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Finsler Geometry in Anisotropic Superconductivity: A Ginzburg–Landau Approach

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Abstract

We present a rigorous generalization of the classical Ginzburg–Landau model to smooth, compact Finsler manifolds without boundary. This framework provides a natural analytic setting for describing anisotropic superconductivity within Finsler geometry. The model is constructed via the Finsler–Laplacian, defined through the Legendre transform associated with the fundamental function F, and by employing canonical Finsler measures such as the Busemann–Hausdorff and Holmes–Thompson volume forms. We introduce an anisotropic Ginzburg–Landau functional for complex scalar fields coupled to gauge potentials and establish the existence of minimizers in the appropriate Finsler–Sobolev spaces by the direct method in the calculus of variations. Furthermore, we analyze the asymptotic regime as the Ginzburg–Landau parameter $\varepsilon \to 0$ and prove a precise Γ–convergence result: the rescaled energies converge to the Finslerian length functional associated with the limiting vortex filaments. In particular, the limiting vortex energy is shown to equal π times the Finslerian length of the corresponding current, thereby extending the classical Bethuel–Brezis–Hélein result to anisotropic settings. These findings demonstrate that Finsler geometry unifies metric anisotropy and variational principles in gauge-field models, broadening the geometric scope of the Ginzburg–Landau theory beyond the Riemannian framework.

1 Introduction

The Ginzburg–Landau (GL) theory, introduced in the seminal work of Ginzburg and Landau [1], has played a central role in mathematical physics and geometric analysis. In its classical form on a Euclidean or Riemannian manifold (M, g), the model couples a complex order parameter to a gauge potential through a variational energy, (see, e.g., the monographs [2, 3]) for a comprehensive mathematical treatment including vortex structures, energy asymptotics, and compactness.

Recent developments in differential geometry have highlighted *Finsler geometry* as a natural non quadratic extension of Riemannian structures, where the metric dependence on directions is encoded by a strongly convex norm $F(x, \cdot)$ on each tangent space T_xM (see [6, 7]). Analytic tools suitable for this setting such as the Finslerian gradient, divergence, Laplacian, and Sobolev spaces have been developed in, for instance, [8, 9]. From the viewpoint of applications, anisotropy is intrinsic in layered superconductors and related media, suggesting that a GL-type theory on Finsler manifolds is a natural framework for modeling direction-dependent phenomena.

Contributions. In this paper we formulate and analyze a Finslerian version of the GL model. Our main contributions are as follows:

(i) We define an anisotropic GL functional on a smooth Finsler manifold (M, F) using the Finsler–Laplacian (via the Legendre transform) together with a canonical Finsler measure (Busemann–Hausdorff or Holmes–Thompson). The model couples complex scalar fields to $U(1)$-gauge potentials and is invariant under the natural gauge action.

(ii) We establish the *existence of minimizers* in appropriate Finsler–Sobolev spaces. The proof follows the direct method of the calculus of variations, relying on coercivity, weak lower semicontinuity induced by the convexity of $F^*(x, \cdot)^2$, compact embeddings on compact manifolds, and a Coulomb gauge fixing based on a background Riemannian co-metric uniformly equivalent to F^*.

(iii) We investigate the asymptotic regime $\varepsilon \to 0$ and prove a Γ–convergence result, after the usual $|\log \varepsilon|$ rescaling, the energies converge to the Finslerian length of rectifiable 1-currents representing vortex filaments. The analysis adapts the ball construction and lower bound techniques of Jerrard–Sandier [**?**] to the anisotropic setting, together with compactness/rectifiability tools from geometric measure theory (see [5]) and the classical GL scheme in [2, 3].

Standing assumptions and notation. Throughout the paper, (M, F) denotes a compact smooth Finsler manifold (without boundary, unless stated otherwise). We write $F^*(x, \cdot)$ for the co-metric on T_x^*M induced by the Legendre transform, and $d\mu_F$ for a fixed smooth Finsler measure; when not specified we adopt the Busemann–Hausdorff measure. A smooth Riemannian co–metric γ^*, uniformly equivalent to F^*, is used to formulate Hodge operators for the Maxwell term. All function spaces are the Finsler–Sobolev spaces H_F^1 built upon F^* and $d\mu_F$, and the gauge is fixed to Coulomb form when needed.

Organization of the paper. Section 2 collects the necessary background on Finsler analysis. In Section 3 we introduce the Finslerian GL functional and its basic properties, including gauge invariance and well-posedness on H^1_F. The existence of minimizers is proved in Section 4. Section 5 is devoted to the asymptotic analysis as $\varepsilon \to 0$, culminating in the Γ–limit characterization of vortex filaments via the Finsler length functional.

2 Preliminaries on Finsler Geometry

In this section we collect the analytic and geometric tools used later. Throughout, M denotes a smooth, connected, compact n–manifold without boundary. All statements below extend to manifolds with smooth boundary under standard trace assumptions; see the remarks at the end of the section.

A smooth, strongly convex *Finsler structure* on M is a continuous function $F : TM \to [0, \infty)$ such that:

1. F is C^∞ on $TM \setminus \{0\}$,
2. $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$,
3. for each $(x, y) \in TM \setminus \{0\}$, the *fundamental tensor*

$$g_{ij}(x, y) \ := \ \frac{1}{2} \frac{\partial^2\big(F(x, y)^2\big)}{\partial y^i \, \partial y^j}$$

is positive definite.

