Moving boundary problems for a novel extended mKdV equation. Application of Ermakov-Painlevé II symmetry reduction.

Colin Rogers ¹, Adriana C. Briozzo ²

- ¹ School of Mathematics and Statistics, The University of New South Wales, Sydney NSW 2052, Australia
- ² Depto. Matemática, FCE, Univ. Austral, Paraguay 1950-CONICET S2000FZF Rosario, Argentina.

Email: c.roqers@unsw.edu.au, abriozzo@austral.edu.ar

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Abstract

A novel extension of the canonical solitonic mKdV equation is introduced which admits hybrid Ermakov-Painlevé II symmetry reduction. Application of the latter is made to obtain exact solution of Airy-type to a class of moving boundary problems of Stefan kind for this extended mKdV equation. A reciprocal transformation is then applied to the latter to generate an associated exactly solvable class of moving boundary problems for an extension of a base Casimir member of a compacton hierarchy. The extended mKdV equation is shown to be embedded in a range of nonlinear evolution equations with temporal modulation as determined via the action of a class of involutory transformations with origin in Ermakov theory. Associated temporal modulation for the hybrid mKdV and KdV equation as embedded in the classical solitonic Gardner equation is delimited.

1 Introduction

The mKdV equation has diverse physical applications, notably, in the analysis of nonlinear Alfvén waves in collisionless plasma [18] and of acoustic wave propagation in an anharmonic lattice [70]. Its connection to the canonical Korteweg-de Vries equation of soliton theory due to Miura [22] was shown in [26] to constitute the spatial part of Bäcklund transformation with classical geometric origins. The property of invariance under Bäcklund transformations and application of iterated associated nonlinear superposition principles has established importance in modern soliton theory [36]. In a geometric context, the mKdV equation can be derived in connection with the motion of an extensible curve of zero torsion.

Moving boundary problems of Stefan-type in continuum mechanics arise importantly in the analysis of the melting of solids and freezing of liquids (qv. [2, 14, 15, 17, 20, 55, 64] and literature cited therein). The heat balance condition on the moving boundary which separates the phases characteristically provides a nonlinear boundary boundary condition. Reciprocal-type transformations have been applied in [28] to derive novel analytic solutions to moving boundary problems associated with heat conduction in a range of metals as detailed by Storm [61] which have temperature-dependent specific heat and thermal conductivity. Conditions for the onset of melting in such metals subject to applied boundary flux may be thereby determined [31]. The threshold melting conditions as previously derived by Tarzia [63] and Solomon et al [60] for analogous moving boundary problems for the classical linear heat equation were thereby extended.

In modern soliton theory, a Painlevé II symmetry reduction was applied in [41] to derive exact solution to a class of moving boundary problems for the canonical Harry Dym equation [65]. The latter arises notably in connection with analysis of the evolution of the interface in Hele-Shaw problems [16]. A novel sequence of analytically solvable moving boundary problems with interface of the type $x = \gamma t^n$ was generated in [41] via iterated action of a Bäcklund transformation. Exact solutions were derived in terms of Yablonski-Vorob'ov polynomials [66, 69]. Therein the index n adopted a sequence of values of the Painlevé II parameter. In [45] such Yablonski-Vorob'ov polynomials solutions for moving boundary problems were shown to extended to a generalised solitonic Dym equation. The latter was derived in a geometric context in [59] and has physical application to the analysis of peakon solitonic phenomena in hydrodynamics [11].

In subsequent work [51] -[54] a series of moving boundary problems of Stefan-type has been shown to be amenable to exact solution via Painlevé II symmetry reduction. In [53] it was established that action of a reciprocal transformation links the solitonic Korteweg- de Vries equation to a novel nonlinear evolution equation which incorporates a source term. Moving boundary problems for the latter were thereby shown to admit exact solution. It is remarked that in [10] a reciprocal transformation has been applied to reduce to canonical form a class of moving boundary problems involving a source term relevant in a soil mechanics context.

Reciprocal transformations as introduced in a modern solitonic setting in [21] constitute a class of auto- Bäcklund transformations which act on admitted conservation laws. In [21] conjugation with the classical geometric nonlinear superposition principle (permutability theorem) of Bianchi was set down which allows iterative generation of multi-soliton solutions in an algorithmic manner.

Reciprocal transformations have been applied in [27] in the linkage of the canonical AKNS and WKI inverse scattering schemes of [1] and [68] respectively. Invariance of the 1+1-dimensional Dym solitonic hierarchy under a class of reciprocal transformations was established in [29]. Reciprocal transformations in 2+1 dimensions [30] were shown in [24] to connect the Kadomtsev-Petviashvili, modified Kadomtsev-Petviashvili and 2+1-dimensional Dym triad of S-integrable hierarchies.

