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# On discrete Painlevé equations associated with the affine Weyl group $E_7$

*Alfred Ramani*[1], *Basil Grammaticos*[1], *Ralph Willox*[2] *and Adrian Stefan Carstea*[3]

[1] *Université Paris-Saclay, CNRS/IN2P3, IJCLab, 91405 Orsay, France and Université de Paris, IJCLab, 91405 Orsay France*

[2] *Graduate School of Mathematical Sciences, the University of Tokyo, 3-8-1 Komaba, Meguro-ku, 153-8914 Tokyo, Japan*

[3] *Department of Theoretical Physics, NIPNE, Magurele 077125, Romania*

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**Abstract**

We derive a class of discrete Painlevé equations associated with the affine Weyl group $E_7^{(1)}$. The method used is the deautonomisation of a QRT mapping belonging to the canonical form VI (according to the classification of said mappings). An equation of such a form was the first instance of a symmetric – in QRT parlance – discrete analogue of the Painlevé VI equation. In this paper we present an exhaustive derivation of all the discrete Painlevé equations of this class. This is made possible thanks to previous studies that established the proper lengths of singularity patterns that are compatible with integrablity, and which were already successfully applied to the study of discrete Painlevé equations associated to the affine Weyl group $E_8^{(1)}$. Given that, from the latter, one can obtain by degeneration the equations related to $E_7^{(1)}$, we decided to link the results of the present study to those of the aforementioned ones. It turns out that a bridge from $E_8^{(1)}$ to $E_7^{(1)}$ exists in almost all cases, with one exception where, while in the former case a discrete Painlevé equation does exist, in the latter we find a mapping with only periodic coefficients, devoid of secular dependence.

Keywords: Integrability, deautonomisation, singularities, discrete Painlevé equations

# 1 Introduction

Among all the discrete analogues of the Painlevé equations the discrete analogue to the Painlevé VI equation proved to be particularly elusive. While the remaining discrete Painlevé equations, analogues to $\mathrm{P_{III}}$, $\mathrm{P_{IV}}$ and $\mathrm{P_{V}}$ were promptly derived [1] once it became clear that they were objects of interest, initially, no analogue was identified for $\mathrm{P_{VI}}$. In fact its derivation took quite some time, despite some early progress based on the desired singularity structure [2]. The first discrete analogue to $\mathrm{P_{VI}}$ was identified by Jimbo and Sakai [3] who analysed in the proper way an equation known as the 'asymmetric' $q$-$\mathrm{P_{III}}$ (already derived in [1]). The term asymmetric here is a direct reference to the QRT [4] terminology (see the appendix in [5] for a definitive classification of said mappings). In this terminology, mappings that can be written as a single equation, involving one dependent variable, are dubbed 'symmetric', while the asymmetric ones involve two dependent variables. In the initial, pioneering, studies on discrete Painlevé equations it was (unjustifiably) assumed that any term proportional to $(-1)^n$ would not play any role in the continuum limit and was therefore discarded (resulting also in a parameter loss). The correct approach would have been to consider the even and odd numbered variables as two distinct variables and thus cast the equation into an asymmetric form. If one takes $q$-$\mathrm{P_{III}}$ keeping the $(-1)^n$-dependent terms, casting the equation into an asymmetric form, one obtains indeed $q$-$\mathrm{P_{VI}}$ as derived by Jimbo and Sakai.

Starting from the equation

$$y_n y_{n-1} = \frac{a_3 a_4 (x_n - \lambda^n b_1)(x_n - \lambda^n b_2)}{(x_n - b_3)(x_n - b_4)} \tag{1a}$$

$$x_{n+1} x_n = \frac{b_3 b_4 (y_n - \lambda^n a_1)(y_n - \lambda^n a_2)}{(y_n - a_3)(y_n - a_4)}, \tag{1b}$$

where $a_1, \ldots, b_4$ are constants constrained by $a_1 a_2 / a_3 a_4 = \lambda b_1 b_2 / b_3 b_4$, Jimbo and Sakai introduced a continuum limit and showed that the system went over to the sixth Painlevé equation. Equation (1) can be obtained through the application of the singularity confinement criterion [6] for the non-autonomous mapping

$$x_{n+1} x_n = \frac{g_n (x_n - a_n)(x_n - b_n)}{(x_n - c_n)(x_n - d_n)}. \tag{2}$$

It turns out that the coefficients $a_n, b_n, c_n, d_n$ have an even-odd dependence and by rewriting the equation as a system for variables of even and odd indices one recovers equation (1).

Once it became clear that any even-odd dependence of the parameters of a discrete Painlevé must be dealt with by casting the equation into asymmetric form, it was straightforward to derive more analogues to $\mathrm{P_{VI}}$. In particular, in [7] we presented the equation

$$(x_{n+1} + y_n)(y_n + x_n) = \frac{(y_n - a)(y_n - b)(y_n - c)(y_n - d)}{(y_n - z_n - \alpha/2)^2 - e^2} \tag{3a}$$

$$(y_n + x_n)(x_n + y_{n-1}) = \frac{(x_n + a)(x_n + b)(x_n + c)(x_n + d)}{(x_n - z_n)^2 - f^2} \tag{3b}$$

with a constraint $a+b+c+d=0$, and $z_n=\alpha n+\beta$, which is a difference analogue of $P_{VI}$ (Jimbo and Sakai's equation being a multiplicative one). Introducing the continuum limit $a=1/2+\epsilon\alpha$, $b=1/2-\epsilon\alpha$, $c=-1/2+\epsilon\beta$, $d=-1/2-\epsilon\beta$, $e=\epsilon\gamma$, $f=\epsilon\delta$, $x=w-1/2$, $z=\zeta+1/2$, $y=w(\zeta-1)/(w-\zeta)+1/2+\epsilon u$ we find that at the limit $\epsilon\to 0$ (3) goes over to the equation

$$\frac{d^2w}{d\zeta^2}=\frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-\zeta}\right)\left(\frac{dw}{d\zeta}\right)^2-\left(\frac{1}{\zeta}+\frac{1}{\zeta-1}+\frac{1}{w-\zeta}\right)\frac{dw}{d\zeta}+\frac{w(w-1)(w-\zeta)}{2\zeta^2(\zeta-1)^2}\left(A+\frac{B\zeta}{w^2}+\frac{C(\zeta-1)}{(w-1)^2}+\frac{D\zeta(\zeta-1)}{(w-\zeta)^2}\right) \quad (4)$$

with $A=4\gamma^2$, $B=-4\alpha^2$, $C=4\beta^2$, $D=1-4\delta^2$, i.e. precisely Painlevé VI.

While at this point the matter of finding a discrete version of $P_{VI}$ could be considered settled, it was frustrating that no 'symmetric' form for the $P_{VI}$ analogue could be derived. Finally, guided by some work on folding transformations [8], two of the present authors derived the equation [9]

$$\frac{(x_nx_{n+1}-z_nz_{n+1})(x_nx_{n-1}-z_nz_{n-1})}{(x_nx_{n+1}-1)(x_nx_{n-1}-1)}=\frac{(x_n-az_n)(x_n-z_n/a)(x_n-bz_n)(x_n-z_n/b)}{(x_n-c)(x_n-1/c)(x_n-d)(x_n-1/d)}, \quad (5)$$

where $z_n=z_0\lambda^n$ for some arbitrary constants $z_0$ and $\lambda$. Putting $\lambda=e^\epsilon$, $a=-e^{\epsilon\alpha}$, $b=e^{\epsilon\beta}$, $c=-e^{\epsilon\gamma}$, $d=e^{\epsilon\delta}$ we obtained at $\epsilon\to 0$ the continuum limit

$$\frac{d^2x}{dz^2}=\frac{1}{2}\left(\frac{1}{x+1}+\frac{1}{x-1}+\frac{1}{x+z}+\frac{1}{x-z}\right)\left(\frac{dx}{dz}\right)^2-\left(\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z+1}+\frac{1}{x-z}-\frac{1}{x+z}\right)\frac{dx}{dz}+\frac{(x^2-z^2)(x^2-1)}{z^2(z^2-1)}\left(\frac{(\alpha^2-1/4)z}{(x+z)^2}-\frac{(\beta^2-1/4)z}{(x-z)^2}-\frac{\gamma^2}{(x+1)^2}+\frac{\delta^2}{(x-1)^2}\right), \quad (6)$$

which is indeed Painlevé VI albeit in non canonical form. (It suffices to introduce the change of variables $z=(1+\sqrt{\zeta})/(1-\sqrt{\zeta})$ and $x=(\sqrt{\zeta}+w)/(\sqrt{\zeta}-w)$ in order to bring (6) to the canonical $P_{VI}$ form).

In the same work, it was shown that the additive equivalent of (5),

$$\frac{(x_n+x_{n+1}-z_n-z_{n+1})(x_n+x_{n-1}-z_n-z_{n-1})}{(x_n+x_{n+1})(x_n+x_{n-1})}=\frac{(x_n-z_n-a)(x_n-z_n+a)(x_n-z_n-b)(x_n-z_n+b)}{(x_n-c)(x_n+c)(x_n-d)(x_n+d)}, \quad (7)$$

where $z_n=\alpha n+\beta$, does not have $P_{VI}$ as continuum limit but is rather a discrete analogue of $P_V$.

With hindsight, it is clear that the delay in discovering a symmetric form of the Painlevé VI discrete analogue was due to the absence of a proper classification of the QRT canonical forms. Once the latter were obtained [10] it was clear that the derivation of discrete Painlevé equations, through deautonomisation, should not have been limited to forms that led to analogues of $P_{III}$, $P_{IV}$ and $P_V$ but should have been extended to mappings of the classes V and VI [5], namely equations of the forms (7) and (5).

Equation (5) is known to possess the so-called self-duality property [11]. For a large class of discrete Painlevé equations (which includes all the equations of difference-type) the evolution equations in the independent variable, on the one hand, and the evolution equations for the parameters generated by the Schlesinger transformations of the discrete

Painlevé equations on the other hand, are the same. The explanation of self-duality for difference Painlevé equations can be given from their relation to the continuous Painlevé equations as, in fact, some difference Painlevé equation are obtained from the Schlesinger transformations of the parameters of a continuous one. When the latter possesses several parameters, which, in general, play the same role, one can consider the evolution as defined by the contiguity relations along any of these parameters while the Schlesinger transformations for the remaining parameters carry over as Schlesinger transformationss for the discrete equation. Since all these transformations are equivalent, it is natural to obtain the same difference equation as a recurrence from the application of the Schlesinger's. However there exist difference Painlevé equations which are not contiguity relations of continuous ones, having more parameters than the maximum number that can be associated to a continuous equation. Still these equations possess the property of self-duality [12] . The self-duality of $q$-Painlevé equations is more of a surprise and, in fact not all of $q$-discrete Painlevé equations are self-dual [13].