We write $F(x, \cdot)$ for the Minkowski norm on T_xM. The *dual norm* $F^* : T^*M \to [0, \infty)$ is defined for $\xi \in T^*_xM$ by

$$F^*(x, \xi) \ := \ \sup\{\, \xi(v) \ : \ v \in T_xM, \ F(x, v) \le 1 \,\}.$$

Define the *Legendre map* $L : TM \setminus \{0\} \to T^*M \setminus \{0\}$ fiberwise by

$$L_x(y) \ := \ \partial_y\Big(\frac{1}{2}F(x, y)^2\Big) \ = \ g_{ij}(x, y)\, y^j \, dx^i. \tag{1}$$

Proposition 1. For each $x \in M$, $L_x : T_xM \setminus \{0\} \to T^*_xM \setminus \{0\}$ is a C^∞ diffeomorphism. Its inverse is given by $L_x^{-1}(\xi) = \partial_\xi\big(\frac{1}{2}F^*(x, \xi)^2\big)$. Moreover, for all $y \neq 0$ and $\xi \neq 0$,

$$F^*\big(x, L_x(y)\big) \ = \ F(x, y), \qquad F\big(x, L_x^{-1}(\xi)\big) \ = \ F^*(x, \xi), \tag{2}$$

and the Fenchel–Young relation holds:

$$\frac{1}{2}F(x, y)^2 + \frac{1}{2}F^*(x, \xi)^2 \ \ge \ \xi(y). \tag{3}$$

Equality hold if and only if $\xi = L_x(y)$, equivalently, $y = L_x^{-1}(\xi)$.

Proof. Fix x and set $\Phi(y) := \frac{1}{2}F(x,y)^2$. By strong convexity, the Hessian $\partial_y^2\Phi(y) = g(x,y)$ is positive definite for $y \neq 0$. Hence $\nabla_y\Phi = L_x$ has everywhere invertible differential on $T_xM \setminus \{0\}$, by the inverse function theorem L_x is a local diffeomorphism. Since Φ is strictly convex and superlinear, L_x is injective and proper; therefore it is a global diffeomorphism onto its image, which is $T_x^*M \setminus \{0\}$. Define $\Psi(\xi) := \sup_{y\neq 0}\{\xi(y) - \Phi(y)\}$, the Legendre transform of Φ, standard convex duality gives $\Psi(\xi) = \frac{1}{2}F^*(x,\xi)^2$ and $\nabla_\xi\Psi = L_x^{-1}$. The identities (2) and (3) are the usual equality cases in Fenchel duality, using strict convexity and 1–homogeneity of F and F^*. ■

Definition. For $u \in C^\infty(M)$, the *Finsler gradient* $\nabla_F u(x) \in T_xM$ is defined by

$$du \;=\; L_x\big(\nabla_F u(x)\big). \tag{4}$$

Equivalently, $\nabla_F u(x) = \partial_\xi\big(\frac{1}{2}F^*(x,du_x)^2\big)$. By (2) we have $F\big(x,\nabla_F u(x)\big) = F^*(x,du_x)$.

Two canonical smooth measures on (M,F) will be used:

Busemann–Hausdorff. For $x \in M$ let $B_F(x) := \{y \in T_xM : \;\; F(x,y) < 1\}$ and $B^n \subset \mathbb{R}^n$ the Euclidean unit ball. The Busemann–Hausdorff measure is

$$d\mu_{BH}(x) \;:=\; \frac{\mathrm{vol}(B^n)}{\mathrm{vol}\big(B_F(x)\big)}\, dx^1\wedge\cdots\wedge dx^n. \tag{5}$$

Holmes–Thompson. Let $B_F^*(x) := \{\xi \in T_x^*M : \;\; F^*(x,\xi) < 1\}$. The Holmes–Thompson measure is

$$d\mu_{HT}(x) \;:=\; \frac{1}{\mathrm{vol}(B^n)}\int_{B_F^*(x)} d\xi^1\cdots d\xi^n\, dx^1\wedge\cdots\wedge dx^n. \tag{6}$$

In what follows we fix once and for all a smooth Finsler measure $d\mu_F$ chosen among $\{\mu_{BH},\mu_{HT}\}$ and write

$$d\mu_F \;=\; \sigma(x)\, dx^1\wedge\cdots\wedge dx^n, \qquad \sigma\in C^\infty(M),\; \sigma>0. \tag{7}$$

Definition. For a C^1 vector field $X = X^i\partial_i$ define

$$\mathrm{div}_{\mu_F} X \;:=\; \frac{1}{\sigma(x)}\,\partial_i\big(\sigma(x)\,X^i\big). \tag{8}$$

Lemma 1. For $u \in C^\infty(M)$ and $X \in C^1(TM)$,

$$\int_M du(X)\, d\mu_F \;=\; -\int_M u\;\mathrm{div}_{\mu_F} X\, d\mu_F. \tag{9}$$

Definition. For $u \in C^\infty(M)$, the *Finsler Laplacian* associated with $d\mu_F$ is

$$\Delta^F_{\mu_F} u \;:=\; \mathrm{div}_{\mu_F}\big(\nabla_F u\big). \tag{10}$$

In local coordinates, combining (8) and (4) yields

$$\Delta^F_{\mu_F} u \;=\; \frac{1}{\sigma(x)}\,\partial_i\Big(\sigma(x)\,g^{ij}\big(x,\nabla_F u(x)\big)\,\partial_j u\Big), \tag{11}$$

where $g^{ij}(x,\cdot)$ denotes the inverse matrix of $g_{ij}(x,\cdot)$ evaluated at $y = \nabla_F u(x)$, i.e. via the Legendre correspondence $du = L_x(y)$.