Hybrid Ermaov-Painlevé II systems were originally derived in [40] via a symmetry reduction of an n+1-dimensional Manakov-type NLS system. Therein, in particular, analysis of certain transverse wave motions in a generalised Mooney-Rivlin hyperelastic material was shown to lead to a novel base canonical Ermakov-Painlevé II reduction. The latter has been subsequently derived and applied in such diverse areas as cold plasma physics [47], Korteweg capillarity theory [45] and in connection with Dirichlet-type boundary value problems for a multi-ion Nernst-Planck electrolytic system [3]. A link with the classical Painlevé XXXIV equation was established in [43]. Here, a novel extension of the solitonic mKdV equation is introduced which admits Ermakov-Painlevé II symmetry reduction. The latter is applied to derive exact solution to a class of moving boundary problems for this generalised mKdV equation. A reciprocal transformation is then used to derive an extension of a base Casimir member of the compacton hierarchy as set down in [25]. A class of reciprocally associated exactly solvable moving boundary problems for the extended Casimir reciprocal associate is delimited.

$\mathbf{2}$ An Extended mKdV Equation: Ermakov-Painlevé II Symmetry Reduction

Here, a novel mKdV equation with temporal modulation is introduced, namely

$$u_t - 6u^2 u_x + u_{xxx} + \lambda (t+a)^{\mu} u^{-4} u_x = 0, \quad \lambda, \mu \in \mathbb{R}$$
 (1)

and a symmetry reduction to the canonical Ermakov-Painlevé II equation established via the ansatz

$$u = (t+a)^m \Psi\left(\frac{x}{(t+a)^n}\right). \tag{2}$$

Thus, insertion of the latter into (1), on reduction, produces

$$m\Psi - n\xi\Psi' - 6(t+a)^{2m-n+1}\Psi^2\Psi' + (t+a)^{-3n+1}\Psi''' + \lambda(t+a)^{\mu-4m-n+1}\Psi^{-4}\Psi' = 0, \ (3)$$

whence $m = -\frac{1}{3}$, $n = \frac{1}{3}$ together with $\mu = -2$ so that

$$\Psi''' - 6\Psi^2 \Psi' - \frac{1}{3} (\xi \Psi)' + \lambda \Psi^{-4} \Psi' = 0 \tag{4}$$

where, in the preceding, $\xi = x/(t+a)^n$.

Integration of the latter yields

$$\Psi'' - 2\Psi^3 - \frac{1}{3}\xi\Psi - \frac{\lambda}{3}\Psi^{-3} = \zeta,\tag{5}$$

with $\zeta \in \mathbb{R}$. On introduction of the scalings $\Psi = \delta w^*$, $\xi = \epsilon \zeta$ with

$$\delta^2 \epsilon^2 = 1, \quad \epsilon^3 / 3 = 1 \tag{6}$$

the canonical Ermakov-Painlevé II equation

$$w_{zz}^* = 2w^{*3} + zw^* + \frac{\delta}{w^{*3}} \tag{7}$$

results with $\delta = -3\lambda$ if $\zeta = 0$.

3 A Class of Moving Boundary Problems

Here a class of Stefan-type moving boundary problems for the extended mKdV equation (1) is introduced, namely

$$u_t - 6u^2 u_x + u_{xxx} + \lambda(t+a)^{-2} u^{-4} u_x = 0, \quad 0 < x < S(t), \quad t > 0,$$
(8)

$$u_{xx} - 2u^3 - \frac{\lambda}{3}(t+a)^{-2}u^{-3} = L_m \dot{S}S^i$$
, on $x = S(t)$, $t > 0$, (9)

$$u = P_m S^j$$
, on $x = S(t), t > 0$, (10)

together with

$$u_{xx} - 2u^3 - \frac{\lambda}{3}(t+a)^{-2}u^{-3} = H_0(t+a)^k$$
, on $x = 0, t > 0$, (11)

$$S(0) = S_0. (12)$$

In the sequel, the moving boundary $x = S(t) = \gamma(t+a)^{1/3}$ is adopted so that the initial condition requires $S_0 = \gamma a^{1/3}$.