In the approach presented in [11] and [14] we showed that the discrete Painlevé equations can be described by a single $\tau$-function which is a function of several variables. Moreover, in perfect parallel to Okamoto's Toda equation for $\tau$-sequences [15], the evolution of the $\tau$-function is given by a non-autonomous Hirota-Miwa equation [16], [17], which is the fully discrete analogue of the Toda equation. What is most important in this approach is the fact that one can describe the evolution of the (multivariable) $\tau$-function in purely geometrical terms. The affine Weyl groups, which play an important role in the description of continuous Painlevé equations, are present also here.

Sakai's monumental work [18] on the classification of a class of rational surfaces associated with affine root systems allowed one to put this approach on a more rigorous basis, linking the geometry of the discrete Painlevé equations to that of affine Weyl groups and allowing an organisation of the equations in the degeneration cascade pictured in Figure 1.

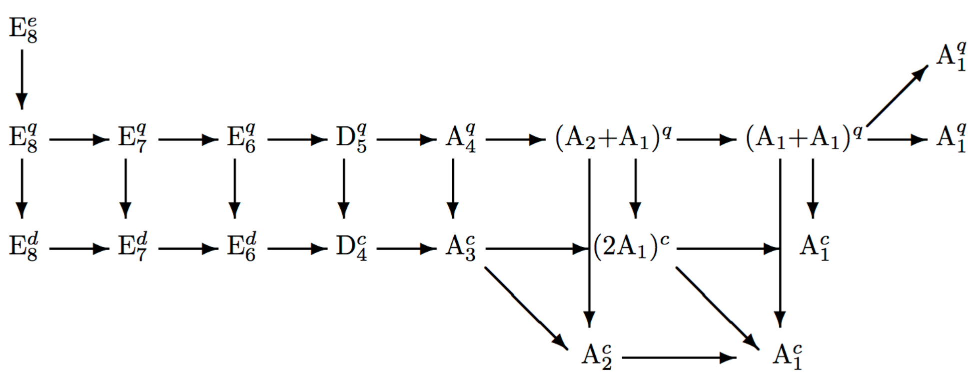


**Figure 1**. *The degeneration cascade of affine Weyl groups associated to discrete Painlevé equations.*

Following Sakai's work, the point of view concerning discrete Painlevé equations shifted markedly. The latter were no longer attached to their differential brethren but were considered as the more fundamental entities. To put it naively, since the discrete Painlevé equations could involve up to 8 parameters, compared to the 4 contained in the richest differential system, the sixth Painlevé, any continuum limit would entail an impoverishment. Thus a new definition of what is a discrete Painlevé equation was introduced, one that did away with references to differential systems:

*a discrete Painlevé equation is a birational mapping on $\mathbb{P}^1\times\mathbb{P}^1$ obtained by translations on the periodic repetition of a non-closed pattern on a lattice contained in the weight lattice associated to one of the affine Weyl groups belonging to the degeneration cascade starting from $E_8^{(1)}$.*

Moreover, this geometrical description made it clear that the number of discrete Painlevé equations was infinite [19], and in fact, there exist infinitely many associated with every affine Weyl group in the degeneration cascade (except, of course, for the four $\mathrm{A}_1^{(1)}$ where there is not sufficient freedom). Finding all the discrete Painlevé equations is therefore not something conceivable. However, finding equations representative of a given class (in the QRT sense) and associated with some affine Weyl group is a well-defined task and an interesting application of the deautonomisation procedure. In this paper, we shall derive $q$-discrete Painlevé equations associated to the affine Weyl group $\mathrm{E}_7^{(1)}$ of the class VI QRT canonical form, i.e. an equation the left-hand side of which is identical to that of (5). In [20] we addressed a similar question but the approach used there did not allow an exhaustive search. Moreover, the application of the deautonomisation was based on ansätze that did not always allow to obtain the full complement of parameters (namely 7) in the equation. These are points that will be remedied in the present work.

## 2 The Method

The main bulk of the discrete Painlevé equations were derived using a method called deautonomisation. Starting with an integrable autonomous mapping, this method consists in treating the free parameters in the mapping as functions of the independent variable, the precise expressions of which are to be determined with the help of a suitable criterion for integrability (usually the singularity confinement criterion). The standard practice is to require that singularities be confined at the very first opportunity, a ‘late’ confinement leading to non-integrable systems. In [21] some of the present authors, in collaboration with T. Mase, used an algebro-geometrical analysis and showed through some selected examples of discrete Painlevé equations, how their regularisation through blow-up yields exactly the same conditions on the parameters in the mapping as the singularity confinement criterion. A rigorous justification of the approach called ‘full-deautonomisation’, which uses the singularity structure of a mapping to deduce its dynamical degree was obtained by some of the present authors in collaboration with T. Mase and A. Stokes in [22].

One important conclusion of [21] was that the regularisation of a generic discrete Painlevé equation (of the QRT-symmetric type, involving a single dependent variable) requires precisely 8 blow-ups in $\mathbb{P}^1\times\mathbb{P}^1$. And linking the number of blow-ups to the overall length of the singularity patterns made it possible to address systematically the question of derivation of discrete Painlevé equations associated to the affine Weyl group $\mathrm{E}_8^{(1)}$. In [23] two of the authors, in collaboration with Y. Ohta, studied two mappings aiming at the derivation, by means of the deautonomisation approach, of the first example of an explicit elliptic discrete Painlev ´e equation, the existence of which had been shown by Sakai who had only presented an example as a bi-rational map on $\mathbb{P}^2$. Both mappings were of the form that the authors called 'trihomographic'

$$\frac{x_{n+1}-a_n}{x_{n+1}-b_n}\frac{x_{n-1}-c_n}{x_{n-1}-d_n}\frac{x_n-e_n}{x_n-f_n}=1 \tag{8}$$

The two mappings had two singularities with pattersn of lengths 2 and 6 for the first and 4 and 4 for the second. For the first we found the form

$$\frac{x_{n+1}-(4t_n-\alpha+a_n)^2}{x_{n+1}-(\alpha+b_n)^2}\frac{x_{n-1}-(4t_n+\alpha+c_n)^2}{x_{n-1}-(\alpha+d_n)^2}\frac{x_n-(2t_n+e_n)^2}{x_n-(6t_n+f_n)^2}=1, \tag{9}$$

where $t_n=\alpha n+\beta$. The $a_n,\cdots,f_n$ are given by

$$a_n=2\omega_{n-1}+\omega_{n+1}+\omega_{n+2},\ b_n=\omega_{n+2}-\omega_{n+1},\ c_n=\omega_{n-1}+\omega_n+2\omega_{n+2},$$
$$d_n=\omega_n-\omega_{n-1},\ e_n=\omega_n+\omega_{n+1},\ f_n=2\omega_{n-1}+\omega_n+\omega_{n+1}+2\omega_{n+2} \tag{10}$$

where

$$\omega_n=\phi_4(n)+\phi_5(n). \tag{11}$$

We have introduced here the function $\phi_m$ with period $m$, i.e. $\phi_m(n+m)=\phi_m(n)$,

$$\phi_m(n)=\sum_{l=1}^{m-1}\delta_l^{(m)}\exp\left(\frac{2i\pi ln}{m}\right) \tag{12}$$

(Note that the sum starts at 1 instead of 0, excluding 1 from the roots of unity, and introducing $m-1$ degrees of freedom). For the second we had

$$\frac{x_{n+1}-(2t_n-\alpha+a_n)^2}{x_{n+1}-(2t_n-\alpha+b_n)^2}\frac{x_{n-1}-(2t_n+\alpha+c_n)^2}{x_{n-1}-(2t_n+\alpha+d_n)^2}\frac{x_n-(4t_n+e_n)^2}{x_n-(4t_n+f_n)^2}=1. \tag{13}$$

The $a_n,\cdots,f_n$ are given by

$$a_n=\omega_n+\psi_n,\ b_n=\omega_n-\psi_n,\ c_n=\omega_{n+1}+\psi_n,\ d_n=\omega_{n+1}-\psi_n,\ e_n=\omega_n+\omega_{n+1}-\psi_n,$$
$$f_n=\omega_n+\omega_{n+1}+\psi_n \tag{14}$$

where

$$\omega_n=\phi_2(n)+\phi_3(n),\quad \text{and}\quad \psi_n=\gamma+\phi_4(n). \tag{15}$$

Having obtained two equations with total length of singularity patterns 8, split into 2+6 and 4+4, led readily to the question about the existence of integrable cases with singularity pattern lengths split into 1+7 and 3+5. It turned out that both such systems exist [24]. For the first we obtained the equation

$$\frac{x_{n+1}-(u_{n+1}+2u_n+2u_{n-1})^2}{x_{n+1}-u_{n+1}^2}\frac{x_{n-1}-(2u_{n+1}+2u_n+u_{n-1})^2}{x_{n-1}-u_{n-1}^2}\frac{x_n-u_n^2}{x_n-(2u_{n+1}+3u_n+2u_{n-1})^2}=1, \tag{16}$$

where

$$u_n=t_n+\omega_n,\quad \text{and}\quad \omega_n=\phi_2(n)+\phi_3(n)+\phi_5(n). \tag{17}$$

For the second we found

$$\frac{x_{n+1}-(u_{n+2}+u_{n+1}-u_n+2u_{n-2})^2}{x_{n+1}-(u_n+u_{n+1}-u_{n+2})^2}\frac{x_{n-1}-(2u_{n+2}-u_n+u_{n-1}+u_{n-2})^2}{x_{n-1}-(u_n+u_{n-1}-u_{n-2})^2}$$

$$\times \frac{x_n-(u_{n+1}+u_n+u_{n-1})^2}{x_n-(2u_{n+2}+u_{n+1}-u_n+u_{n-1}+2u_{n-2})^2}=1, \quad (18)$$

where

$$u_n=t_n+\omega_n \quad \text{and} \quad \omega_n=\phi_2(n)+\phi_7(n). \quad (19)$$

