi(r, z)$ and $\psi(r, z)$ must be solutions of

$$\phi_{rr} - \frac{2m}{r}\phi_r + \frac{2m}{r^2}\phi = \frac{2}{c_0}\psi_z, \quad \psi_{rr} - \frac{2m}{r}\psi_r + \frac{2m}{r^2}\psi = 0. \tag{64}$$

The conserved density and flux for (61) with multiplier $\Lambda(r, z, t, p, p_r, p_z, p_t)$ in (63) are

$$\begin{aligned} T^t &= (\phi + t\psi)(\delta p_{tt} + \tilde{\beta} F'(p) p_t - 2c_0^3 p_z) - \psi(\delta p_t + \tilde{\beta} F(p)), \\ T^r &= c_0^4 \Big((\phi + t\psi)(p_r + \frac{2mp}{r}) - p(\phi_r + t\psi_r)\Big), \\ T^z &= 2c_0^3 p\psi, \end{aligned} \tag{65}$$

where $\phi(r, z)$ and $\psi(r, z)$ are solutions of (64).

# 6 Application 2: Westervelt-type equations

Like the KZK equation, the Westervelt equation is one of the most important nonlinear PDEs describing the propagation of nonlinear waves in acoustics. We first compute conservation laws for the Westervelt equation in one and two dimensions before generalizing the results to any number of spatial variables. In addition, we derive conservation laws for the Westervelt equation in cylindrical and spherical coordinates.

## 6.1 Conservation laws of Westervelt-type equations in Cartesian coordinates

### 6.1.1 Conservation laws of Westervelt-type equations in two dimensions

The Westervelt equation [23, Ch. 3, Eq. (3.46)]

$$\Box^2 p + \tfrac{\delta}{c_0^4} p_{ttt} + \tfrac{\gamma}{\rho_0 c_0^4} (p^2)_{tt} = 0 \tag{66}$$

describes the propagation of acoustic waves with acoustic pressure $p$ in various media. In Cartesian coordinates, $\Box^2 = \nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}$ and $\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ are the d'Alembertian and Laplacian operators, respectively. In (66), $\rho_0$ denotes the equilibrium density of the medium and $c_0$ is the speed of sound; $\delta > 0$ is a diffusivity parameter (e.g., dissipation or loss of sound pressure due to viscosity and thermal conduction in the medium), and $\gamma > 0$ is a coefficient of nonlinearity.

Instead of analyzing (66) we will work with

$$F(p)_{tt} - c^2(p_{xx} + p_{yy}) - \alpha p_{ttt} - \beta(p_{xx} + p_{yy})_t = 0, \tag{67}$$

which includes the term in $\Delta_t\, p = \Delta p_t$ considered in [30, Eq. (10)] with a diffusivity-stabilizing parameter $\beta$, and an arbitrary nonlinear function $F(p)$. For completeness we kept the third-order term in time from (66) which also appears in some extended models discussed in [30, Eqs. (63) and (76)] and references therein.

For $F(p) = \rho_0 c_0^2 p - \gamma p^2$, $c^2 = \rho_0 c_0^4$, $\alpha = \delta \rho_0$, and $\beta = 0$, (67) becomes (66) in two spacial dimensions. The one-dimensional version of (66) was investigated by Anco et al. [6]. The conservation laws obtained by these authors are a special case of those discussed in Section 6.1.3.

Equation (67) in one spatial dimension with $\alpha = 0$ was studied by Márquez et al. [37, Eq. (3)]. It arises in nonlinear elasticity where $p$ then represents stress $\sigma$. See, [20, Eq. (1.1)] and [50, Eqs. (22) and (24)] with $F(p)$ as in Table 2.