Boundary conditions

(I)

$$u_{xx}(S(t),t) - 2u^{3}(S(t),t) - (\lambda/3)(t+a)^{-2}u^{-3}(S(t),t) = L_{m}S^{i}(t)\dot{S}(t), \quad t > 0$$

Insertion of the symmetry ansatz (2) into the preceding yields

$$\Psi''(\gamma) - 2\Psi^{3}(\gamma) - (\lambda/3)\Psi^{-3}(\gamma) = L_{m}S^{i}(t)\dot{S}(t)(t+a) = \frac{1}{3}L_{m}\gamma^{i+1}(t+a)^{(i+1)/3}$$
 (13)

whence i = -1 together with

$$\Psi''(\gamma) - 2\Psi^{3}(\gamma) - (\lambda/3)\Psi^{-3}(\gamma) = \frac{1}{3}L_{m},$$
(14)

so that $L_m = \gamma \Psi(\gamma)$ by virtue of (5) with $\zeta = 0$.

$$u(S(t),t) = P_m S^j(t), \quad t > 0$$

Accordingly,

$$(t+a)^{-1/3}\Psi(\gamma) = P_m \gamma^j (t+a)^{j/3}, \quad t > 0$$
(15)

so that j = -1 and

$$P_m = \gamma \Psi(\gamma). \tag{16}$$

(III)

120

$$u_{xx}(0,t) - 2u^3(0,t) - (\lambda/3)(t+a)^{-2}u^{-3}(0,t) = H_0(t+a)^k, \quad t > 0.$$

This yields

$$\Psi''(0) - 2\Psi^{3}(0) - (\lambda/3)\Psi^{-3}(0) = H_0(t+a)^{k+1}$$
(17)

so that k = -1 and H_0 is determined by

$$\Psi''(0) - 2\Psi^{3}(0) - (\lambda/3)\Psi^{-3}(0) = H_{0}.$$
(18)

Accordingly $H_0 = 0$ in view of Ermakov-Painlevé II reduction (5) with $\zeta = 0$ and $\xi = 0$.

An Airy reduction

The Ermakov-Painlevé II equation (5) with the alternative scalings $\Psi = \delta w^*$, $\xi = \epsilon z$ wherein

$$\delta^2 = -\frac{c_3}{2\epsilon^2}, \quad \epsilon^3 = -3c_2, \quad \sigma = \frac{\lambda}{3}\epsilon^2\delta^{-4} \tag{19}$$

produces symmetry reduction

$$w_{zz}^* + c_3 w^{*3} + c_2 z w^* - \frac{\sigma}{w^{*3}} = \alpha^*$$
 (20)

with $\alpha^* = \varsigma \epsilon^2 / \delta \neq 0$ if $\zeta \neq 0$.

This avatar of the Ermakov-Painlevé II equation has been applied in a Korteweg capillarity system context in [45] with zero parameter α^* . It is remarked that a detailed analysis of aspects of the canonical Painlevé II equation with zero Painlevé parameter was conducted in [9].

On setting $w^* = \rho^{1/2}$ and with $\alpha^* = 0$ in (20), there results

$$\rho_{zz} = \frac{1}{2\rho} (\rho_z)^2 - 2c_3 \rho^2 - 2c_2 z\rho + \frac{2\sigma}{\rho}$$
(21)

which with the specifications [46]

$$c_2 = -\frac{1}{2}, \quad c_3 = -1, \quad \sigma = -\frac{1}{4} \left(\alpha + \frac{1}{2}\right)^2$$
 (22)

admits the solution

$$\rho(z) = w_z + w^2 + \frac{1}{2}z\tag{23}$$

whence

$$\Psi = \delta w^* = \delta [w_z + w^2 + \frac{1}{2}z]^{1/2}$$
(24)

with w(z) governed by the classical Painlevé II equation

$$w_{zz} = 2w^3 + zw + \alpha. \tag{25}$$

It is remarked that (21) is equivalent to the classical integrable Painlevé XXXIV equation in ρ on appropriate re-scaling.

The latter admits an important particular class of solutions when $\alpha = 1/2$, namely

$$w = -\frac{\Phi'(z)}{\Phi(z)} \tag{26}$$

with $\Phi(z)$ governed by de Airy equation

$$\Phi'' + \frac{1}{2}z\Phi = 0\tag{27}$$

whence (24) yields

$$\Psi = \delta \left[2 \left(\frac{\Phi'(z)}{\Phi(z)} \right)^2 + z/2 \right]^{1/2} \tag{28}$$

with

$$\Phi = aAi(-2^{-\frac{1}{3}}z) + bBi(-2^{-\frac{1}{3}}z), \quad a, b \in \mathbb{R}.$$
 (29)

It is remarked that, in [8] the seed Airy-type solution (26) and the subsequent class of solutions of (25) generated by the iterated action of an admitted Bäcklund transformation has been applied in the solution of certain boundary value problems for the classical Nernst-Planck electrolytic system. Here, use of the Airy-type representation of Ψ in the moving boundary problem determines the parameters L_m , P_m and H_0 in the boundary conditions.