While it was shown in [25] that all discrete Painlevé equations can be cast into a trihomographic form provided one introduces a sufficient number of auxiliary variables, the use of coupled trihomographic forms can easily become cumbersome. Fortunately, an alternative approach does exist and it made the study of $\mathrm{E}_8^{(1)}$ associated discrete Painlevé equations perfectly tractable. It is based on the introduction of what we called an 'ancillary' variable [26], which replaces the dependent variable and allows a compact expression for the general $\mathrm{E}_8^{(1)}$ discrete Painlevé equation. Let us show how this works in the case of the general additive (difference) symmetric discrete Painlevé equation associated with the affine Weyl group $E_8^{(1)}$. In [14] we derived the form

$$\frac{(x_n-x_{n+1}+(z_n+z_{n+1})^2)(x_n-x_{n-1}+(z_n+z_{n-1})^2)+4x_n(z_n+z_{n+1})(z_n+z_{n-1})}{(z_n+z_{n-1})(x_n-x_{n+1}+(z_n+z_{n+1})^2)+(z_n+z_{n+1})(x_n-x_{n-1}+(z_n+z_{n-1})^2)}=R(x_n), \quad (20)$$

where $z_n$ is equal to $\alpha n+\beta$ and $R$ is a ratio of two specific polynomials of $x$, quartic in the numerator and cubic in the denominator. Its precise form is

$$R(x_n)=2\frac{x_n^4+S_2x_n^3+S_4x_n^2+S_6x_n+S_8}{S_1x_n^3+S_3x_n^2+S_5x_n+S_7}, \quad (21)$$

where the $S_k$ are the elementary symmetric functions of the quantities $z_n+\kappa_n^i$, in eight parameters $\kappa^i$ which are, generically, functions of the independent variable. Introducing the substitution

$$x_n=\xi_n^2, \quad (22)$$

and the quantity $\Pi(\xi_n)=\prod_{i=1}^8(z_n+\kappa_n^i+\xi_n)$ one finds that

$$R(x_n)=2\xi_n\frac{\Pi(\xi_n)+\Pi(-\xi_n)}{\Pi(\xi_n)-\Pi(-\xi_n)}. \quad (23)$$

Rearranging equation (20) so that the ratio of $\Pi(-\xi_n)/\Pi(\xi_n)$ appears on the right-hand side, we find finally

$$\frac{x_{n+1}-(\xi_n-z_n-z_{n+1})^2}{x_{n+1}-(\xi_n+z_n+z_{n+1})^2}\,\frac{x_{n-1}-(\xi_n-z_n-z_{n-1})^2}{x_{n-1}-(\xi_n+z_n+z_{n-1})^2}=\frac{\prod_{i=1}^8(\kappa_n^i+z_n-\xi_n)}{\prod_{i=1}^8(\kappa_n^i+z_n+\xi_n)}. \quad (24)$$

Note that both the left and right hand sides of (24) are expressed in a factorised form thanks to the introduction of the ancillary variable $\xi$.

Ancillary variable substitutions exist also for multiplicative- and elliptic-type equations. In the former case the substitution is $x=\xi+1/\xi$ while the latter involves theta functions: $x=\theta_1^2(\xi)/\theta_0^2(\xi)$.

At this point one may wonder why we are insisting on the $\mathrm{E}_8^{(1)}$ case since the aim of the present paper is to study a selection of $\mathrm{E}_7^{(1)}$-associated systems. The answer is simple. From the degeneration cascade one sees readily that the $E_7^{(1)}$-associated equations can be obtained as a limit of the $E_8^{(1)}$-associated ones. Now, as explained above, the $E_8^{(1)}$ equations were studied systematically and we believe that the obtained results are complete and exhaustive. Thus they would allow a verification of the results we intend to obtain, by deautonomisation, for the $E_7^{(1)}$-associated systems by checking whether they correspond indeed to a limit of an already studied $E_8^{(1)}$-associated equation with the same singularity pattern.

Having explained the method we shall employ, we can now proceed to the study of the $P_{VI}$ $q$-discrete analogues.

## 3 The $E_7$ equations

Before presenting the calculations that will lead to the various discrete Painlevé equations we must introduce a most convenient function. In section 2 above we introduced the periodic function $\phi_m(n)$ with period $m$, which was suitable in the case of difference equations. Since in what follows we shall deal with multiplicative equations, we must introduce the 'multiplicative' analogue of $\phi_m(n)$, by defining $\varphi_m(n) = \exp(\phi_m(n))$. Notice that while $\phi_2(n) + \phi_2(n+1) = 0$ we have now $\varphi_2(n)\varphi_2(n+1) = 1$ and analogous relations for the higher periods.

*A. The quartic over quartic case*

As explained in [20] the generic $E_7^{(1)}$-associated multiplicative symmetric equation is obtained from the deautonomisation of the class-VI QRT mapping

$$\frac{(x_n x_{n+1} - z^2)(x_n x_{n-1} - z^2)}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \frac{(x_n - a)(x_n - b)(x_n - c)(x_n - d)}{(x_n - p)(x_n - r)(x_n - s)(x_n - t)}, \tag{25}$$

where the parameters appearing in the equation are subject to the constraint $abcd = z^4 prst$. Note that (25) is invariant under the transformations

$$x \to \frac{1}{x}, \quad z \to \frac{1}{z}, \quad a \to \frac{1}{a}, \cdots t \to \frac{1}{t},$$

and

$$x \to \frac{z}{x}, \quad z \to z, \quad a \to \frac{z}{p}, \cdots t \to \frac{z}{d}.$$

The deautonomisation of (25) is readily obtained if one assumes that the singularity pattern has lengths (1,1,1,1,1,1,1,1), which is shorthand way to indicate that there exist 8 singularity patterns with length 1 each. We start from

$$\frac{(x_n x_{n+1} - z_n z_{n+1})(x_n x_{n-1} - z_n z_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \frac{(x_n - a_n)(x_n - b_n)(x_n - c_n)(x_n - d_n)}{(x_n - p_n)(x_n - r_n)(x_n - s_n)(x_n - t_n)}. \tag{26}$$

The constraint

$$\frac{a_n b_n c_n d_n}{p_n r_n s_n t_n} = z_{n-1} z_n^2 z_{n+1} \tag{27}$$

ensures that $x_n = 0$ is not a singularity. Requiring that all patterns have length 1 means that if one enters the singularity at some step one must exit it at the very next step.

Supposing we enter a singularity at $x_n = a_n$, we find readily that $x_{n+1} = z_n z_{n+1}/a_n$ which must then be equal to $a_{n+1}$. So for the terms on the numerator we have confinement conditions of the form

$$a_n a_{n+1} = z_n z_{n+1}, \quad b_n b_{n+1} = z_n z_{n+1}, \quad c_n c_{n+1} = z_n z_{n+1}, \quad d_n d_{n+1} = z_n z_{n+1}. \tag{28}$$

Similarly, for the denominators, if we enter a singularity through $x_n = p_n$ we find $x_{n+1} = 1/p_n$ which must be equal to $p_{n+1}$ in order to exit the singularity in one step. Thus for the factors in the denominator we have the confinement condition

$$p_n p_{n+1} = 1, \quad r_n r_{n+1} = 1, \quad s_n s_{n+1} = 1, \quad t_n t_{n+1} = 1. \tag{29}$$

Multiplying the constraint (27) by its upshift we find

$$\frac{a_n a_{n+1} b_n b_{n+1} c_n c_{n+1} d_n d_{n+1}}{p_n p_{n+1} r_n r_{n+1} s_n s_{n+1} t_n t_{n+1}} = z_{n-1} z_n^3 z_{n+1}^3 z_{n+2}, \tag{30}$$

and, using the confinement conditions we find that $z_n$ must obey

$$z_n z_{n+1} = z_{n-1} z_{n+2}. \tag{31}$$

The solution of (31) is $z_n = \kappa\lambda^n \varphi_2(n)$ but the period-2 term can be neglected here since only the products $z_n z_{n\pm1}$ appear in (26) where the periodic function cancels out. The solution of the confinement conditions is straightforward. For the terms of the numerator we have $a_n = z_n \varphi_2(n)$ and similarly for $b, c, d$ which introduce four distinct period-2 functions $\varphi_2(n)$, while for the denominator we find that $p, r, s, t$ are just period-2 functions. Clearly all 8 $\varphi_2(n)$ functions are not free. First, from (27) we find that the product of the $\varphi_2(n)$ of the numerator must be equal to the product of those of the denominator. Second, a gauge in $x_n$ by a period-2 term is always possible since it leaves the left-hand side invariant. But such a gauge allows to absorb one of the periodic functions $\varphi_2(n)$. Thus out of a total of 8 functions only 6 are really independent, introducing 6 genuine degrees of freedom and since $z_n$ has one genuine degree of freedom, $\kappa$, we have in all 7 degrees of freedom, as is expected for an equation associated to the group $E_7^{(1)}$. Thus the generic $E_7^{(1)}$-associated discrete Painlevé equation is richer than the form (5). As one can readily understand, the form of the latter has been chosen so as to contain just the five degrees of freedom that survive at the continuum limit towards $P_{VI}$ in the most symmetric form possible.

While (26) gives a generic form of the $E_7^{(1)}$-associated multiplicative symmetric discrete Painlevé equation it is, and by far, not the only one. All the equations obtained by successive simplifications of the right-hand side of (25) and subsequent deautonomisation possess 7 degrees of freedom and are also associated to $E_7^{(1)}$ (and could be shown to possess $P_{VI}$ as continuum limit).

*B. The cubic over cubic case*

When one factor of the right-hand side of (25) is simplified out one gets the mapping,

$$\frac{(x_n x_{n+1} - z^2)(x_n x_{n-1} - z^2)}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)} = \frac{(x_n - a)(x_n - b)(x_n - c)}{(x_n - p)(x_n - r)(x_n - s)}, \tag{32}$$

where the parameters obey the constraint $abc = z^4prs$. Two singularity patterns are possible in this case with corresponding lengths (1,1,1,1,2,2) and (1,1,1,1,1,3). We shall refer to them as cases B.i and B.ii. We deautonomise (32) to

$$\frac{(x_nx_{n+1}-z_nz_{n+1})(x_nx_{n-1}-z_nz_{n-1})}{(x_nx_{n+1}-1)(x_nx_{n-1}-1)} = \frac{(x_n-a_n)(x_n-b_n)(x_n-c_n)}{(x_n-p_n)(x_n-r_n)(x_n-s_n)}. \tag{33}$$

where the constraint

$$\frac{a_nb_nc_n}{p_nr_ns_n} = z_{n-1}z_n^2z_{n+1} \tag{34}$$

guarantees that $x_n = 0$ is not a singularity.