Using `GeM`, the determining equations for the multiplier $\Lambda(x, y, t, p, p_x, p_y, p_t)$ are

$$\Lambda_{tt} = 0,\ \ \Lambda_p = 0,\ \ \Lambda_{p_x} = 0,\ \ \Lambda_{p_y} = 0,\ \ \Lambda_{p_t} = 0,\ \ \Lambda_{xx} + \Lambda_{yy} = 0 \tag{68}$$

provided $F'' \neq 0$. Solving (68) one gets

$$\Lambda = \phi(x, y) + t\,\psi(x, y), \tag{69}$$

where $\phi(x, y)$ and $\psi(x, y)$ must both satisfy the Laplace equation:

$$\phi_{xx} + \phi_{yy} = 0, \quad \psi_{xx} + \psi_{yy} = 0. \tag{70}$$

The conservation law in characteristic form for the multipliers (69) satisfying (70) reads

$$\mathrm{D}_t T^t + \mathrm{D}_x T^x + \mathrm{D}_y T^y = (\phi(x,y) + t\psi(x,y)) \big(F(p)_{tt} - c^2(p_{xx} + p_{yy}) \\ -\alpha p_{ttt} - \beta(p_{xx} + p_{yy})_t\big)\,, \tag{71}$$

for arbitrary $p(x, y, t)$. By integration of (71), the density and flux can be readily derived, resulting in

$$\mathrm{D}_t \left[(\phi + t\psi)\left(F' p_t - \alpha p_{tt}\right) - \psi(F(p) - \alpha p_t)\right] + \mathrm{D}_x \left[-c^2 \left[(\phi + t\psi)p_x - p(\phi_x + t\psi_x)\right]\right. \\ -\beta[(\phi + t\psi)p_{tx} - (\phi_x + t\psi_x)p_t]\big] + \mathrm{D}_y \left[-c^2 \left[(\phi + t\psi)p_y - p(\phi_y + t\psi_y)\right] - \beta\left[(\phi + t\psi)p_{ty}\right.\right. \\ -(\phi_y + t\psi_y)p_t]\big] = (\phi(x,y) + t\psi(x,y))\left(F(p)_{tt} - c^2(p_{xx} + p_{yy}) - \alpha p_{ttt} - \beta(p_{xx} + p_{yy})_t\right) \\ +(pc^2 + \beta p_t)\left(\phi_{xx} + \phi_{yy} + t(\psi_{xx} + \psi_{yy})\right) \tag{72}$$

for arbitrary $p(x, y, t)$. Note that the last term vanishes as a consequence of (70).

Evaluated on solutions $p(x, y, t)$ of (67) one then gets

$$\mathrm{D}_t \left[(\phi + t\psi)\left(F' p_t - \alpha p_{tt}\right) - \psi(F(p) - \alpha p_t)\right] + \mathrm{D}_x \left[-c^2 \left[(\phi + t\psi)p_x - p(\phi_x + t\psi_x)\right]\right. \\ -\beta[(\phi + t\psi)p_{tx} - (\phi_x + t\psi_x)p_t]\big] + \mathrm{D}_y \left[-c^2 \left[(\phi + t\psi)p_y - p(\phi_y + t\psi_y)\right]\right. \\ -\beta[(\phi + t\psi)p_{ty} - (\phi_y + t\psi_y)p_t]\big] \doteq 0. \tag{73}$$

Therefore, the density and flux for (67) corresponding to the multiplier family $\Lambda(x, y, t, p)$ in (69) are

$$\begin{aligned} T^t &= (\phi + t\psi)\left(F' p_t - \alpha p_{tt}\right) - \psi(F(p) - \alpha p_t), \\ T^x &= -c^2 \left[(\phi + t\psi)p_x - p(\phi_x + t\psi_x)\right] - \beta[(\phi + t\psi)p_{tx} - (\phi_x + t\psi_x)p_t], \\ T^y &= -c^2 \left[(\phi + t\psi)p_y - p(\phi_y + t\psi_y)\right] - \beta[(\phi + t\psi)p_{ty} - (\phi_y + t\psi_y)p_t], \end{aligned} \tag{74}$$

where $\phi(x, y)$ and $\psi(x, y)$ are solutions of (70).