An Extended Casimir Equation with Temporal Modulation. A Class of Reciprocal Moving Boundary Problems

Moving boundary problems of Stefan-type have recently been shown to admit exact solution via Painlevé II symmetry reduction for a range of canonical solitonic equations [41, 44, 51, 52, 53, 54]. The original investigation of such nonlinear nonlinear moving boundary problems was motivated by aspects of the classical Saffman -Taylor problem with surface tension [57] and a link to the solitonic Dym equation. Moving boundary

problems of a Stefan kind for an extended Dym equation which arises in both hydrodynamics and geometric contexts [59] have subsequently been shown to be solvable by Painlevé II reduction.

Here, a reciprocal transformation is introduced, namely

$$dx^* = udx + \left[-u_{xx} + 2u^3 + \frac{\lambda}{3}(t+a)^{-2}u^{-3}\right]dt, \quad t^* = t, \quad u^* = \frac{1}{u}, \quad \mathbb{R}^*$$
 (30)

which is applied to the extended mKdV equation (1) and to the class of moving boundary problems (8)-(12). Thus, under \mathbb{R}^* there results

$$dx = u^* dx^* + \left[\frac{\partial}{\partial x^*} \left(\frac{1}{u^*} \frac{\partial}{\partial x^*} \right) \left(\frac{1}{u^*} \right) - \frac{2}{u^{*2}} - \frac{\lambda}{3} u^{*4} (t^* + a)^{-2} \right] dt^*, \tag{31}$$

which compatibility condition

$$\frac{\partial u^*}{\partial t^*} = \frac{\partial}{\partial x^*} \left[\frac{\partial}{\partial x^*} \left(\frac{1}{u^*} \frac{\partial}{\partial x^*} \right) \left(\frac{1}{u^*} \right) - \frac{2}{u^{*2}} - \frac{\lambda}{3} u^{*4} (t^* + a)^{-2} \right],\tag{32}$$

By virtue of the reciprocal connection of the latter via \mathbb{R}^* to the extended mKdV equation (1), it inherits a variant of its Airy-type symmetry reduction. The reciprocal associate (32) of the extended mKdV equation (1) constitutes a novel extendion of the base Casimir member of the compacton hierarchy of [25]. In [52] the reciprocal version of the moving boundary problem (8) for the standard Casimir equation corresponding to $\lambda = 0$ has been solved via Painlevé II symmetry reduction. Here, the class of moving boundary problems determined by application of \mathbb{R}^* to (8)-(12) is determined by the reciprocal system

$$\frac{\partial u^*}{\partial t^*} = \frac{\partial}{\partial x^*} \left[\frac{\partial}{\partial x^*} \left(\frac{1}{u^*} \frac{\partial}{\partial x^*} \left(\frac{1}{u^*} \right) \right) - \frac{2}{u^{*2}} - \frac{\lambda}{3} u^{*4} (t^* + a)^{-2} \right], \quad x^* \big|_{x=0} < x^* < x^* \big|_{x=S(t)}, \quad t^* > 0,$$

$$\frac{\partial}{\partial x^*} \left(\frac{1}{u^*} \frac{\partial}{\partial x^*} \left(\frac{1}{u^*} \right) \right) - \frac{2}{u^{*2}} - \frac{\lambda}{3} u^{*4} (t^* + a)^{-2} = L_m S^i \dot{S}, \quad \text{on} \quad x^*|_{x = S(t)}, \quad t^* > 0, (33)$$

$$\frac{1}{u^*} = P_m S^j(t)$$
, on $x^*|_{x=S(t)}$, $t^* > 0$,

$$\frac{\partial}{\partial x^*} \left(\frac{1}{u^*} \frac{\partial}{\partial x^*} \left(\frac{1}{u^*} \right) \right) - \frac{2}{u^{*2}} - \frac{\lambda}{3} u^{*4} (t^* + a)^{-2} = H_0(t^* + a)^k, \quad \text{on} \quad x^*|_{x=0}, \quad t^* > 0$$

wherein $S(t) = \gamma (t+a)^{1/3}$. In the preceding

$$dx^*\big|_{x=0} = \left[-u_{xx} + 2u^3 + (\lambda/3)u^{-3}(t+a)^{-2} \right] dt\big|_{x=0}$$

$$= (t+a)^{-1} \left[-\Psi'' + 2\Psi^3 + (\lambda/3)\Psi^{-3} \right] dt\big|_{x=0} = 0$$
(34)

by virtue of (5) with $\zeta = 0$ corresponding to the Ermakov-Painlevé II reduction (20). Accordingly, $x^*|_{x=0}$ is constant. In addition

$$dx^*|_{x=S(t)} = dx^*|_{x=\gamma(t+a)^{1/3}} = [(t+a)^{-1/3}\Psi(\gamma)\dot{S}(t) - L_m S^i \dot{S}(t)]dt$$
$$= (t+a)^{-1}[\Psi(\gamma) - L_m \gamma^{-1}]\frac{\gamma}{3}dt$$
(35)