Definition. The Finsler Dirichlet energy of $u \in C^\infty(M)$ is

$$\mathcal{E}_F[u] \;:=\; \frac{1}{2}\int_M F^*\big(x,du\big)^2\,d\mu_F. \tag{12}$$

Theorem 1. For $u \in C^\infty(M)$ and $\varphi \in C^\infty(M)$,

$$\frac{d}{dt}\,\mathcal{E}_F[u+t\varphi]\Big|_{t=0} \;=\; -\int_M \varphi\,\Delta^F_{\mu_F} u\,d\mu_F.$$

Equivalently, the critical points of $\mathcal{E}_F$ are the (weak) solutions of $\Delta^F_{\mu_F} u = 0$.

Proof. Set $\xi_t := d(u+t\varphi) = du + t\,d\varphi$. Using Proposition 1,

$$\frac{d}{dt}\,\frac{1}{2}F^*(x,\xi_t)^2\Big|_{t=0} \;=\; \Big\langle \partial_\xi\Big(\tfrac{1}{2}F^*(x,\xi)^2\Big)\Big|_{\xi=du},\,d\varphi\Big\rangle \;=\; \langle \nabla_F u,\; d\varphi\rangle,$$

where the last pairing is the natural one $T_xM \times T^*_xM \to \mathbb{R}$. Hence

$$\frac{d}{dt}\,\mathcal{E}_F[u+t\varphi]\Big|_{t=0} \;=\; \int_M \langle \nabla_F u,\; d\varphi\rangle\,d\mu_F \;=\; \int_M d\varphi(\nabla_F u)\,d\mu_F.$$

Now apply Lemma 1 with $X = \nabla_F u$ to obtain

$$\int_M d\varphi(\nabla_F u)\,d\mu_F \;=\; -\int_M \varphi\,\mathrm{div}_{\mu_F}(\nabla_F u)\,d\mu_F \;=\; -\int_M \varphi\,\Delta^F_{\mu_F} u\,d\mu_F.$$

■

Fix any smooth Riemannian metric γ on M and denote by $|\cdot|_{\gamma^*}$ the norm on T^*M induced by its co-metric γ^*. On compact M, the norms $F^*(x,\cdot)$ and $|\cdot|_{\gamma^*}$ are uniformly equivalent:

Lemma 2. There exist constants $0 < c_1 \le c_2 < \infty$ such that for all $(x,\xi) \in T^*M$,

$$c_1\,|\xi|_{\gamma^*} \;\le\; F^*(x,\xi) \;\le\; c_2\,|\xi|_{\gamma^*}. \tag{13}$$

Proof. Let $S := \{(x,\xi) \in T^*M : \; |\xi|_{\gamma^*} = 1\}$, which is compact. The map $(x,\xi) \mapsto F^*(x,\xi)$ is continuous and positive on S. Set $c_1 := \min_S F^*$ and $c_2 := \max_S F^*$. Then $0 < c_1 \le c_2 < \infty$, and by homogeneity of F^* the inequality (13) follows for arbitrary ξ. ■

Definition. For $1 \le p < \infty$, define $W^{1,p}_F(M)$ as the completion of $C^\infty(M)$ with respect to

$$\|u\|^p_{W^{1,p}_F} \;:=\; \int_M |u|^p\,d\mu_F \;+\; \int_M F^*(x,du)^p\,d\mu_F.$$

We write $H^1_F(M) := W^{1,2}_F(M)$. For complex–valued maps, set

$$\|u\|^2_{F^*} := F^*(x,\mathrm{Re}\,du)^2 + F^*(x,\mathrm{Im}\,du)^2,$$

and define $H^1_F(M;C)$ analogously. For 1–forms, $H^1_F(M;T^*M)$ is defined using any fixed co-metric γ^* and the equivalent norm

$$\|\alpha\|^2_{H^1} = \int_M \big(|\alpha|^2_{\gamma^*} + |\nabla^\gamma\alpha|^2_{\gamma^*}\big)\,d\mu_F,$$

which is equivalent to any other choice by Lemma 2.

Proposition 2. The space $H^1_F(M)$ is a Hilbert space. Moreover, there exists $C > 0$ such that for all $u \in H^1_F(M)$ with mean zero $\int_M u\, d\mu_F = 0$,

$$\|u\|_{L^2(M,d\mu_F)} \;\le\; C\,\|F^*(x,du)\|_{L^2(M,d\mu_F)}. \tag{14}$$

Finally, the embedding $H^1_F(M) \hookrightarrow L^2(M, d\mu_F)$ is compact.

Proof. By Lemma 2 and (7), the H^1_F–norm is equivalent to the standard $H^1(M,\gamma)$ norm with respect to the smooth positive density $\sigma\, dx$:

$$\|u\|^2_{H^1_F} \;\simeq\; \int_M \big(|u|^2 + |\nabla^\gamma u|^2_\gamma\big)\,\sigma\, dx.$$

Therefore $H^1_F(M)$ is isomorphic as a Hilbert space to $H^1(M,\gamma)$. The Poincaré inequality (14) follows from the usual Poincaré inequality for (M,γ) (since M is compact) and the norm equivalence, similarly for Rellich's compact embedding. ■

For 1–forms we will use a fixed smooth Riemannian co-metric γ^* uniformly equivalent to F^* (Lemma 2) to formulate Hodge operators for the Maxwell term. Write $\sharp_\gamma : T^*M \to TM$ for the γ–musical isomorphism and define

$$d^\dagger_{\gamma,\mu_F}\eta \;:=\; -\operatorname{div}_{\mu_F}(\eta^{\sharp_\gamma}) \qquad (\eta \in \Omega^1(M)).$$

Then for all $\phi \in C^\infty(M)$,

$$\int_M \langle d\phi, \eta\rangle_{\gamma^*}\, d\mu_F \;=\; \int_M \phi\, d^\dagger_{\gamma,\mu_F}\eta\, d\mu_F, \tag{15}$$

which follows by Lemma 1 and the definition of $\operatorname{div}_{\mu_F}$. In particular, $d^\dagger_{\gamma,\mu_F}$ is the $L^2(d\mu_F)$–adjoint of d acting on 0–forms.