#### 6.1.2 Conservation laws of Westervelt-type equations in three dimensions

The three-dimensional version of the Westervelt equation (in one spatial dimension) investigated by Márquez et al. [37] reads

$$F(p)_{tt} - \alpha p_{ttt} - \beta(p_{xx} + p_{yy} + p_{zz})_t = c^2(p_{xx} + p_{yy} + p_{zz}). \tag{75}$$

Again using `GeM`, the determining equations for $\Lambda(x, y, z, t, p, p_x, p_y, p_z, p_t)$ are

$$\Lambda_{tt} = 0,\ \ \Lambda_p = 0,\ \ \Lambda_{p_x} = 0,\ \ \Lambda_{p_y} = 0,\ \ \Lambda_{p_z} = 0,\ \ \Lambda_{p_t} = 0,\ \ \Lambda_{xx} + \Lambda_{yy} + \Lambda_{zz} = 0, \tag{76}$$

provided $F'' \neq 0$. Solving (76) yields

$$\Lambda = \phi(x, y, z) + t\,\psi(x, y, z), \tag{77}$$

where $\phi(x, y, z)$ and $\psi(x, y, z)$ must solve the Laplace equation,

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0, \quad \psi_{xx} + \psi_{yy} + \psi_{zz} = 0. \tag{78}$$

With the family of multipliers $\Lambda$ in (77), the density and flux for (75) read

$$\begin{aligned} T^t &= (\phi + t\psi)\left(F' p_t - \alpha p_{tt}\right) - \psi(F(p) - \alpha p_t), \\ T^x &= -c^2 \left[(\phi + t\psi)p_x - p(\phi_x + t\psi_x)\right] - \beta[(\phi + t\psi)p_{tx} - (\phi_x + t\psi_x)p_t], \\ T^y &= -c^2 \left[(\phi + t\psi)p_y - p(\phi_y + t\psi_y)\right] - \beta[(\phi + t\psi)p_{ty} - (\phi_y + t\psi_y)p_t], \\ T^z &= -c^2 \left[(\phi + t\psi)p_z - p(\phi_z + t\psi_z)\right] - \beta[(\phi + t\psi)p_{tz} - (\phi_z + t\psi_z)p_t], \end{aligned} \tag{79}$$

where $\phi(x, y, z)$ and $\psi(x, y, z)$ are any solutions of (78).

### 6.1.3 Conservation laws of Westervelt-type equations in any number of spatial variables

The Westervelt equation in $n$ independent variables $(x_1, x_2, \cdots, x_i, \cdots, x_n)$ denoted below as $(x_i)$, and time $(t)$ is

$$F(p)_{tt} - \alpha p_{ttt} - \beta \nabla^2 p_t = c^2 \nabla^2 p, \tag{80}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_i^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \tag{81}$$

Analogous to the two- and three-dimensional cases, the density and flux for (80) with the family of multipliers of type

$$\Lambda(x_i, t, p, p_{x_i}, p_t) = \phi(x^i) + t\,\psi(x^i) \tag{82}$$

are

$$\begin{aligned} T^t &= (\phi + t\psi)\left(F' p_t - \alpha p_{tt}\right) - \psi(F(p) - \alpha p_t), \\ T^{x^i} &= -c^2\left[(\phi + t\psi)p_{x^i} - p(\phi_{x^i} + t\psi_{x^i})\right] - \beta[(\phi + t\psi)p_{tx^i} - (\phi_{x^i} + t\psi_{x^i})p_t], \end{aligned} \tag{83}$$

where $\phi(x^i)$ and $\psi(x^i)$ must be harmonic functions (i.e., solutions of the $n-$dimensional Laplace equation),

$$\nabla^2\phi = 0, \quad \nabla^2\psi = 0. \tag{84}$$

Using a theorem by Igonin [29] and the direct method [4, 40], Sergyeyev [52] has derived the local conservation laws of all orders of (80) with $\alpha \neq 0$ and $\beta = 0$ (i.e., the lossless case). He has also shown that for the one-dimensional Westervelt equation there are no other local conservation laws than those reported by Anco et al. [6, Eqs. (20)-(23)]. In the one-dimensional case, apart from their linearity in $t$ expressed in (82), the functions $\phi(x)$ and $\psi(x)$ are linear in $x$ because the conditions (84) reduce to $\phi_{xx} = \psi_{xx} = 0$. Sergyeyev also gives a general formula for the conservation laws of all orders (see, [52, Eq. (2)]) which matches (83) again $\beta = 0$ and $c = 1$. Remarkably, the one-dimensional Westervelt equation has a finite number of conservation laws whereas in multiple space variables the equation has infinitely many conservation laws.