whence

$$x^*|_{x=S(t)} = \frac{\gamma}{3} [\Psi(\gamma) - L_m \gamma^{-1}] \ln|t^* + a| = S^*(t^*)$$
(36)

upto an additive constant and the reciprocal initial condition on the moving boundary becomes

$$S^*(0) = (\gamma/3)[\Psi(\gamma) - L_m \gamma^{-1}] \ln |a|. \tag{37}$$

4 Modulation

In [49], combined action of a reciprocal and integral-type transformation have been applied sequentially to solve a class of Stefan-type moving boundary problems involving spatial heterogeneity. The latter arise notably as a model of percolation of liquid through porous media in soil mechanics [32]. Physical systems which incorporate spatial or temporal modulation arise naturally in both physics and continuum mechanics (qv [5, 6, 42, 71] and [7, 12, 19, 50] respectively together with literature cited therein). Thus, in physics such modulated systems have importance notably in the theory of Bose-Einstein condensates and Bloch wave propagation. In continuum mechanics, modulated systems arise 'inter alia' in elastodynamics [19], visco-elastodynamics [7] and the analysis of crack problems in elastostatics [12, 50].

In recent work [50], spatially modulated coupled systems of sine-Gordon, Demoulin and Manakov-type have been systematically reduced to their unmodulated canonical counterparts via classes of involutory transformations. The temporal analogue of the latter to be applied here had their origin in a procedure introduced in [4], in connection with the Ermakov-Ray-Reid coupled systems [39]. The latter has extensive physical applications [48]. The transformations are of the type

$$dt^* = \rho^{-2}(t)dt, \qquad u^* = \rho^{-1}(t)u, \qquad T^*$$
 (38)

and augmented by the relation $\rho^* = \rho^{-1}$ admit the key involutory property $T^{**} = I$. Application of T^* to (1) results in a wide novel class of extended mKdV equations with temporal modulation, namely

$$\frac{\partial}{\partial t^*} \left(\frac{u^*}{\rho^*} \right) - 6\rho^{*-5} u^{*2} u_{x*}^* + \rho^{*-3} u_{x*x*x*}^* + \lambda (t^* + a)^{\mu} \rho^* u^{*-4} u_{x*}^* = 0$$
 (39)

with

$$dt = \rho^{*-2}dt^*. (40)$$

Application of T^* to the moving boundary problems (8)-(12) produces associated Stefantype problems for (39) which inherit the key property of exact solution via Ermakov-Painlevé II symmetry reduction.

In [50] modulated versions of established solitonic systems were derived via the spatial analogue of the involutory transformation (38). Therein modulation with ρ determined by hybrid Ermakov-Painlevé II, Ermakov-Painlevé III or Ermakov-Painlevé IV as set down in

[44] were applied. Here, the modulation term $\rho(t)$ in the class of involutary transformations T^* is taken to be determined by the classical Ermakov equation

$$\rho_{tt} + w(t)\rho = k/\rho^3 \tag{41}$$

which admits the nonlinear superposition principle

$$\rho = \sqrt{c_1 \Omega_1^2 + 2c_2 \Omega_1 \Omega_2 + c_3 \Omega_2^2} \tag{42}$$

wherein Ω_1 , Ω_2 constitute a pair of linearly independent solutions of the auxiliary linear equation

$$\Omega_{tt} + w(t)\Omega = 0 \tag{43}$$

with constants k together with c_i , i = 1, 2, 3 such that

$$c_1 c_3 - c_2^2 = \frac{k}{W^2} \tag{44}$$

where $W = \Omega_1 \Omega_{2t} - \Omega_{1t} \Omega_2$ is the constant Wronskian of Ω_1, Ω_2 . The nonlinear superposition principle (42) can be derived via Lie group invariance as in [33]. Therein, application was made to the analysis of moving boundary shoreline evolution hydrodynamics with an underlying rigid basin. In general terms, the classical Ermakov equation (41) has diverse physical applications, notably, 'inter alia', in the nonlinear elastodynamics of boundary-loaded hyperelastic tubes [58, 34], oceanographic pulsrodon eddy evolution [35], magnetogasdynamics [37] and the analysis of rotating gas cloud phenomena [38].

5 An Extended Solitonic Gardner Equation with Temporal Modulation

A hybrid mKdV and KdV equation, namely the Gardner equation was originally introduced in a now classical paper [23]. It has subsequently had diverse physical applications, notably in plasma physics [56], optical lattice theory [67] and most recently elastodynamics [13].

In [62], Slyunyaav and Palinovsky established a novel link between the mKdV equation and the canonical Gardner equation

$$v_{\tau} + 6v(1 - v)v_y + v_{yyy} = 0. (45)$$

Thus, on introduction of

$$u = v - \frac{1}{2} \tag{46}$$

together with

$$x = -(3/2)\tau + y, \qquad t = \tau \tag{47}$$

the mKdV equation

$$u_t - 6u^2 u_x + u_{xxx} = 0 (48)$$

results.