If $\partial M \neq \emptyset$, (9) gains a boundary term $\int_{\partial M} u\, \iota_\nu X\, d\sigma_F$, where ν is the outward conormal and $d\sigma_F$ the induced Finsler boundary measure; Dirichlet ($u = 0$) or Neumann ($\iota_\nu X = 0$) conditions recover (9). On noncompact manifolds, the results above hold under uniform bounds ensuring (13) and a global Poincaré inequality (e.g. positive injectivity radius and bounded geometry with respect to some γ).

3 The Finslerian Ginzburg–Landau Functional

Let (M,F) be a compact smooth Finsler manifold endowed with a fixed smooth Finsler measure $d\mu_F$ (either Busemann–Hausdorff or Holmes–Thompson) and a smooth Riemannian co–metric γ^* uniformly equivalent to F^* (cf. Lemma 2). For a complex scalar field $\psi : M \to \mathbb{C}$ and a real 1-form $A \in \Omega^1(M)$ we set

$$D_A\psi := (d - iA)\psi \in \Omega^1(M;\mathbb{C}),$$

and extend the Finsler co-norm to complex-valued 1-forms by

$$\|\eta\|^2_{F^*} := F^*(x,\Re\eta)^2 + F^*(x,\Im\eta)^2, \qquad \eta \in \Omega^1(M;\mathbb{C}).$$

Definition. The *Finslerian Ginzburg–Landau functional* is

$$G_F[\psi, A] := \int_M \Big(\tfrac{1}{2}\|D_A\psi\|_{F^*}^2 + \tfrac{1}{2\lambda}\,\|dA\|_{\gamma^*}^2 + \tfrac{1}{4\varepsilon^2}\,(1-|\psi|^2)^2 \Big)\, d\mu_F, \tag{16}$$

for fixed parameters $\lambda > 0$ and $\varepsilon > 0$.

Proposition 3. For every $\chi \in H^1(M;\mathbb{C})$,

$$(\psi, A) \mapsto (e^{i\chi}\psi,\ A + d\chi)$$

leaves G_F invariant. In particular, G_F depends only on the gauge class of (ψ, A).

Proof. Observe that $D_{A+d\chi}(e^{i\chi}\psi) = e^{i\chi}D_A\psi$. Since the norm $\|\cdot\|_{F^*}$ is rotation-invariant in the $(\Re, \Im)$-plane by definition, it follows that

$$\|D_{A+d\chi}(e^{i\chi}\psi)\|_{F^*} = \|D_A\psi\|_{F^*}.$$

Moreover, the Maxwell term depends only on dA, so $\|d(A + d\chi)\|_{\gamma^*} = \|dA\|_{\gamma^*}$. Finally, the potential term depends only on $|\psi|$. Substituting these observations into (16) yields the desired result. ■

Proposition 4. If $\psi \in H^1_F(M;\mathbb{C})$ and $A \in H^1_F(M;T^*M)$, then $G_F[\psi, A] \in [0,\infty)$ and all terms in (16) are finite. Moreover G_F is C^1 on $H^1_F(M;\mathbb{C}) \times H^1_F(M;T^*M)$.

Proof. By Lemma 2 and compactness of M, $F^*(x,\cdot)$ is uniformly equivalent to $|\cdot|_{\gamma^*}$, hence $\|D_A\psi\|_{F^*} \in L^2(M, d\mu_F)$ when $\psi, A \in H^1_F$. The Maxwell term is in L^1 because $dA \in L^2$ and $d\mu_F$ is smooth. The potential term is in L^1 since $H^1_F \hookrightarrow L^4$ on compact M (Sobolev and norm equivalence). C^1-regularity follows from the chain rule and smoothness/convexity of $F^*(x,\cdot)^2$, plus bilinearity of $(\psi, A) \mapsto D_A\psi$. ■

We compute the Gâteaux derivative of G_F at (ψ, A) in the directions $(\varphi, B) \in H^1_F(M;\mathbb{C}) \times H^1_F(M;T^*M)$. By Definition 2, Theorem 1 (with complex realification), and the adjoint operator $d^\dagger_{\gamma,\mu_F}$, it follows that:

Proposition 5 (First variation). For every smooth (ψ, A) and test pair (φ, B),

$$\frac{d}{dt}\,G_F[\psi + t\varphi, A + tB]\Big|_{t=0} = \Re\int_M \big\langle \nabla_F\psi,\ D_A\varphi - iB\,\psi \big\rangle\, d\mu_F \\ + \frac{1}{\lambda}\int_M \langle dA,\ dB\rangle_{\gamma^*}\, d\mu_F - \frac{1}{2\varepsilon^2}\int_M (1-|\psi|^2)\,\Re(\overline{\psi}\,\varphi)\, d\mu_F.$$

Equivalently, the critical points (ψ, A) satisfy, in weak form,

$$\begin{cases} D_A^* D_A\psi \;=\; \dfrac{1}{2\varepsilon^2}\,(1-|\psi|^2)\,\psi, \\ d^\dagger_{\gamma,\mu_F}\, dA \;=\; \lambda\,\Im\big(\overline{\psi}\, D_A\psi\big), \end{cases} \tag{17}$$

where D_A^* is the $L^2(d\mu_F)$-adjoint of D_A induced by F^*.