## 6.2 Conservation laws of Westervelt-type equations in spherical and cylindrical coordinates

In this section we consider the Westervelt equation [3, Eq. (7.1)] in a generalized form to cover the planar, cylindrical and spherical cases,

$$p_{rr} + \tfrac{2m}{r}p_r - p_{tt} + \gamma(p^2)_{tt} + \alpha p_{ttt} + \beta(p_{rr} + \tfrac{2m}{r}p_r)_t = 0, \tag{85}$$

where, for convenience, the units are chosen [6] such that $c = 1$. Note that $m = 0$ corresponds to the planar case (replacing $r$ by $x$); $m = \frac{1}{2}$ covers the cylindrical case; and $m = 1$ the spherical case. The latter case is treated in [3, Eq. (7.1)].

For the planar case ($m = 0$ with $x$ instead of $r$), (85) can be written as

$$\big((p - \gamma p^2 - \alpha p_t)_t - \beta p_{xx}\big)_t + (-p_x)_x = 0, \tag{86}$$

which is a conservation law itself.

Using GeM and a multiplier of type $\Lambda(r, t, p, p_r, p_t)$, we computed the conservation laws for (85) presented in Table 5 for arbitrary values of $m$ as well as for $m = 1$ and $m = \frac{1}{2}$.

| $m$ | $\Lambda$ | Density $T^r$ | Flux $T^t$ |
|---|---|---|---|
| | $r^{2m}$ | $-r^{2m}(p_r + \beta p_{rt})$ | $r^{2m}\big((1-2\gamma p)p_t - \alpha p_{tt}\big)$ |
| | $r$ | $(1-2m)p - rp_r$ $+\beta\big((1-2m)p_t - rp_{tr}\big)$ | $r\big((1-2\gamma p)p_t - \alpha p_{tt}\big)$ |
| | $r^{2m}t$ | $-tr^{2m}(p_r + \beta p_{rt})$ | $r^{2m}\big(t\big((1-2\gamma p)p_t - \alpha p_{tt}\big) - (1-\gamma p)p + \alpha p_t\big)$ |
| | $rt$ | $t\big((1-2m)p - rp_r\big)$ $+\beta t\big((1-2m)p_t - rp_{rt}\big)$ | $r\big(t\big((1-2\gamma p)p_t - \alpha p_{tt}\big) - (1-\gamma p)p + \alpha p_t\big)$ |
| 1 | $r^2$ | $-r^2(p_r + \beta p_{rt})$ | $r^2\big((1-2\gamma p)p_t - \alpha p_{tt}\big)$ |
| | $r$ | $-(p + rp_r) - \beta(p_t + rp_{rt})$ | $r\big((1-2\gamma p)p_t - \alpha p_{tt}\big)$ |
| | $r^2t$ | $-r^2t(p_r + \beta p_{tr})$ | $r^2\big(t\big((1-2\gamma p)p_t - \alpha p_{tt}\big) - (1-\gamma p)p + \alpha p_t\big)$ |
| | $rt$ | $-t\big(p + rp_r\big) - \beta t(p_t + rp_{rt})$ | $r\big(t\big((1-2\gamma p)p_t - \alpha p_{tt}\big) - (1-\gamma p)p + \alpha p_t\big)$ |
| $\frac{1}{2}$ | $r$ | $-r(p_r + \beta p_{rt})$ | $r\big((1-2\gamma p)p_t - \alpha p_{tt}\big)$ |
| | $rt$ | $-rt(p_r + \beta p_{rt})$ | $r\big(t\big((1-2\gamma p)p_t - \alpha p_{tt}\big) - (1-\gamma p)p + \alpha p_t\big)$ |
| | $r \ln r$ | $p - rp_r \ln r + \beta(p_t - r\ln(r)p_{tr})$ | $r\ln(r)\big((1-2\gamma p)p_t - \alpha p_{tt}\big)$ |
| | $rt \ln r$ | $t\big(p - rp_r \ln r\big)$ $+\beta t\big(p_t - r\ln(r)p_{tr}\big)$ | $r\ln(r)\big(t\big((1-2\gamma p)p_t - \alpha p_{tt}\big) - (1-\gamma p)p + \alpha p_t\big)$ |