B.i) For the singularities of odd number of steps this means that if one enters the singularity at the numerator (respectively the denominator) of the right-hand side one must exit again at the numerator (respectively the denominator). An even number of steps means that one enters the numerator/denominator and exits at the opposite side. Following these considerations we find the confinement constraints

$$r_nr_{n+1} = 1, \quad s_ns_{n+1} = 1, \quad b_nb_{n+1} = z_nz_{n+1}, \quad c_nc_{n+1} = z_nz_{n+1}, \\ a_{n+2} = z_{n+1}z_{n+2}p_n, \quad a_n = z_nz_{n+1}p_{n+2}. \tag{35}$$

Multiplying the constraint (34) by its upshift we find, after simplifications,

$$a_na_{n+1} = z_{n-1}z_nz_{n+1}z_{n+2}p_np_{n+1}. \tag{36}$$

From the last two constraints in (35), multiplied by their own downshift, we find $a_{n+1}a_{n+2} = z_nz_{n+1}^2z_{n+2}p_{n-1}p_n$ and $a_{n-1}a_n = z_{n-1}z_n^2z_{n+1}p_{n+1}p_{n+2}$. Comparing these to the up- and downshift of (36), we find the equation for $z_n$

$$z_nz_{n+1} = z_{n-2}z_{n+3}, \tag{37}$$

with solution $z_n = \kappa\lambda^n\varphi_2(n)\varphi_3(n)$.

Going back to equation (36), we can solve it as $a_n = z_{n-1}z_{n+1}p_n\tilde{\varphi}_2(n)$ (where the tilde indicates that this is a period-2 function different from the previously introduced $\varphi_2(n)$) and multiplying it by its double upshift we can compare the result to the product of the two last constraints in (35) and find the relation $\tilde{\varphi}_2(n)^2\varphi_2(n+1)^2 = \varphi_2(n)^2$. Since only the product $z_nz_{n\pm1}$ appears in the equation we can freely choose $\varphi_2(n) = 1$, which, in the present case, entails $\tilde{\varphi}_2(n) = 1$. Next, multiplying the two confinement conditions involving $a_n$ and $p_n$ we find $a_{n+2}p_{n+2}/z_{n+2} = a_np_n/z_n$, which can be integrated to $a_np_n = \gamma^2z_n\tilde{\tilde{\varphi}}_2(n)^2$ In order to solve for $a_n$ and $p_n$ we introduce the square root of $z_n$ through, $z_n = q_n^2$. We can now readily solve for $a_n$, obtaining $a_n = \gamma q_{n-1}q_nq_{n+1}\tilde{\tilde{\varphi}}_2(n)$ and remark that the period-3 term drops out from the right-hand side. Once $a_n$ is obtained we can solve for $p_n$ and find $p_n = \gamma q_n\tilde{\tilde{\varphi}}_2(n)/(q_{n-1}q_{n+1})$.

Counting the degrees of freedom we find that $b, c, r, s$ introduce four period-2 functions but only three genuine degrees of freedom since (34) must be satisfied. One more degree of freedom is introduced by $\tilde{\tilde{\varphi}}_2(n)$. However a gauge of $x_n$ allows to reduce the number of parameters by one. Finally we have 2 degrees of freedom associated to $\varphi_3(n)$ to which one must add $\kappa$ and $\gamma$. This gives a total of 7 degrees of freedom as expected.

B.ii) The confinement constraints are

$$p_np_{n+1}=1,\quad r_nr_{n+1}=1,\quad s_ns_{n+1}=1,\quad b_nb_{n+1}=z_nz_{n+1},\quad c_nc_{n+1}=z_nz_{n+1},$$
$$a_na_{n+3}=z_nz_{n+1}z_{n+2}z_{n+3}. \tag{38}$$

From (34) we obtain $a_na_{n+1}=z_{n-1}z_nz_{n+1}z_{n+2}$ and solving we find $a_n=z_{n-1}z_{n+1}\varphi_2(n)$. Using the expression of $a_n$ in the confinement constraint we obtain for $z_n$ the equation

$$z_nz_{n+3}=z_{n-1}z_{n+4}, \tag{39}$$

the solution of which is $z_n=\kappa\lambda^n\varphi_4(n)$. However, since, as we pointed out, a $\varphi_2$ factor does not play any role in $z_n$ it is better to give the expression of $z_n$ as $z_n=\kappa\lambda^n\psi_4(n)$, where $\psi_{2m}$ is a periodic function obeying the equation $\psi_{2m}(n+m)\psi_{2m}(n)=1$. It has period $2m$ while involving only $m$ parameters and can be expressed in terms of roots of unity as $\psi_{2m}=\exp(\chi_{2m}(n))$ where $\chi_{2m}(n)$ is the periodic function introduced in [27] by

$$\chi_{2m}(n)=\sum_{\ell=1}^{m}\eta_\ell^{(m)}\exp\left(\frac{i\pi(2\ell-1)n}{m}\right). \tag{40}$$

and obeying the relation $\chi_{2m}(n+m)+\chi_{2m}(n)=0$. Thus $z_n$ introduces three degrees of freedom $\kappa$ and two parameters of $\psi_4$. We have 6 more parameters introduced by the $\varphi_2$ of $a,...,s$ but they are constrained by the condition (34). Finally, given the form of the equation, it is possible to introduce a gauge and eliminate one, say the $\varphi_2$ factor of $a_n$, and, in the end, only 7 genuine degrees of freedom survive.

*C. The quadratic over quadratic case*

When two factors of the right-hand side of (25) are simplified out one gets the mapping,

$$\frac{(x_nx_{n+1}-z^2)(x_nx_{n-1}-z^2)}{(x_nx_{n+1}-1)(x_nx_{n-1}-1)}=\frac{(x_n-a)(x_n-b)}{(x_n-p)(x_n-r)}, \tag{41}$$

where the parameters obey the constraint $ab=z^4pr$. We have now five possible singularity patterns with lengths (1,1,1,5), (1,1,2,4), (1,1,3,3), (1,2,2,3) and (2,2,2,2). We shall refer to them as cases C.i to C.v. We deautonomise (41) to

$$\frac{(x_nx_{n+1}-z_nz_{n+1})(x_nx_{n-1}-z_nz_{n-1})}{(x_nx_{n+1}-1)(x_nx_{n-1}-1)}=\frac{(x_n-a_n)(x_n-b_n)}{(x_n-p_n)(x_n-r_n)}. \tag{42}$$

where the constraint that guarantees that $x_n=0$ is not a singularity becomes now

$$\frac{a_nb_n}{p_nr_n}=z_{n-1}z_n^2z_{n+1}. \tag{43}$$

C.i) The singularity confinement constraints in the case (1,1,1,5) are

$$p_np_{n+1}=1,\quad r_nr_{n+1}=1,\quad b_nb_{n+1}=z_nz_{n+1},\quad a_na_{n+5}=z_nz_{n+1}z_{n+2}z_{n+3}z_{n+4}z_{n+5}. \tag{44}$$

Multiplying the constraint (43) by its upshift we find

$$a_na_{n+1}=z_{n-1}z_n^2z_{n+1}^2z_{n+2}, \tag{45}$$

which can be readily integrated to $a_n = \varphi_2(n) z_{n-1} z_n z_{n+1}$. Combining this expression with the confinement constraint for $a_n$ we find that $z_n$ must satisfy

$$z_{n+2} z_{n+3} = z_{n-1} z_{n+6}, \tag{46}$$

and integrating the latter we find $z_n = \kappa \lambda^n \varphi_3(n) \psi_4(n)$, neglecting as usual, a period-2 factor. From (44) we find that $b_n = z_n \tilde{\varphi}_2(n)$ and two more period-2 functions are introduced by $p_n$ and $r_n$. However the four period-2 functions are constrained by (45) and, moreover, a gauge of $x_n$ allows to eliminate one more. So in the end there remain precisely 7 degrees of freedom.

C.ii) The singularity confinement constraints with singularity lengths (1,1,2,4) are

$$p_n r_{n+1} = 1, \quad b_n b_{n+1} = z_n z_{n+1}, \quad r_n z_{n+1} z_{n+2} = a_{n+2}, \quad a_n = z_n z_{n+1} z_{n+2} z_{n+3} p_{n+4}. \tag{47}$$

From the second of these constraints we find readily $z_n = b_n \tilde{\varphi}_2(n)$, but the period-2 freedom is again immaterial. Replacing $z_n$ by $b_n$ in the constraint (43) we have $a_n = b_{n-1} b_n b_{n+1} p_n r_n$. Eliminating $a_n$ we obtain for $p_n, r_n$ the relations $r_n = b_{n+3} p_{n+2} r_{n+2}$ and $p_n r_n = b_{n+2} b_{n+3} p_{n+4} / b_{n-1}$, which can be further simplified using the relation $p_n = 1/r_{n+1}$. We remark that the former can be solved for $b_n$ yielding $b_n = r_n r_{n-3} / r_{n-1}$. We find finally that $r_n$ must satisfy

$$r_{n-4} r_{n+5} = r_{n-2} r_{n+3}, \tag{48}$$

with solution $r_n = \kappa \lambda^n \varphi_7(n) \varphi_2(n)$. A simpler expression for $a_n$ involving only $r_n$ is $a_n = r_n r_{n-3} r_{n-4}$. Thus $a_n$, $b_n$ and $p_n$ inherit a $\varphi_2(n)$ from $r_n$. As a consequence this common period-2 term can be removed by a gauge of $x_n$ and only 7 parameters remain.

The case C.iii is more complicated. In fact there are two distinct ways to satisfy confinement for singularities of lengths (1,1,3,3): either the long and the short patterns are related to singularities in the numerator and the denominator respectively (case a) or they concern singularities present in both the top and bottom parts of the right-hand side (case b).