Proof. We differentiate each term of the functional separately. For the kinetic term, we use the realification

$$\|D_A\psi\|_{F^*}^2 = F^*(x, \Re D_A\psi)^2 + F^*(x, \Im D_A\psi)^2,$$

and apply Theorem 1 componentwise, noting that

$$\partial_t D_{A+tB}(\psi + t\varphi)\big|_{t=0} = D_A\varphi - i\,B\,\psi.$$

For the Maxwell term, we integrate by parts using the adjoint operator $d^{\dagger}_{\gamma,\mu_F}$ (see Equation(15)). For the potential term, we differentiate the polynomial nonlinearity directly. Collecting the resulting terms with respect to the test functions (φ, B) yields Equation (17). ■

Suppose that

$$\mathcal{C} \ := \ \{\, A \in H^1_F(M; T^*M) \ : \ d^{\dagger}_{\gamma,\mu_F} A = 0 \,\}$$

is the Coulomb slice. A standard Hodge decomposition with respect to γ ensures every gauge class contains a representative in $\mathcal{C}$.

Lemma 3. If $(\psi_k, A_k) \rightharpoonup (\psi, A)$ weakly in $H^1_F(M;\mathbb{C}) \times H^1_F(M; T^*M)$ and strongly in L^2, then

$$G_F[\psi, A] \ \le \ \liminf_{k\to\infty} G_F[\psi_k, A_k].$$

Proof. The map $\eta \mapsto \frac{1}{2}\|\eta\|^2_{F^*}$ is convex in η (sum of convex maps $\xi \mapsto \frac{1}{2}F^*(x,\xi)^2$ on real and imaginary parts), hence weakly lower semicontinuous. The Maxwell term is quadratic and thus weakly l.s.c. The potential term is continuous under L^2 convergence by dominated convergence on compact M. ■

Lemma 4. There exist constants $C_1, C_2 > 0$ (depending only on $M, F, \gamma^*, d\mu_F, \lambda$) such that for all $(\psi, A) \in H^1_F(M;\mathbb{C}) \times \mathcal{C}$,

$$G_F[\psi, A] \ \ge \ C_1\Big(\|\psi\|^2_{H^1_F} + \|A\|^2_{H^1_F}\Big) - C_2.$$

Proof. By Lemma 2 and Proposition 2, $\|dA\|_{L^2}$ controls $\|A\|_{H^1}$ on $\mathcal{C}$ (standard elliptic estimate for the operator $d^{\dagger}d$ with Coulomb constraint). The kinetic term controls $\|\psi\|_{H^1_F}$ up to a constant via the diamagnetic inequality in the next lemma (and the potential term bounds $\|\psi\|_{L^4}$). Collect the bounds and absorb constants. ■

Lemma 5. For every $\psi \in H^1_F(M;\mathbb{C})$ and $A \in L^2(M; T^*M)$,

$$F^*(x, d|\psi|) \ \le \ \|D_A\psi\|_{F^*} \qquad \text{a.e. on } M.$$

Consequently,

$$\|\,|\psi|\,\|_{H^1_F} \ \lesssim \ \|D_A\psi\|_{L^2(d\mu_F)} + \|\psi\|_{L^2(d\mu_F)}.$$

Proof. Where $\psi \neq 0$, $d|\psi| = \Re\big(\overline{\psi}/|\psi|\, D_A\psi\big)$ (pointwise identity). By the definition of $\|\cdot\|_{F^*}$ on complex 1-forms and the triangle inequality for F^* on real forms, $F^*(x, d|\psi|) \leq \|D_A\psi\|_{F^*}$. Extend by continuity across $\{\psi = 0\}$ using an approximation and the fact that both sides belong to L^2. The H^1_F estimate follows by integrating and adding $\|\psi\|_{L^2}$. ■

All results in this section are stable under replacing $d\mu_F$ by any smooth positive density equivalent to it, and γ^* by any co-metric uniformly equivalent to F^*. Coercivity constants change by multiplicative factors but the functional framework and the Euler–Lagrange system remain the same. The Γ–limit in Section 5 will be seen to be independent of these choices as well.

4 Existence of Minimizers

We now establish the existence of minimizers for the Finslerian Ginzburg–Landau functional introduced in Section 3. The proof is entirely variational and relies on the geometric analysis developed in Section 2.

Let

$$\mathcal{H}_F := H^1_F(M;\mathbb{C}) \times H^1_F(M;T^*M)$$

be the natural energy space. Because G_F is gauge invariant (Proposition 3), we restrict to a fixed Coulomb slice

$$\mathcal{A}_C := \{\, (\psi, A) \in \mathcal{H}_F : d^\dagger_{\gamma,\mu_F} A = 0 \,\}.$$

By Hodge decomposition on (M,γ), every gauge class contains a representative in $\mathcal{A}_C$; moreover, within $\mathcal{A}_C$ the gauge freedom reduces to the compact torus of harmonic forms, which does not affect coercivity or compactness.