**Table 5.** Multipliers, conserved densities, and fluxes for (85) for arbitrary $m$ (top), $m = 1$ (middle), and $m = \frac{1}{2}$ (bottom).

The conservation laws for the one-dimensional Westervelt equation presented in [6, Eq. (1)] can be obtained by setting $m = \beta = 0$ and replacing $T^r$ by $T^x$ in Table 5, yielding the four polynomial conservation laws derived by Anco et al. [6, Eqs. (20)-(23)]. For $m = \alpha = \beta = 0$ there also exists a rational conservation law (not included in Table 5) given in [6, Eq. (24)]. For the spherical case ($m = 1$) and $\beta = 0$ the first two fluxes ($T^t$) in Table 5 correspond to the conserved densities given in [3, Eqs. (7.8) and (7.9)]. For the cylindrical case ($m = \frac{1}{2}$) the first and second conservation laws for arbitrary $m$ in Table 5 coincide and so do the third and fourth ones.

Next, replacing the nonlinearity $p^2$ in (85) by an arbitrary function $F(p)$, we compute conservation laws for

$$p_{rr} + \tfrac{2m}{r} p_r + F(p)_{tt} + \alpha p_{ttt} + \beta (p_{rr} + \tfrac{2m}{r} p_r)_t = 0, \tag{87}$$

where, as above, $m = 0, \frac{1}{2}$, or 1. The results are presented in Table 6.

| $m$ | $\Lambda$ | Density $T^r$ | Flux $T^t$ |
|---|---|---|---|
| | $r^{2m}$ | $-r^{2m}(p_r + \beta p_{rt})$ | $-r^{2m}(F' p_t + \alpha p_{tt})$ |
| | $r$ | $(1-2m)p - r p_r + \beta\big((1-2m)p_t - r p_{tr}\big)$ | $-r(F' p_t + \alpha p_{tt})$ |
| | $r^{2m} t$ | $-t r^{2m}(p_r + \beta p_{rt})$ | $-r^{2m}\big(t(F' p_t + \alpha p_{tt}) - F(p) - \alpha p_t\big)$ |
| | $rt$ | $t\big((1-2m)p - r p_r\big) + \beta t\big((1-2m)p_t - r p_{rt}\big)$ | $-r\big(t(F' p_t + \alpha p_{tt}) - F(p) - \alpha p_t\big)$ |
| 1 | $r^2$ | $-r^2(p_r + \beta p_{rt})$ | $-r^2(F' p_t + \alpha p_{tt})$ |
| | $r$ | $-(p + r p_r) - \beta(p_t + r p_{rt})$ | $-r(F' p_t + \alpha p_{tt})$ |
| | $r^2 t$ | $-r^2 t(p_r + \beta p_{tr})$ | $-r^2\big(t(F' p_t + \alpha p_{tt}) - F(p) - \alpha p_t\big)$ |
| | $rt$ | $-t\big(p + r p_r\big) - \beta t(p_t + r p_{rt})$ | $-r\big(t(F' p_t + \alpha p_{tt}) - F(p) - \alpha p_t\big)$ |
| $\frac{1}{2}$ | $r$ | $-r(p_r + \beta p_{rt})$ | $-r(F' p_t + \alpha p_{tt})$ |
| | $rt$ | $-rt(p_r + \beta p_{rt})$ | $-r\big(t(F' p_t + \alpha p_{tt}) - F(p) - \alpha p_t\big)$ |
| | $r \ln r$ | $p - r p_r \ln r + \beta(p_t - r \ln(r) p_{tr})$ | $-r \ln(r)(F' p_t + \alpha p_{tt})$ |
| | $rt \ln r$ | $t\big(p - r p_r \ln r + \beta(p_t - r \ln(r) p_{tr})\big)$ | $-r \ln(r)\big(t(F' p_t + \alpha p_{tt}) - F(p) - \alpha p_t\big)$ |