C.iii.a) The singularity confinement constraints are

$$p_n p_{n+1} = 1, \quad r_n r_{n+1} = 1, \quad a_n a_{n+3} = z_n z_{n+1} z_{n+2} z_{n+3}, \quad b_n b_{n+3} = z_n z_{n+1} z_{n+2} z_{n+3}. \tag{49}$$

Multiplying the constraint (43) by its upshift we find

$$a_n a_{n+1} b_n b_{n+1} = z_{n-1} z_n^3 z_{n+1}^3 z_{n+2}. \tag{50}$$

Next, we introduce the auxiliary quantities $A_n = a_n b_n$ and $B_n = a_n / b_n$. Constraint (50) can be rewritten as $A_n A_{n+1} = z_{n-1} z_n^3 z_{n+1}^3 z_{n+2}$, whereas from (49) we have $A_n A_{n+3} = z_n^2 z_{n+1}^2 z_{n+2}^2 z_{n+3}^2$. Combining the two, we obtain for $z_n$ the equation

$$z_{n+1} z_{n+2} = z_{n-1} z_{n+4}, \tag{51}$$

with solution $z_n = \kappa \lambda^n \varphi_3(n)$, neglecting, as usual, a period-2 factor. Dividing the last two constraints in (49) we find $B_n B_{n+3} = 1$ with solution $B_n = \psi_6(n)$. Integrating (50) we find $A_n = z_{n-1} z_n^2 z_{n+1} \varphi_2(n)$ and using the expression of $B_n$ we obtain finally $a_n^2 = z_{n-1} z_n^2 z_{n+1} \varphi_2(n) \psi_6(n)$ from which we can obtain $a_n$, whereupon $b_n = a_n / \psi_6(n)$.

Counting the degrees of freedom, we find that $p_n = \tilde{\varphi}_2(n)$ and $r_n = \tilde{\tilde{\varphi}}_2(n)$ introduce two, $z_n$ introduces three more and the $\psi_6(n)$ appearing in $a_n$ and $b_n$ brings three more free parameters. A gauge of $x_n$ is always possible and can be used to remove one superfluous parameter bringing the total down to 7, as expected.

C.iii.b) The singularity confinement constraints now become

$$r_n r_{n+1} = 1, \quad b_n b_{n+1} = z_n z_{n+1}, \quad p_n p_{n+3} z_{n+1} z_{n+2} = 1, \quad a_n a_{n+3} = z_n z_{n+1} z_{n+2} z_{n+3}. \tag{52}$$

Multiplying the constraint (43) by its upshift we find

$$\frac{a_n a_{n+1}}{p_n p_{n+1}} = z_{n-1} z_n^2 z_{n+1}^2 z_{n+2}, \tag{53}$$

From (52) we find readily $a_n a_{n+3}/(p_n p_{n+3}) = z_n z_{n+1}^2 z_{n+2}^2 z_{n+3}$ and combining it with (53) we obtain for $z_n$ the equation

$$z_{n+1} z_{n+2} = z_{n-1} z_{n+4}, \tag{54}$$

which is identical to (51), and leads to the same solution $z_n = \kappa\lambda^n \varphi_3(n)$, where we neglected a period-2 factor. Next we introduce the auxiliary quantity $C_n = a_n p_n / z_n$ and, using (52), find that it satisfies $C_n C_{n+3} = 1$. The solution of the latter is $C_n = \psi_6(n)$ and integrating (53) to $a_n/p_n = z_{n-1} z_n z_{n+1} \varphi_2(n)$ which leads to $a_n^2 = z_{n-1} z_n^2 z_{n+1} \psi_6(n) \varphi_2(n)$. This is just a local relation from which we obtain $a_n$ whereupon $p_n$ is given by $p_n = a_n/(z_{n-1} z_n z_{n+1} \varphi_2(n))$. From (52) we have $b_n = z_n \tilde{\varphi}_2(n)$ and $r_n = \tilde{\tilde{\varphi}}_2(n)$. The constraint (43) leads to the relation $\tilde{\tilde{\varphi}}_2(n) = \varphi_2(n) \tilde{\varphi}_2(n)$, meaning that we have only 8 degrees of freedom, one of which can be removed by the proper gauge of $x_n$ bringing the total down to 7, as expected.

C.iv) The singularity confinement constraints with singularity lengths (1,2,2,3) are

$$r_n r_{n+1} = 1, \quad a_n = z_n z_{n+1} p_{n+2}, \quad a_{n+2} = z_{n+1} z_{n+2} p_n, \quad b_n b_{n+3} = z_n z_{n+1} z_{n+2} z_{n+3}, \tag{55}$$

and from the constraint (43) we find

$$\frac{a_n a_{n+1} b_n b_{n+1}}{p_n p_{n+1}} = z_{n-1} z_n^3 z_{n+1}^3 z_{n+2}, \tag{56}$$

We introduce again the quantity

$$C_n = a_n p_n / z_n$$

and, using (55), find that it satisfies $C_n = C_{n+2}$. Thus

$$a_n p_n = \gamma z_n \varphi_2(n)$$

Using (56) and its upshifts we can eliminate $b_n$ and obtain $a_n a_{n+3}/(p_n p_{n+3}) = z_{n-1} z_n z_{n+3} z_{n+4}$. Using the latter together with (55), which gives directly the ratio $a_n a_{n+2}/(p_n p_{n+2})$, we obtain $a_n a_{n+1}/(p_n p_{n+1}) = z_n^3 z_{n+1}^3/(z_{n-2} z_{n+3})$ and finally the equation for $z_n$:

$$z_n z_{n+3} = z_{n-2} z_{n+5}. \tag{57}$$

The solution of the latter is $z_n = \kappa\lambda^n \varphi_5(n)$, where a period-2 term has been neglected. From equation relating $a_n, p_n$ to their upshifts we find, using (57), $a_n/p_n = z_{n-1} z_n^2 z_{n+1}/(z_{n-2} z_{n+2})$

and since $a_np_n = \gamma z_n\varphi_2(n)$ we have $a_n^2 = \gamma z_{n-1}z_n^3z_{n+1}\varphi_2(n)/(z_{n-2}z_{n+2})$ from which one can compute $a_n$. Finally from (43) we can obtain $b_n = r_nz_{n-2}z_{n+2}$ where $r_n = \tilde{\varphi}_2(n)$. Again, a gauge on $x_n$ can be used in order to eliminate one parameter and bring the total down to 7.

C.v) The final case for a quadratic over quadratic right-hand side corresponds to singularity lengths (2,2,2,2) leading to singularity constraints

$$a_n = z_nz_{n+1}p_{n+2}, \quad a_{n+2} = z_{n+1}z_{n+2}p_n, \quad b_n = z_nz_{n+1}r_{n+2}, \quad b_{n+2} = z_{n+1}z_{n+2}r_n. \tag{58}$$

Multiplying the constraint (43) by its double upshift we find

$$\frac{a_na_{n+2}b_nb_{n+2}}{p_np_{n+2}r_nr_{n+2}} = z_{n-1}z_n^2z_{n+1}^2z_{n+2}^2z_{n+3}. \tag{59}$$

Integrating the relations between $a_n, b_n$ and $p_n, r_n$ we find $a_np_n = \gamma z_n\tilde{\varphi}_2(n)$ and $b_nr_n = \delta z_n\tilde{\tilde{\varphi}}_2(n)$. Combining (58) and (59) we obtain the equation for $z_n$:

$$z_{n-2}z_{n+2} = z_n^2, \tag{60}$$

with solution $z_n = \kappa\lambda^n\varphi_2(n)^n$ (neglecting a period-2 term, for the usual reasons).

Multiplying the first two conditions (58) we find $a_na_{n+2} = \gamma z_nz_{n+1}z_{n+2}\tilde{\varphi}_2(n)$, after elimination of $p_n$. The condition can be integrated to $a_n = z_{n-1}^{1/4}z_nz_{n+1}^{1/4}(\gamma\tilde{\varphi}_2(n))^{1/2}\psi_4(n)$. A similar expression can be obtained for $b_n$ involving an a priori different period-4 function $\tilde{\psi}_4(n)$. However using the constraint (43) we find readily that $\tilde{\psi}_4(n)\psi_4(n) = 1$. This leaves 8 parameters $(\kappa, \varphi_2(n), \tilde{\varphi}_2(n), \tilde{\tilde{\varphi}}_2(n), \gamma, \delta$ and the 2 parameters of $\psi_4(n)$, but a gauge of $x_n$ removes one and the total is 7 as expected.

*D. The linear over linear case*

When three factors are simplified in the right-hand side of (25) we find the mapping

$$\frac{(x_nx_{n+1} - z^2)(x_nx_{n-1} - z^2)}{(x_nx_{n+1} - 1)(x_nx_{n-1} - 1)} = \frac{x_n - a}{x_n - p}, \tag{61}$$

where the parameters obey the constraint $a = z^4p$. There are four possible singularity patterns with lengths (1,7), (2,6), (3,5) and (4,4) to which we shall refer as cases D.i to D.iv. We deautonomise (61) to

$$\frac{(x_nx_{n+1} - z_nz_{n+1})(x_nx_{n-1} - z_nz_{n-1})}{(x_nx_{n+1} - 1)(x_nx_{n-1} - 1)} = \frac{x_n - a_n}{x_n - p_n}. \tag{62}$$

and the constraint that guarantees that $x_n = 0$ is not a singularity is simply

$$\frac{a_n}{p_n} = z_{n-1}z_n^2z_{n+1}. \tag{63}$$

D.i) The singularity confinement constraints in the case (1,7) are

$$p_np_{n+1} = 1, \quad a_na_{n+7} = z_nz_{n+1}z_{n+2}z_{n+3}z_{n+4}z_{n+5}z_{n+6}z_{n+7}, \tag{64}$$

while the condition (63), combined with its upshift leads to

$$a_na_{n+1} = z_{n-1}z_n^3z_{n+1}^3z_{n+2}. \tag{65}$$

Iterating (65) and using (64) we obtain for $z_n$ the equation

$$z_{n+9}z_{n+8}z_{n+1}z_n = z_{n+6}z_{n+5}z_{n+4}z_{n+3}, \tag{66}$$

the solution of which is $z_n = \kappa\lambda^n\varphi_5(n)\varphi_3(n)$ and where a period-2 term has been neglected, as usual. From (64) we have $p_n = \varphi_2(n)$ and integrating (65) we obtain $a_n = z_{n-1}z_n^2 z_{n+1}\varphi_2(n)$ where the period-2 term is the same for both $a_n, p_n$. This leads to a parametrisation of (62) involving precisely 7 parameters once the adequate gauge of $x_n$ has been implemented.