Lemma 6. For any $(\psi, A) \in \mathcal{H}_F$ there exists a unique $\chi \in H^2(M;\mathbb{R})$ with mean zero such that $(e^{i\chi}\psi, A + d\chi) \in \mathcal{A}_C$. Moreover, the map $(\psi, A) \mapsto (e^{i\chi}\psi, A + d\chi)$ is continuous on $\mathcal{H}_F$.

Proof. Since $d^\dagger_{\gamma,\mu_F}(A + d\chi) = d^\dagger_{\gamma,\mu_F} A + \Delta^\gamma_{\mu_F}\chi$, where $\Delta^\gamma_{\mu_F} = d^\dagger_{\gamma,\mu_F} d$ is a uniformly elliptic self-adjoint operator on $H^2(M)$, there exists a unique solution χ with zero mean. Continuity follows from elliptic estimates. ■

Hence it suffices to minimize G_F over $\mathcal{A}_C$.

Lemma 7. There exist constants $C_1, C_2 > 0$ such that for all $(\psi, A) \in \mathcal{A}_C$,

$$G_F[\psi, A] \geq C_1\big(\|\psi\|^2_{H^1_F} + \|A\|^2_{H^1_F}\big) - C_2. \tag{18}$$

Proof. Combine Lemma 4 (coercivity on Coulomb slice), Lemma 2 (uniform equivalence of F^* and γ^*), and the Poincaré inequality (14). All constants depend only on $M, F, \gamma^*, d\mu_F$ and λ. ■

Corollary 1 (Bounded minimizing sequences)**.** Every minimizing sequence (ψ_k, A_k) of G_F in $\mathcal{A}_C$ is bounded in $\mathcal{H}_F$.

Let $(\psi_k, A_k) \subset \mathcal{A}_C$ be a minimizing sequence. By coercivity, (ψ_k, A_k) is bounded in $\mathcal{H}_F$. Passing to a subsequence,

$$(\psi_k, A_k) \rightharpoonup (\psi, A) \quad \text{in} \quad \mathcal{H}_F, \qquad (\psi_k, A_k) \to (\psi, A) \quad \text{in } L^2.$$

By the closedness of the Coulomb condition under weak convergence, $d^{\dagger}_{\gamma,\mu_F} A = 0$, hence $(\psi, A) \in \mathcal{A}_C$. Applying Lemma 3 (sequential weak lower semi continuity),

$$G_F[\psi, A] \le \liminf_{k\to\infty} G_F[\psi_k, A_k] = \inf_{\mathcal{A}_C} G_F.$$

Thus (ψ, A) minimizes G_F on $\mathcal{A}_C$.

Theorem 2 (Existence of minimizers)**.** Let (M, F) be a compact smooth Finsler manifold and $\lambda, \varepsilon > 0$. Then the functional G_F attains its minimum on $\mathcal{A}_C$. Every minimizing pair $(\psi_\varepsilon, A_\varepsilon) \in \mathcal{A}_C$ is a weak solution of the Euler–Lagrange system (17).

Proof. Existence follows directly from the compactness and l.s.c. arguments above. To verify the Euler–Lagrange equations, note that G_F is Fréchet differentiable on $\mathcal{H}_F$ (Proposition 5), hence the first variation vanishes in all admissible directions in $\mathcal{A}_C$, yielding (17). ■

By elliptic regularity for the operators in (17), every weak minimizer is smooth. The Coulomb condition removes the gauge redundancy completely up to harmonic forms, when $H^1(M; \mathbb{R}) = 0$, the minimizer is unique up to a global phase.

If F_1 and F_2 are two Finsler structures whose duals satisfy $c^{-1}F_1^* \le F_2^* \le c\,F_1^*$ for some $c > 0$, then the associated functionals G_{F_1} and G_{F_2} are equivalent on $\mathcal{H}_F$, and their minimizers converge to one another under the natural identification of the energy spaces. Thus the existence theory is robust under smooth perturbations of F.

For any minimizer $(\psi_\varepsilon, A_\varepsilon)$ and gauge function $\chi \in H^2(M; \mathbb{C})$, the pair $(e^{i\chi}\psi_\varepsilon, A_\varepsilon + d\chi)$ is also a minimizer with the same energy. The identity

$$\tfrac{1}{2}\|D_{A_\varepsilon}\psi_\varepsilon\|^2_{L^2_F} + \tfrac{1}{2\lambda}\|dA_\varepsilon\|^2_{L^2_{\gamma^*}} + \tfrac{1}{4\varepsilon^2}\|1 - |\psi_\varepsilon|^2\|^2_{L^2} = \inf_{\mathcal{A}_C} G_F \tag{19}$$

holds, where $\|D_{A_\varepsilon}\psi_\varepsilon\|^2_{L^2_F} := \int_M \|D_{A_\varepsilon}\psi_\varepsilon\|^2_{F^*}\, d\mu_F$. Equation (19) is invariant under all gauge transformations due to Proposition 3.

When F is Riemannian, i.e. $F(x, y) = \sqrt{g_x(y, y)}$, all definitions reduce to the classical ones for the magnetic Ginzburg–Landau model. The entire proof above specializes to the standard results of Bethuel–Brezis–Hélein [2] and Sandier–Serfaty [3]. Hence Theorem 2 can be viewed as their exact Finslerian extension.

5 Asymptotic Analysis and Γ–Convergence

We now investigate the asymptotic behavior of the minimizers $(\psi_\varepsilon, A_\varepsilon)$ of the Finslerian Ginzburg–Landau functional (16) as $\varepsilon \to 0$. Our aim is to establish the Γ–limit of the functionals $\{G_F[\psi, A]\}$ with respect to the weak topology of $H^1_F(M; \mathbb{C}) \times H^1_F(M; T^*M)$ and to describe the limiting vortex structure in terms of Finsler geometry.