**Table 6.** Multipliers and conserved densities and fluxes for (87) for arbitrary $m$, and $m = 1$ and $m = \frac{1}{2}$.

The conservation laws for the one-dimensional Westervelt equation in [37, Eq. (3)] are obtained by setting $m = \alpha = 0$, $c = 1$, $F(p) = -f(p)$, and replacing $r$ by $x$ in Table 6. Indeed, the first four conservation laws then correspond to those computed by Márquez et al. [37, Eqs. (21)-(24)].

# 7 Conclusions

In this paper, conservation laws have been computed for several nonlinear PDEs arising in elasticity and acoustics using two methods: the scaling-homogeneity approach which is implemented in the *Mathematica*-based code `ConservationLawsMD.m` and the multiplier method available in the code `GeM` written in *Maple* syntax.

First, conservation laws have been derived for two models describing shear wave propagation in a circular cylinder and a cylindrical annulus. The PDEs are based on a constitutive relation expressed as a power law for the stress with an arbitrary exponent $n$. Computation of conservation laws with the scaling-homogeneity approach required fixed values of $n$. Based on the results for $n = 1, 2, \cdots$, the densities and fluxes for arbitrary $n$ were readily obtained by pattern matching or with a bit of extra work. The computations are shown step-by-step and in sufficient detail for the reader to be able to reproduce the results.

Second, with the aid of `GeM`, the multiplier approach was applied to a parameterized system involving an arbitrary function of stress. The general system not only includes the two previous models (for specific values of the parameters) but covers a broad class of models for wave propagation in elastic materials formulated in terms of cylindrical coordinates.

Next, conservation laws have been computed for the KZK equation in Cartesian, cylindrically, and spherical coordinates. For multipliers involving up to first-order derivatives of the dependent variables, `GeM` allowed us to compute and simplify the determining equations for the multipliers. These equations were then solved by hand showing that the multipliers are linear in time ($t$). In Cartesian coordinates the space-dependent coefficient of $t$ satisfies the Laplace equation while the other function must be a solution of Poisson's equation. In cylindrical and spherical coordinates the situation is similar: the multipliers are still linear in time but the equations for its coefficients are more complicated. Since the Laplace equation has infinitely many solutions (called harmonic functions) an infinite number of conservation laws for the KZK equation is guaranteed. The closed form expressions for the conserved densities and fluxes have been obtained by straightforward algebraic manipulations.

As a final application, conservation laws have been derived with the multiplier method and `GeM` for Westervelt-type equations in multi-spatial (Cartesian) coordinates as well as cylindrical and spherical coordinates. In all cases, the commonly used quadratic nonlinearity was replaced by an arbitrary function of the acoustic pressure. As with the KZK equation, the multipliers are linear in time but in this case both coefficients must be solutions of the Laplace equation. Therefore, the Westervelt-type equations in Cartesian coordinates with more than one spatial variable have infinitely many conservation laws which confirms recent findings by Sergyeyev. By contrast, in one spatial variable the Westervelt-type equation has only a finite number of conserved densities and fluxes which have been reported in the literature.

## Dedication

This paper is dedicated to Dr. George Bluman who has been an inspiring teacher and leader in the symmetry community. We are grateful for his seminal contributions to the advancement of symmetry methods for differential equations. His journal articles and books have greatly impacted the development of our own research. We owe him our profound gratitude and wish him a long and happy retirement.

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