D.ii) The case of the singularity with lengths (2,6) is special. The confinement constraints are

$$a_n = z_n z_{n+1} p_{n+2}, \quad a_{n+6} = z_{n+1}z_{n+2}z_{n+3}z_{n+4}z_{n+5}z_{n+6}p_n, \tag{67}$$

and we can use (63) to eliminate $a_n$ in terms of $p_n$. From the confinement conditions we obtain for $z_n$ the equation

$$z_{n-1}z_n z_{n+6}z_{n+7} = 1 \tag{68}$$

the solution of which is $z_n = \psi_{14}(n)\varphi_2(n)$ where the period-2 factor must be neglected since $z_n$ enters only through products involving consecutive indices. We thus have 7 genuine parameters. We remark that not only there is no secular dependence on $n$ and thus the equation cannot be a discrete Painlevé, but there is not even a constant term in $z_n$. Thus the resulting integrable system makes sense only as a non-autonomous QRT mapping with periodic coefficients [28].

D.iii) In this case the singularity lengths are (3,5), leading to the confinement constraints

$$p_n p_{n+3} z_{n+1} z_{n+2} = 1, \quad a_n a_{n+5} = z_n z_{n+1} z_{n+2} z_{n+3} z_{n+4} z_{n+5}. \tag{69}$$

Again, since the variable $z_m$ enters only through the combination $z_m z_{m+1}$, the first of these conditions allows to express $z_n$ in terms of $p_n$ as $z_n z_{n+1} = 1/(p_{n-1}p_{n+2})$. Using condition (63) we can express $a_n$ in terms of $p_n$ as $a_n = p_n/(p_{n+2}p_{n+1}p_{n-1}p_{n-2})$. From the second condition (69) we find that $p_n$ must satisfy the equation

$$p_{n-2}p_{n+7} = p_n p_{n+5}. \tag{70}$$

the solution of which is $p_n = \kappa\lambda^n\varphi_2(n)\varphi_7(n)$, introducing 8 parameters, one of which can be removed by a choice of gauge for $x_n$.

D.iv) The last case corresponds to singularities with lengths (4,4), leading to the confinement conditions

$$a_n = z_n z_{n+1} z_{n+2} z_{n+3} p_{n+4}, \quad a_{n+4} = z_{n+1}z_{n+2}z_{n+3}z_{n+4}p_n, \tag{71}$$

Again, using (63), we eliminate $a_n$ and use the two constraints for $p_n$ to obtain the equation for $z_n$. We find

$$z_{n-1}z_n z_{n+4}z_{n+5} = z_{n+1}z_{n+2}^2 z_{n+3}, \tag{72}$$

with solution $z_n = \kappa\lambda^n\varphi_3(n)\varphi_2(n)^n$ (where we have neglected a period-2 term for the usual reasons). Introducing the auxiliary quantity $C_n = a_n p_n/z_n$ and combining the two relations (71) we find the constraint $C_{n+4} = C_n$ that can be integrated to $C_n = \gamma\varphi_4(n)$. This allows, using (63), to compute $a_n$ from $a_n^2 = C_n z_{n-1} z_n^3 z_{n+1}$ and then $p_n = C_n z_n/a_n$. Eight parameters enter the solution, one of which can be removed by the adequate gauge of $x_n$.

# 4 From $\mathrm{E}_8$ to $\mathrm{E}_7$ equations

From the degeneration pattern presented in Figure 1, it is clear that it should be possible to link the equations we have just obtained to equations associated to the $\mathrm{E}_8^{(1)}$ group, obtained from them by some limiting procedure. For reasons that will become clear later in this section, we shall work with the most general, QRT-asymmetric form of the generic, *multiplicative*, $q$-equation associated to $\mathrm{E}_8^{(1)}$.

We introduce the notation $Z_n = \kappa\lambda^n$ and $H_n = Z_n/\sqrt{\lambda}$ and eight constants $D_i$. The $M_1, \cdots, M_8$ are the elementary symmetric functions constructed from the $D_i$ and we assume $M_8 = \prod_i D_i = 1$. The equation is:

$$\frac{(Y_{n+1}H_{n+1}Z_n - X_n)(Y_nH_nZ_n - X_n) - (H_{n+1}^2Z_n^2 - 1)(H_n^2Z_n^2 - 1)}{(Y_{n+1}/(H_{n+1}Z_n) - X_n)(Y_n/(H_nZ_n) - X_n) - (1 - 1/(H_{n+1}^2Z_n^2))(1 - 1/(H_n^2Z_n^2))} =$$
$$\frac{X_n^4 - M_1Z_nX_n^3 + (M_2Z_n^2 - 3 - Z_n^8)X_n^2 + (M_7Z_n^7 - M_3Z_n^3 + 2M_1Z_n)X_n + Z_n^8 - M_6Z_n^6 + M_4Z_n^4 - M_2Z_n^2 + 1}{X_n^4 - M_7X_n^3/Z_n + (M_6/Z_n^2 - 3 - 1/Z_n^8)X_n^2 + (M_1/Z_n^7 - M_5/Z_n^3 + 2M_7/Z_n)X_n + 1/Z_n^8 - M_2/Z_n^6 + M_4/Z_n^4 - M_6/Z_n^2 + 1} \quad (73a)$$

$$\frac{(X_{n-1}H_nZ_{n-1} - Y_n)(X_nH_nZ_n - Y_n) - (H_n^2Z_{n-1}^2 - 1)(H_n^2Z_n^2 - 1)}{(X_{n-1}/(H_nZ_{n-1}) - Y_n)(X_n/(H_nZ_n) - Y_n) - (1 - 1/(H_n^2Z_{n-1}^2))(1 - 1/(H_n^2Z_n^2))} =$$
$$\frac{Y_n^4 - M_7H_nY_n^3 + (M_6H_n^2 - 3 - H_n^8)Y_n^2 + (M_1H_n^7 - M_5H_n^3 + 2M_7H_n)Y_n + H_n^8 - M_2H_n^6 + M_4H_n^4 - M_6H_n^2 + 1}{Y_n^4 - M_1Y_n^3/H_n + (M_2/H_n^2 - 3 - 1/H_n^8)Y_n^2 + (M_7/H_n^7 - M_3/H_n^3 + 2M_1/H_n)Y_n + 1/H_n^8 - M_6/H_n^6 + M_4/H_n^4 - M_2/H_n^2 + 1} \quad (73b)$$

In order to go from this equation to an $\mathrm{E}_7^{(1)}$-associated one we introduce $X = \Omega x$, $Y = \Omega y$, and take the limit $\Omega \to \infty$. The terms containing the secular dependence, $Z_n$ and $H_n$ remain finite, but for uniformity, we shall represent them as $z_n$ and $\eta_n$. Among the 8 quantities $D_i$ we assume that four (say, $D_1$ to $D_4$) will go to infinity like $\Omega$, the remaining four ($D_5$ to $D_8$) going to zero like $1/\Omega$ so as to ensure that $M_8 = 1$. Keeping just the dominant terms, the first three symmetric functions $M_1$, $M_2$, $M_3$ divided by $\Omega^1$, $\Omega^2$, $\Omega^3$ respectively, become, at the limit $\Omega \to \infty$ the elementary symmetric functions $m_1$, $m_2$, $m_3$ of the four $D_i/\Omega$, $(i = 1, \cdots, 4)$. Similarly $M_7$, $M_6$ $M_5$ divided by $\Omega^1$, $\Omega^2$, $\Omega^3$ respectively, become, at the limit the elementary functions $p_1$, $p_2$, $p_3$ of the *inverse* of the four $\Omega D_i$, $(i = 5, \cdots, 8)$. The function $M_4$ divided by $\Omega^4$, at the limit, gives rise to $m_4 = p_4$, i.e. the common value of $\prod_1^4 D_i/\Omega$ and $\prod_5^8(\Omega D_i)^{-1}$.

At the limit $\Omega \to \infty$ we keep only the dominant terms and (73) becomes

$$\frac{(y_{n+1}\eta_{n+1}z_n - x_n)(y_n\eta_nz_n - x_n)}{(y_{n+1}/(\eta_{n+1}z_n) - x_n)(y_n/(\eta_nz_n) - x_n)} = \frac{x_n^4 - m_1z_nx_n^3 + m_2z_n^2x_n^2 - m_3z_n^3x_n + m_4z_n^4}{x_n^4 - p_1x_n^3/z_n + p_2x_n^2/z_n^2 - p_3x_n/z_n^3 + p_4/z_n^4} \quad (74a)$$

$$\frac{(x_{n-1}\eta_nz_{n-1} - y_n)(x_n\eta_nz_n - y_n)}{(x_{n-1}/(\eta_nz_{n-1}) - y_n)(x_n/(\eta_nz_n) - y_n)} = \frac{y_n^4 - p_1\eta_ny_n^3 + p_2\eta_n^2y_n^2 - p_3\eta_n^3y_n + p_4\eta_n^4}{y_n^4 - m_1y_n^3/\eta_n + m_2y_n^2/\eta_n^2 - m_3y_n/\eta_n^3 + m_4/\eta_n^4} \quad (74b)$$

We remark that this form of the $\mathrm{E}_7^{(1)}$-associated equation is one that, as we have shown, can be obtained by deautonomisation of a QRT mapping belonging to the class VI′ of the classification [10] . However, as is well-known, a VI′ form can be transformed to a VI one [29]. To this end we introduce the change of variables $x_n \to x_n/z_n$, $y_n \to \eta_n/y_n$ and find, using the fact that $\eta_n^2 = z_nz_{n-1}$, and after inverting both sides of the equation obtained from (74b):

$$\frac{(x_ny_{n+1} - z_n^2\eta_{n+1}^2)(x_ny_n - z_n^2\eta_n^2)}{(x_ny_{n+1} - 1)(x_ny_n - 1)} = \frac{x_n^4 - m_1z_n^2x_n^3 + m_2z_n^4x_n^2 - m_3z_n^6x_n + m_4z_n^8}{x_n^4 - p_1x_n^3 + p_2x_n^2 - p_3x_n + p_4} \quad (75a)$$

$$\frac{(x_{n-1}y_n - z_{n-1}^2\eta_n^2)(x_ny_n - z_n^2\eta_n^2)}{(x_{n-1}y_n - 1)(x_ny_n - 1)} = \frac{m_4y_n^4 - m_3\eta_n^2y_n^3 + m_2\eta_n^4y_n^2 - m_1\eta_n^6y_n + \eta_n^8}{p_4y_n^4 - p_3y_n^3 + p_2y_n^2 - p_1y_n + 1} \tag{75b}$$

It is possible to give an even more interesting form to these equations by deciding that the four $D_i$ that go to infinity are those for $i = 1, \dots, 4$ and introduce $D_i = d_i\Omega$, while those for $i = 5, \dots, 8$ go to zero, so we put $D_i = d_i/\Omega$. We find thus that the right-hand side of (75) factorises leading to

$$\frac{(x_ny_{n+1} - z_n^2\eta_{n+1}^2)(x_ny_n - z_n^2\eta_n^2)}{(x_ny_{n+1} - 1)(x_ny_n - 1)} = \frac{(x_n - d_1z_n^2)(x_n - d_2z_n^2)(x_n - d_3z_n^2)(x_n - d_4z_n^2)}{(x_n - 1/d_5)(x_n - 1/d_6)(x_n - 1/d_7)(x_n - 1/d_8)} \tag{76a}$$

$$\frac{(x_{n-1}y_n - z_{n-1}^2\eta_n^2)(x_ny_n - z_n^2\eta_n^2)}{(x_{n-1}y_n - 1)(x_ny_n - 1)} = \frac{(y_n - \eta_n^2/d_1)(y_n - \eta_n^2/d_2)(y_n - \eta_n^2/d_3)(y_n - \eta_n^2/d_4)}{(y_n - d_5)(y_n - d_6)(y_n - d_7)(y_n - d_8)} \tag{76b}$$

where we have used the fact that $\prod_i d_i = 1$. Equation (76) is precisely the equation we introduced in [9] under the name of asymmetric $q$-P$_{\rm VI}$.