Moving boundary problems of Stefan-type for both the solitonic Gardner equation and a reciprocally associated 3rd order nonlinear evolution incorporating a source term have recently been shown to be amenable to exact solution via a Painlevé II symmetry reduction [54]. In the present context, the result of [62] may be used to establish that the extended Gardner equation

$$v_{\tau} + 6v(1 - v)v_y + v_{yyy} + \lambda(\tau + a)^{-2}(v - \frac{1}{2})^{-4}v_y = 0$$
(49)

together with the temporally-modulated class in which it is embedded via application of the involutory transformations T^* admit Ermakov-Painlevé II symmetry reduction. Application of the latter can be made to construct exact solution of associated nonlinear moving boundary problems with temporal modulation.

6 Conclusion

Investigation of exact solution of Stefan- type moving boundary problems for a range of canonical 1+1- dimensional solitonic equations via Painlevé II symmetry reduction as initiated in [41] has been detailed in [45], [51]-[54]. In the present work, a class of extended mKdV equations has been introduced which admit solution via hybrid Ermakov-Painlevé II symmetry reduction. Currently under investigation is exact solution of novel moving boundary problems by means of the latter procedure for certain 2+1- dimensional solitonic equations and their reciprocal associates.

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References

- [1] Ablowitz M J, Kaup D J, Newell A C and Segur H, The inverse scattering transform-Fourier analysis for nonlinear problems, *Stud. Appl. Math.*, 1974, V.53, 249–315.
- [2] Alexiades V, Solomon A D, Mathematical Modeling of Melting and Freezing Processes, Hemisphere Publishing Corp., Washington (1993)
- [3] Amster P and Rogers C, On a Ermakov-Painlevé II reduction in three-ion electrod-iffusion. A Dirichlet boundary value problem, *Discrete and Continuous Dynamical Systems*, 2015, V. 35, 3277–3292.
- [4] Athorne C, Rogers C, Ramgulam U and Osbaldestin A, On linearization of the Ermakov system, *Phys. Lett*, 1990, A 143, 207–212.
- [5] Belmonte-Beitia J, Perez-Garcia V M, Vekslerchik V and Torres PL, Lie symmetries and solitons in nonlinear systems with spatially inhomogeneous nonlinearities, *Phys. Rev. Lett.* 2007 V.98, 064102.

- [6] Belmonte-Beitia J, Perez-Garcia V M, Vekslerchik V and Konotop V V,Localized nonlinear waves in systems with time- and space-modulated nonlinearities, *Phys. Rev. Lett.* 2008 V.100, 164102-1-064102-4.
- [7] Barclay D W, Moodie T B and Rogers C, Cylindrical impact waves in inhomogeneous viscoelastic media, *Acta Mechanica*, 1978, V.29, 93–11.
- [8] Bass L, Nimmo J J C, Rogers C and Schief W K, Electrical structures of interfaces: A Painleve II Model, *Proc. Roy. Soc. London* 2010 A 466, 2117–2135.
- [9] Bassom A P, Clarkson P A, Law C K and McLeod J B, Application of Uniform Asymptotics to the Second Painlevé Transcendent, Arch Rational Mech Anal 1998 V. 143, 241–271.
- [10] Briozzo A C, Rogers C and Tarzia D A. A class of moving boundary problems with a source term: Application of a reciprocal transformation, *Acta Mechanica*, 2023, V.234, 1889—1900.
- [11] Camassa R and Holm D. An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* 1993, V. 71, 1661–1664.
- [12] Clements D L, Atkinson C and Rogers C, Antiplane crack problems for an inhomogeneous elastic material, *Acta Mech.*, 1978V.29,199—211
- [13] Coclite G M, Maddalena F, Puglisi G, Romano M and Saccomandi G, The Gardner equation in elastodynamics, SIAM Journal on Applied Mathematics, 2021, V.81, 2346–2361.
- [14] Crank J, Free and Moving Boundary Problems, Clarendon Press, Oxford, 1987.
- [15] Elliot C M and Ockendon J R, Weak and Variational Methods for Moving Boundary Problems, Wiley, New York, 1982.
- [16] Fokas A S and Tanveer S, A Hele-Shaw problem and the second Painlevé transcendent, Math. Proc. Camb. Phi. Soc., 1998, V. 124, 169–191.
- [17] Friedman A, Variational Principles and Free-Boundary Problems, Wiley, New York 1982.
- [18] Kakutani T and Ono H, Weak non-linear hydromagnetic waves in a cold collision-free plasma, J. Phys. Soc. Japan, 1969, V.26, 1305–1318.
- [19] Karal FC and Keller J B, Elastic wave propagation in homogeneous and inhomogeneous media, J. Acoust. Soc. Am., 1959, V.31, 694–705.
- [20] Keller J B, Melting and freezing at constant speed, Phys. Fluids 1986 V. 92, 2013.
- [21] Kingston J G and Rogers C, Reciprocal Bäcklund transformations of conservation laws, *Phys. Lett.*, 1982, V. 92A, 261–264.
- [22] Miura R M, Korteweg-de Vries equation and generalizations I. A remarkable explicit transformation J. Math. Phys., 1968, V.9, 1202-1204.