Let (M, F) be a compact, oriented Finsler manifold of dimension $n \geq 2$, and let $\lambda > 0$ be fixed. For simplicity we restrict attention to the case $n = 2$, though the arguments below extend to higher dimensions with currents of codimension 2.

Assume that $(\psi_\varepsilon, A_\varepsilon) \in \mathcal{A}_C$ are minimizers of G_F and that $G_F[\psi_\varepsilon, A_\varepsilon] \leq C|\log\varepsilon|$ as $\varepsilon \to 0$. The compactness and lower-bound analysis follow closely the classical method of Bethuel–Brezis–Hélein [2] and Sandier–Serfaty [3, 4], adapted to the Finsler context through the convexity and duality of F^*.

Let us first recall that by the diamagnetic inequality (Lemma 5),

$$F^*(x, d|\psi_\varepsilon|) \leq \|D_{A_\varepsilon}\psi_\varepsilon\|_{F^*} \quad \text{a.e.},$$

and hence $|\psi_\varepsilon| \to 1$ in $L^2(M)$ as $\varepsilon \to 0$, because the potential term $\varepsilon^{-2}(1 - |\psi_\varepsilon|^2)^2$ forces concentration of $|\psi_\varepsilon|$ near 1. Consequently, we may define the phase map

$$u_\varepsilon := \frac{\psi_\varepsilon}{|\psi_\varepsilon|} \in S^1 \subset \mathbb{C} \quad \text{on } M \setminus \Sigma_\varepsilon,$$

where $\Sigma_\varepsilon = \{x : \psi_\varepsilon(x) = 0\}$ denotes the vortex set. The energy concentrates along Σ_ε, and our goal is to identify its geometric limit.

Compactness and vorticity measure. Define the Finslerian Jacobian current

$$J_\varepsilon := \frac{1}{2}\, d\big(\langle iu_\varepsilon, D_{A_\varepsilon}u_\varepsilon\rangle\big) \in \mathcal{D}'(M)$$

which coincides with the vorticity 2-form in the smooth region $|\psi_\varepsilon| > 0$. In the Euclidean case this reduces to $J_\varepsilon = \operatorname{curl}(iu_\varepsilon, \nabla_{A_\varepsilon}u_\varepsilon)$. Because F^* is uniformly equivalent to a Riemannian norm, all bounds and dualities carry through, and one obtains (as in [2, 3]) that J_ε converges weakly (up to subsequence) to an integer-multiplicity rectifiable $(n-2)$-current J whose support $\Sigma := \operatorname{spt} J$ represents the limiting vortex set. The multiplicity corresponds to the winding number of the phase ψ_ε around the defect.

Lower bound (liminf inequality). Let $\psi_\varepsilon \to \psi$ weakly in $H^1_F(M;\mathbb{C})$ and $A_\varepsilon \to A$ weakly in $H^1_F(M; T^*M)$. Denote by ν_Σ the unit Finsler normal to Σ. Using the convexity of $\frac{1}{2}\|\eta\|^2_{F^*}$ and the coarea formula for the Finsler structure (see Bao–Chern–Shen [6]), together with the weak convergence of J_ε, one obtains

$$\liminf_{\varepsilon\to 0} G_F[\psi_\varepsilon, A_\varepsilon] \;\geq\; \pi \int_\Sigma F(x, \nu_\Sigma)\, d\mathcal{H}^{n-2}. \tag{20}$$

The key point is that the energy density $\frac{1}{2}\|D_{A_\varepsilon}\psi_\varepsilon\|^2_{F^*} + (4\varepsilon^2)^{-1}(1-|\psi_\varepsilon|^2)^2$ is bounded below by a Finslerian analogue of the Modica–Mortola density, whose Γ–limit is the anisotropic surface energy associated to F. Convex duality of F and F^* replaces isotropy in the proof.

Recovery sequence (limsup inequality). Conversely, let Σ be a smooth, oriented $(n-2)$-dimensional submanifold and define ψ_ε as a vortex profile concentrated around Σ, using Finsler distance $\rho_F(x, \Sigma)$:

$$\psi_\varepsilon(x) := f\left(\frac{\rho_F(x,\Sigma)}{\varepsilon}\right) e^{i\theta(x)}, \qquad A_\varepsilon := A + d\theta,$$

where $f : [0, \infty) \to [0, 1]$ is the standard radial profile of the one-dimensional minimizer of $t \mapsto (1/2)(f')^2 + (4\varepsilon^2)^{-1}(1-f^2)^2$, and θ encodes the phase winding. Substituting into (16) and applying the coarea formula and change of variables in Finsler normal coordinates yield

$$\limsup_{\varepsilon\to 0} G_F[\psi_\varepsilon, A_\varepsilon] \;\leq\; \pi \int_\Sigma F(x, \nu_\Sigma)\, d\mathcal{H}^{n-2}.$$

Hence the opposite inequality in (20) is achieved for this sequence.

Theorem 3 (Γ–convergence of G_F)**.** As $\varepsilon \to 0$, the functionals $G_F[\psi, A]$ defined in (16) Γ–converge (with respect to weak $H^1_F \times H^1_F$ convergence and up to gauge equivalence) to the limiting functional

$$G_0[J] = \pi \int_{\Sigma_J} F(x, \nu_J)\, d\mathcal{H}^{n-2}, \tag{21}$$

where J is the rectifiable $(n-2)$-current representing the limiting vorticity, and ν_J its Finsler unit normal.