Since the E$_8^{(1)}$-associated equations have been classified in [26] it is edifying to establish the correspondence between the results of that paper and the ones obtained above. One should bear in mind that the results of [26] were obtained in the ancillary formulation and concerned additive rather than multiplicative equations. Still the limiting process is straightforward: one can consider the limit from E$_8^{(1)}$ to E$_7^{(1)}$ in an additive setting and then transcribe the results to the multiplicative one that we are focusing on here. We are not going to present the details, lest this becomes tedious, and we just give the correspondence. The generic case, equation (26) is obtained from the generic equation (6) of [26]. The two equations B.i and B.ii are obtained from 5.2.1 and 5.1.1 of [26]. The equations of the list C, i to v, are obtained from 4.5.1, 4.4.3, 4.3.1, 4.2.1 and 4.1, in that order. Finally the equations of the list D, corresponding to a linear over linear right-hand side are obtained from the equations we have called trihomographic, D.i coming from 3.1, D.iii from 3.3 and D.iv from 3.4. Another equation was present in [26], obtained from singularity patterns of lengths 6 and 2, just as we posited for the case D.ii and there it led to a discrete Painlevé equation with coefficients with periods 4 and 5. In order to understand why in the case of E$_7^{(1)}$ no discrete Painlevé results from the deautonomisation approach it is instructive to go back to the limit from E$_8^{(1)}$ to E$_7^{(1)}$.

The equations of the subcase D, i.e. those with linear over linear right-hand side are what we call trihomographic equations. For E$_8^{(1)}$-associated equations, starting from the general form

$$\frac{x_{n+1} - a_n}{x_{n+1} - b_n}\frac{x_{n-1} - c_n}{x_{n-1} - d_n}\frac{x_n - e_n}{x_n - f_n} = 1, \tag{8}$$

we have, in the multiplicative case,

$$a_n = k_nz_nz_{n-1} + \frac{1}{k_nz_nz_{n-1}},\ b_n = \frac{z_nz_{n-1}}{k_n} + \frac{k_n}{z_nz_{n-1}},\ c_n = k_nz_nz_{n+1} + \frac{1}{k_nz_nz_{n+1}},$$
$$d_n = \frac{z_nz_{n+1}}{k_n} + \frac{k_n}{z_nz_{n+1}},\ e_n = -\frac{z_n^2z_{n-1}z_{n+1}}{k_n} - \frac{k_n}{z_n^2z_{n-1}z_{n+1}},$$
$$f_n = -k_nz_n^2z_{n-1}z_{n+1} - \frac{1}{k_nz_n^2z_{n-1}z_{n+1}}, \tag{77}$$

corresponding to an equation of the form

$$\frac{(x_{n+1}z_{n+1}z_n - x_n)(x_{n-1}z_{n-1}z_n - x_n) - (z_{n+1}^2 z_n^2 - 1)(z_{n-1}^2 z_n^2 - 1)}{(x_{n+1} - z_{n+1}z_n x_n)(x_n - z_{n-1}z_n x_n) - (z_{n+1}^2 z_n^2 - 1)(z_{n-1}^2 z_n^2 - 1)/(z_{n+1}z_n^2 z_{n-1})} = \frac{x_n + z_{n+1}z_n^2 z_{n-1}(k_n + 1/k_n)}{x_n z_{n+1}z_n^2 z_{n-1} + k_n + 1/k_n}. \tag{78}$$

The $E_7^{(1)}$ case is obtained by assuming that $x_n$ and $k_n$ go to infinity resulting to the equation

$$\frac{x_{n+1} - k_n z_n z_{n-1}}{x_{n+1} - \frac{k_n}{z_n z_{n-1}}} \frac{x_{n-1} - k_n z_n z_{n+1}}{x_{n-1} - \frac{k_n}{z_n z_{n+1}}} \frac{x_n + \frac{k_n}{z_n^2 z_{n-1} z_{n+1}}}{x_n + k_n z_n^2 z_{n-1} z_{n+1}} = 1, \tag{79}$$

the equation thus obtained is of the form VI′

$$\frac{(x_{n+1}z_n z_{n+1} - x_n)}{(x_{n+1} - z_n z_{n+1} x_n)} \frac{(x_{n-1}z_n z_{n-1} - x_n)}{(x_{n-1} - z_n z_{n-1} x_n)} = \frac{x_n + k_n z_{n+1} z_n^2 z_{n-1}}{x_n z_{n+1} z_n^2 z_{n-1} + k_n}. \tag{80}$$

However the limiting procedure must be adapted to the details of every specific realisation of the $E_8^{(1)}$-associated discrete Painlevé equations.

In the paragraph that follows we shall show how to implement the limit while keeping the secular and periodic dependence of the coefficients. Our aim is just to show how this limit does not exist in the D.ii case, but it is instructive to start with cases where the limit does exist. In the case of singularity patterns with lengths (4,4) the $E_8^{(1)}$ discrete Painlevé equation has $z_n z_{n-1} = \kappa\lambda^n \varphi_2(n)\varphi_3(n)$ and $k_n = \gamma\varphi_4(n)$. It suffices in this case to let $\gamma$ and $x_n$ go to infinity together in order to obtain the corresponding $E_7^{(1)}$ discrete Painlevé equation.

$$\frac{x_{n+1} - z_n z_{n-1}\varphi_4(n)}{z_n z_{n-1} x_{n+1} - \varphi_4(n)} \frac{x_{n-1} - z_n z_{n+1}\varphi_4(n)}{z_n z_{n+1} x_{n-1} - \varphi_4(n)} = \frac{x_n + \varphi_4(n) z_{n-1} z_n^2 z_{n+1}}{z_{n-1} z_n^2 z_{n+1} x_n + \varphi_4(n)}, \tag{81}$$

We remark that this leads to an equation of the same form as equation (80) where $k_n$ is replaced by $\varphi_4(n)$.

The case of singularity patterns of lengths (1,7) and (3,5) is more delicate. We have, for instance in the former case, $z_n z_{n-1} = \kappa\lambda^n \varphi_3(n)\varphi_5(n)$ while $k_n$ is a product of a secular term, a term of period 5 *and* a $\varphi_2(n)$. The latter is a term of the form $\gamma^{(-1)^n}$ and the limit to $E_7^{(1)}$ is obtained by taking $\gamma$ together with $x_n$ to infinity. We put $k_n = g_n\varphi_2(n)$ and, using the expressions (77) we find, for $n = 2m$, the terms of the $a_n, \cdots, f_n$ surviving at the limit are the ones having $k_n$ at the numerator, resulting to the equation

$$\frac{x_{2m+1} - g_{2m}z_{2m}z_{2m-1}}{z_{2m}z_{2m-1}x_{2m+1} - g_{2m}} \frac{x_{2m-1} - g_{2m}z_{2m}z_{2m+1}}{z_{2m}z_{2m+1}x_{2m-1} - g_{2m}} = \frac{x_{2m} + g_{2m}z_{2m-1}z_{2m}^2 z_{2m+1}}{z_{2m-1}z_{2m}^2 z_{2m+1}x_{2m} + g_{2m}}. \tag{82}$$

On the other hand, for $n = 2m+1$, the terms of the $a_n, \cdots, f_n$ surviving at the limit are in this case those having $k_n$ at the denominator, resulting in the equation

$$\frac{z_{2m}z_{2m+1}x_{2m+2} - g_{2m+1}}{x_{2m+2} - g_{2m+1}z_{2m}z_{2m+1}} \frac{z_{2m+2}z_{2m+1}x_{2m} - g_{2m+1}}{x_{2m} - g_{2m+1}z_{2m+2}z_{2m+1}} = \frac{z_{2m}z_{2m+1}^2 z_{2m+2}x_{2m+1} + g_{2m+1}}{x_{2m+1} + g_{2m+1}z_{2m}z_{2m+1}^2 z_{2m+2}}. \tag{83}$$

which, once inverted, is just the upshift of (82). Again, this leads to an equation of the form of (80) where $k_n$ is replaced by $g_n$. A similar result can be obtained in the case (3,5).

However the case (2,6) is different. Here the various quantities can be expressed in terms of an auxiliary object of the form $u_n = \kappa\lambda^n\varphi_4(n)\varphi_5(n)$. The quantity $z_nz_{n+1}$ can be expressed as $z_nz_{n-1} = u_{n-1}u_{n+1} = \kappa^2\lambda^{2n}\varphi_4(n-1)\varphi_4(n+1)\varphi_5(n-1)\varphi_5(n+1)$ Note that the periodic function $\varphi_4(n)$ can be rewritten as $\varphi_2(n)\psi_4(n)$ so $z_nz_{n-1} = \kappa^2\lambda^{2n}\varphi_5(n-1)\varphi_5(n+1)/\varphi_2(n)^2$. The expression of $k_n$ in terms of $u_n$ is in this case $k_n = u_{n-1}u_{n+2} = \kappa^2\lambda^{2n+1}\varphi_4(n-1)\varphi_4(n+2)\varphi_5(n-1)\varphi_5(n+2)$ and the $\varphi_2(n)$ "hidden" in $\varphi_4(n)$ disappears in $k_n$. Infinite values for $\psi_4$ or $\varphi_5$ cannot be used to obtain a limit of the form we are seeking. Moreover, in order to obtain an equation like (80), $z_nz_{n-1}$ must remain finite in the limit we are seeking, so $\kappa$ and $\varphi_2(n)$ must remain finite. So, there is no way to implement the limit leading to an $E_7^{(1)}$ equation while preserving the discrete Painlevé character.