- [23] Miura R M, Gardner C S and Kruskal M D, Korteweg-de Vries equation and generalizations II. Existence of conservation laws and constants of motion, J. Math. Phys., 1968, V.9, 1204-1209.
- [24] Oevel W and Rogers C, Gauge transformations and reciprocal links in 2+1-Dimensions. Rev. Math. Phys., 1993, V.5, 299–330.
- [25] Olver P J and Rosenau P, Tri-Hamiltonian duality between solitons and solitary wave solutions having compact support, *Phys. Rev. E*, 1996, V.53, 1900–1906.
- [26] Rogers C and Shadwick W F, Bäcklund Transformations and Their Applications, Academic Press, Mathematics in Science and Engineering Series, Ed. Richard Bellman, New York, 1982.
- [27] Rogers C and Wong P, On reciprocal Bäcklund transformations of inverse scattering schemes, Physica Scripta, 1984 V.30, 10–14.
- [28] Rogers C, Application of a reciprocal transformation to a two phase Stefan Problem, J. Phys. A: Mathematical and General, 1985, V.18, L105–L109.
- [29] Rogers C and Nucci M C, On Reciprocal Bäcklund transformations and the Korteweg-de Vries hierarchy, *Physica Scripta*, 1986, V.33, 289–292.
- [30] Rogers C, Reciprocal transformations in 2+1-dimensions, J. Phys. A: Math. Gen., 1986, V.49, L491–L496.
- [31] Rogers C and Yu Guo B, A note on the onset of melting in a class of simple metals. Condition on the applied boundary flux, *Acta Mathematica Scientia*, 1988, V.8, 425–430.
- [32] Rogers C and Broadbridge P, On a nonlinear moving boundary problem with heterogeneity: application of a reciprocal transformation, *Z. angew. Math. Phys.* 1988 V.39, 122–128.
- [33] Rogers C and Ramgulam U, A nonlinear superposition principle and Lie group invariance. Application in rotating shallow wales theory, Int. J. Nonlinear Mech., 1989, V.24, 229–235.
- [34] Rogers C and Ames W F, Nonlinear Boundary Value Problems in Science and Engineering, Academic Press, New York, 1989.
- [35] Rogers C, Elliptic warm core theory: the pulsrodon, *Phys. Lett.*, 1989, A 138, 267–273.
- [36] Rogers C and Schief W K, Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory, Cambridge Texts in Applied Mathematics, Cambridge University Press, 2002.
- [37] Rogers C and Schief W K, The pulsrodon in 2+1-dimensional magnetogasdynamics. Hamiltonian-Ermakov reduction, *J. Math. Phys.*, 2011, V.52, 08370.

- [38] Rogers C and Schief W K, On the integrability of a Hamiltonian reduction of a 2+1-dimensional non-isothermal rotating gas cloud system, *Nonlinearity*, 2011, V. 24, 3165–3178.
- [39] Rogers C, Malomed B and An H, Ermakov-Ray-Reid reductions of variational aproximations in nonliner optics, Stud. Appl. Math, 2012, V. 129, 389–413.
- [40] Rogers C, A novel Ermakov-Painlevé II System. N+1-dimensional coupled NLS and elastodynamic reductions, *Stud. Appl. Math.*, 2014, V. 133, 214–231.
- [41] Rogers C, Moving Boundary Problems for the Harry Dym equation and its reciprocal associates, *Zeit angew Math. Phys.*, 2015, V.66, 3205–3220.
- [42] Rogers C, Saccomandi G and Vergori L, Ermakov- modulated nonlinear Schrödinger models: Integrable reduction, J. Nonlinear Mathematical Physics, 2016, V.23, 108– 126.
- [43] Rogers C and Schief W K, On Ermakov-Painlevé II systems. Integrable reduction, Meccanica, 2016, V.51, 2967–2974.
- [44] Rogers C, On hybrid Ermakov-Painlevé systems, Integrable reduction, J. Nonlinear Math. Phys., 2017, V.24, 239–249.
- [45] Rogers C, Moving boundary problems for an extended Dym equation. Reciprocal connections, *Meccanica*, 2017, V.52, 3531–3540.
- [46] Rogers C and Clarkson P A, Ermakov-Painlevé II symmetry reduction of a Korteweg capillarity system, Symmetry, Integrability and Geometry: Methods and Applications, 2017, 13, 018.
- [47] Rogers C and Clarkson P A, Ermakov-Painlevé II reduction in cold plasma physics. Application of a Bäcklund transformation, J. Nonlinear Mathematical Physics, 2018, V.25, 247–261.
- [48] Rogers C and Schief W K, Ermakov-type systems in nonlinear physics and continuum mechanics, *Nonlinear Systems and Their Remarkable Mathematical Structures*, Ed. Norbert Euler, CRC Press, 2018, 541–576.
- [49] Rogers C, Moving boundary problems for heterogeneous media. Integrability via conjugation of reciprocal and integral transformations, J Nonlinear Math Phys, 2019, 26, 313–325.
- [50] Rogers C, Schief W K and Malomed B, On modulated coupled systems. Canonical reduction via reciprocal transformations, *Comm. Nonlinear Science and Numerical Simulation*, 2020, V.33, 10509.
- [51] Rogers C, Moving boundary problems for a canonical member of the WKI inverse scattering scheme: conjugation of a reciprocal and Möbius transformation, *Physica Scripta*, 2022, V.97, 095207.