## 5 Conclusion

The systematic derivation of discrete Painlevé equations can be traced back to a paper of two of the present authors in collaboration with J. Hietarinta. In that paper the newfangled discrete integrability criterion of singularity confinement was used in order to produce discrete analogues of the Painlevé equations for I to V. The discrete forms of $P_I$ and $P_{II}$ had already been derived using a different approach, but our study showed that the singularity approach could not only reproduce said forms but, in fact, obtain degrees of freedom in the form of periodic functions that had eluded the previous studies. An interesting, and at the time unexpected, result was the discovery of multiplicative, $q$-discrete, forms of Painlevé equations, in particular for $P_{III}$ and $P_V$.

The method used in that paper was the one that came to be known as 'deautonomisation'. As explained above, the deautonomisation method consists in, starting from an integrable autonomous mapping, to seek non-autonomous extensions that preserve the integrable character. In the case of discrete Painlevé equations the starting point is a mapping belonging to the QRT family. The rationale behind this choice is that the QRT mapping is integrated in terms of elliptic functions. Thus, in parallel to the differential case where the autonomous limit of Painlevé equations are equations integrated by elliptic functions, one expects the deautonomisation of the QRT mapping to lead to discrete analogues of Painlevé equations. This was indeed verified in the study in question (and in subsequent ones) by the direct calculation of the continuum limit of the obtained system, which did indeed lead to differential Painlevé equations.

The deautonomisation approach introduced in that pioneering work made possible the derivation of a slew of integrable systems. The proper way of applying the procedure was presented in an algrebrogeometric setting by some of the present authors in collaboration with T. Mase. The notions of 'late' and 'early' confinement emerged from that study. When applying the singularity confinement criterion it is often possible to eschew the application of the confinement constraints when they first appear only to apply them at some later occurence. This invariably results in non-integrable systems. (The possibility of 'early' confinement is less frequent, possible only when the structure of the mapping allows it. Contrary to the 'timely' confinement the constraints of the early one lead to trivial results, usually periodic mappings).

It is worth pointing out that the analysis of the deautonomisation procedure led to an intriguing discovery. When studying the confinement constraints in the case of late confinement, it turned out that the root of the characteristic polynomial coincided with the dynamical degree of the mapping. This is not a mere coincidence. In [22], some of the present authors in collaboration with T. Mase and A. Stokes, presented a rigorous justification of this. And the extension of the approach to what was dubbed 'full-deautonomisation' led to an integrability criterion, based on singularity confinement, which covers all cases where a naïve application of confinement would lead to false positive results.

The application of deautonomisation for the derivation of discrete Painlevé equations, while undeniably successful, was plagued by a certain empiricism. Given a discrete system, the choice of the 'timely' confinement constraints was, in some cases, a question of experience and/or intuition. This changed with the advent of the algebro-geometric methods which aimed at the regularisation of integrable mappings through a series of blow-ups but also by the push of the study towards more and more complicated systems, culminating with equations associated with the affine Weyl group $\mathrm{E}_8^{(1)}$. The latter systems are pretty complicated and the use of an empirical deautonomisation approach was out of the question. We shall not enter here into any detail concerning our results on $\mathrm{E}_8^{(1)}$-associated systems. (The interested reader is invited to consult our papers [24, 27,26]). The main tool for the study of these equations is one offered by the algebro-geometric analysis which showed that in all discrete Painlevé equations examined the number of blow-ups is equal to 8. While linking the number of blow-ups to the length of the singularity pattern required some leap of faith, it turned out that it was a fair prescription in the case of $\mathrm{E}_8^{(1)}$-associated systems.

In this paper we set out to deautonomise a selection of mappings which are expected to lead, when non-autonomous, to discrete Painlevé equations associated with the affine Weyl group $\mathrm{E}_7^{(1)}$. This work complements (and corrects) results obtained in [20] by the same authors in collaboration with J. Satsuma. Fixing the length of the singularity patterns simplify the calculations to a point that no computer algebra is needed. Given the degeneration pattern that links equations of the affine Weyl groups $\mathrm{E}_8^{(1)}$ and $\mathrm{E}_7^{(1)}$, it was interesting to establish the relations between the equations obtained here and those of [26]. (To be fair, the equations of the latter were of additive type, whereas here we have dealt with multiplicative equations. However this does not constitute a problem. First the results obtained here can be directly transposed to an additive setting, provided one works with a mapping of the form (7) and takes the logarithm of the parameters depending on the independent variable. Vice versa, the results obtained for the $\mathrm{E}_8^{(1)}$ additive equations can be directly transformed to ones idoneous for $q$-systems, by simple exponentiation. One interesting result is that while in the $\mathrm{E}_8^{(1)}$ case all singularity lengths led to discrete Painlevé equations, in the $\mathrm{E}_7^{(1)}$ case we found one instance, equation D.ii, where the application of singularity confinement does not lead neither to secular nor to constant terms, and thus merely yields a QRT mapping with periodic coefficients and not a discrete Painlevé equation.

The method presented in this paper has wide applicability and we expect it to be at the core of future works of ours.

## References

[1] A. Ramani, B. Grammaticos and J. Hietarinta, *Discrete versions of the Painlevé equations*, Phys. Rev. Lett. 67 (1991) 1829.

[2] B. Grammaticos and A. Ramani, *Retracing the Painlevé-Gambier classification for discrete systems*, Meth. and Appl. of An. 4 (1997) 196.

[3] M. Jimbo and H. Sakai, *A q-analog of the sixth Painlevé equation*, Lett. Math. Phys. 38 (1996) 145.

[4] G.R.W. Quispel, J.A.G. Roberts and C.J. Thompson, *Integrable mappings and soliton equations II*, Physica D 34 (1989) 183.

[5] B. Grammaticos and R. Willox, *Full-deautonomisation of a class of second-order mappings in ancillary form*, Open Comm. Nonlin. Math. Phys. 3 (2023) 1.

[6] B. Grammaticos, A. Ramani and V. Papageorgiou, *Do integrable mappings have the Painlevé property?*, Phys. Rev. Lett. 67 (1991) 1825.

[7] *The Painlevé of discrete equations and other stories*, A. Ramani, B. Grammaticos and Y. Ohta, invited talk at the "Atelier sur la théorie des fonctions spéciales non linéaires: les transcendants de Painlevé", Montréal (1996).

[8] A. Ramani, B. Grammaticos and T. Tamizhmani, *Quadratic relations in continuous and discrete Painlevé equations*, J. Phys. A 33 (2000) 3033.

[9] B. Grammaticos and A. Ramani, *On a q-discrete analogue of the Painlevé VI equation*, Phys. Lett. A 257 (1999) 288.

[10] A. Ramani, S. Carstea, B. Grammaticos and Y. Ohta, *On the autonomous limit of discrete Painlevé equations*, Physica A 305 (2002) 437.

[11] A. Ramani, Y. Ohta, J. Satsuma and B. Grammaticos, *Self-duality and Schlesinger chains for the asymmetric d-PII and q-PIII equations*, Comm. Math. Phys. 192 (1998) 67.

[12] A. Ramani and B. Grammaticos, *The Grand Scheme for discrete Painlevé equations*, Lecture at the Toda symposium (1996).

[13] B. Grammaticos and A. Ramani, *The hunting for the discrete Painlevé equations*, Reg. and Chaot. Dyn., 5 (2000) 53.

[14] Y. Ohta, A. Ramani and B. Grammaticos, *An affine Weyl group approach to the 8-parameter discrete Painlevé equations*, J. Phys. A 34 (2001) 10523.

[15] K. Okamoto, *On the $\tau$-function of the Painlevé equations*, Physica D 2 (1981), 525.

[16] R. Hirota, *Discrete Analogue of a Generalized Toda Equation*, J. Phys. Soc. Japan 50 (1981) 3785.

[17] T. Miwa, *On Hirota's difference equations*, Proc. Japan. Acad. 58 (1982) 9.

[18] H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Commun. Math. Phys. 220 (2001) 165.

[19] A. Ramani and B. Grammaticos, *The number of discrete Painlevé equations is infinite*, Phys. Lett. A 373 (2009) 3028.

[20] A. Ramani, R. Willox, B. Grammaticos, A.S. Carstea and J. Satsuma, *Limits and degeneracies of discrete Painlevé equations: a sequel*, Physica A 347 (2005) 1.

[21] T. Mase, R. Willox, B. Grammaticos and A. Ramani, *Deautonomisation by singularity confinement: an algebro-geometric justification*, Proc. R. Soc. A 471 (2015) 20140956.

[22] A. Stokes, T. Mase, R. Willox and B. Grammaticos, *Deautonomisation by singularity confinement and degree growth*, J. Geom. Anal. 35 (2025) 65.

[23] Y. Ohta, A. Ramani and B. Grammaticos, *Elliptic discrete Painlevé equations*, J. Phys. A 35 (2002) L653.

[24] A. Ramani and B. Grammaticos, *Discrete Painlevé equations associated with the affine Weyl group E8*, J. Phys. A 48 (2015) 355204.

[25] B. Grammaticos, A. Ramani, *On a novel representation of discrete Painlevé equations*, J. Math. Phys. 56 (2015) 083507.

[26] A. Ramani and B. Grammaticos, *Singularity analysis for difference Painlevé equations associated with the affine Weyl group E8*, J. Phys. A 50 (2017) 055204.

[27] B. Grammaticos and A. Ramani, *Elliptic and multiplicative discrete Painlevé equations from the affine Weyl group E8*, J. Phys. A 48 (2015) 16FT02.

[28] A. Ramani, B. Grammaticos and R. Willox, *Generalised QRT mappings with periodic coefficients*, Nonlinearity 24 (2011) 113.

[29] A. Ramani and B. Grammaticos, *A new family of discrete Painlevé equations and associated linearisable systems*, J. Nonlin. Math. Phys. 20 (2013) 153.