- [52] Rogers C, On mKdV and associated classes of moving boundary problems: reciprocal connections, *Meccanica* 2023, V.58, 1633–1640.
- [53] Rogers C, On Korteweg-de Vries and associated reciprocal moving boundary problems, Z. angew. Math. Phys., 2025, V.76, 33.
- [54] Rogers C, On moving boundary problems for the solitonic Gardner equation. a reciprocally associated class, Z. angew. Math. Phys., 2025, 76:186.
- [55] Rubinstein L I, The Stefan Problem, Amer. Math. Soc. 1971 Vol. 27.
- [56] Ruderman M, Talipova T and Pelinovsky E, Dynamics of modulationally unstable ion-acoustic wavepackets in plasmas with negative ions, J. Plasma Phys., 2008, V.74, 639–656.
- [57] Saffman P G and Taylor G I, The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid, *Proceedings of the Royal Society of London*, 1958 A 245, 312–329.
- [58] Shahinpoor M and Nowinski J L, Exact solutions to the problem of forced large amplitude radial oscillations in a thin hyperelastic tube, *Int. J. Nonlinear Mech.* 1971 V.6, 193-207.
- [59] Schief W K and Rogers C, Binormal motion of curves of constant curvature and torsion. Generation of soliton surfaces, *Proc. Roy. Soc. London*,1999 455 A, 3163–3188.
- [60] Solomon A D, Wilson D G and Alexiades V, Explicit solutions to phase change problems, 1983 Quart. App. Math., 1983, V. 41, 237–243.
- [61] Storm M L, Heat conduction in simple metals, J. Appl. Phys., 1951 V.22, 940–951.
- [62] Slyunyaev A V and Pelinovski E N, Dynamics of large-amplitude solitons, *J. Experimental Theoretical Physics*, 1999, 89, 173–181.
- [63] Tarzia D A, An inequality for the coefficient σ of the free boundary $s(t) = 2\sigma\sqrt{t}$ of the Neumann solution for the two-phase Stefan problem", Quart. App. Math., 1981, 39, 491–497.
- [64] Tarzia D A, A bibliography on moving-free boundary problems for the heat diffusion equation. The Stefan and related problems, MAT Serie A, Rosario, 2000, 2, 1-297.
- [65] Vassiliou P J, Harry Dym Equation in Encyclopaedia of Mathematics, Springer, 2001.
- [66] Vorob'ov A P, On the rational solutions of the second Painlevé equation, *Differ. Uravn.*, 1965, 79–81.
- [67] Wadati M, Wave Propagation in nonlinear lattice. II. Journal of the Physical Society of Japan, 1975, V.38, 681–686.
- [68] Wadati M, Konno K and Ichikawa V H, New integrable nonlinear evolution equations, J. Phys. Soc. Japan, 1979, V.47, 1698–1700.

- [69] Yablonski A I, On rational solutions of the second Painlevé equations, Vesti Akad. Nauk BSSR, Ser. Fiz.-Tkh. Nauk, 1959, vol. 3, pp. 30-35.
- [70] Zabusky N J, A synergetic approach to problems of nonlinear dispersive wave propagation and interaction, in *Nonlinear Partial Differential Equations*. Ed. W. F. Ames, Academic Press, New York, 223–258.
- [71] Zhang F, Li Y S, Meng J, Wu L and Malomed B A, Matter-wave solitons and finite-amplitude Bloch waves in optical lattices with a spatially modulated nonlinearity, *Phys. Rev. A*, 2010, V.82, 033614.