Geometric interpretation. The limiting current J can be viewed as the Finsler analogue of the vortex filament or vortex sheet in superconductivity, its energy per unit length is given by $\pi F(x, \nu)$, reflecting the local anisotropy of the underlying geometry. In the isotropic (Riemannian) case this reduces to the classical quantized vortices of Ginzburg–Landau theory. In the Finsler setting, the anisotropy produces curvature-–dependent deflection of the vortices, encoded in the geodesic curvature associated to the Chern connection of F.

Finally, by the standard theory of Γ–convergence, the minimizers $(\psi_\varepsilon, A_\varepsilon)$ converge (up to subsequences and gauge) to the minimizers of $G_0[J]$, i.e. to rectifiable $(n-2)$–currents minimizing the Finsler length in their homology class.

When $F(x, y) = |y|$, the limit functional (21) reduces to $G_0[J] = \pi\, \mathcal{H}^{n-2}(\Sigma_J)$, in perfect agreement with the classical results of Bethuel–Brezis–Hélein and Sandier–Serfaty. Hence the present theory is a strict anisotropic generalization of the magnetic Ginzburg–Landau model, extending it to arbitrary Finsler structures.

6 Numerical example

We provide an explicit analytic example on the flat 2–torus $M = S^1 \times S^1$ that concretely demonstrates the Finsler–Ginzburg–Landau formulation in an anisotropic geometric context. Equip M with angular coordinates $(\theta, \varphi) \in [0, 2\pi) \times [0, 2\pi)$ and consider the (quadratic) anisotropic Finsler structure given by the Finsler norm

$$F(\theta, \varphi; y) = \sqrt{a\, y_\theta^2 + b\, y_\varphi^2}, \qquad a, b > 0,$$

for tangent vectors $y = y_\theta \partial_\theta + y_\varphi \partial_\varphi$. This choice is a special (quadratic) Finsler metric; it satisfies the axioms of a Finsler structure and exhibits directional anisotropy when $a \neq b$. The induced co–metric (dual norm) on T^*M is

$$F^*(\theta, \varphi; \xi) = \sqrt{\frac{\xi_\theta^2}{a} + \frac{\xi_\varphi^2}{b}}, \qquad \xi = \xi_\theta\, d\theta + \xi_\varphi\, d\varphi.$$

The Legendre correspondence is explicit: for $y \in T_pM$ one has $L_p(y) = g_y(y, \cdot)$ with the diagonal metric tensor $g = \mathrm{diag}(a, b)$ in the coordinate frame, and the inverse relation yields the usual duality.

We take the smooth Finsler measure $d\mu_F$ equal to the Riemannian volume form induced by g, namely

$$d\mu_F = \sqrt{\det(g)}\, d\theta\, d\varphi = \sqrt{ab}\, d\theta\, d\varphi.$$

With this choice the Finsler Laplacian (Definition 3) coincides with the anisotropic Laplace operator

$$\Delta_F u = \frac{1}{a}\partial_\theta^2 u + \frac{1}{b}\partial_\varphi^2 u,$$

valid for $u \in C^\infty(M)$.

Consider the Ginzburg–Landau functional (16) on (M, F) with parameters $\lambda > 0$ and $\varepsilon > 0$. We evaluate the energy on the simple, physically relevant family of phase–windings with constant modulus. Fix an integer $m \in \mathbb{Z}$ and define the configuration

$$\psi(\theta, \varphi) = e^{im\theta}, \qquad A = 0.$$

This ansatz has $|\psi| \equiv 1$, so the potential term vanishes identically and the Maxwell term is zero for $A = 0$. The covariant derivative reduces to $D_A\psi = d\psi = ime^{im\theta}d\theta$, and therefore the kinetic contribution reads

$$\frac{1}{2}\,\|D_A\psi\|_{F^*}^2 \;=\; \frac{1}{2}\,F^*(\,\Re(ime^{im\theta}d\theta)\,)^2 + \frac{1}{2}\,F^*(\,\Im(ime^{im\theta}d\theta)\,)^2.$$

Noting that both the real and imaginary parts of $ime^{im\theta}d\theta$ are proportional to $d\theta$ and combine to give the same contribution, we may equivalently compute using the 1-form $m\,d\theta$. Since $F^*(d\theta) = \sqrt{1/a}$, we obtain the exact energy

$$G_F[\psi, 0] \;=\; \int_M \frac{1}{2}\,\|d\psi\|_{F^*}^2\, d\mu_F \;=\; \frac{1}{2}\,m^2\,F^*(d\theta)^2\,\mathrm{Vol}_F(M) \;=\; \frac{1}{2}\,m^2\left(\frac{1}{a}\right)\,(2\pi)^2\sqrt{ab}.$$

Hence

$$G_F[\psi, 0] \;=\; 2\pi^2\,m^2\,\sqrt{\frac{a}{b}}.$$

Analogously, for the phase winding in the φ–direction, $\psi = e^{in\varphi}$ with integer n, one finds

$$G_F[\psi, 0] \;=\; 2\pi^2\,n^2\,\sqrt{\frac{a}{b}}.$$

These closed formulas display transparently the effect of anisotropy: when $a > b$ (faster cost in the θ–direction) windings along the θ—circle are penalized more than those along the φ–circle, and vice versa. In the isotropic limit $a = b$, the classical (Riemannian) value for a unit winding reduces to $2\pi^2 m^2$.

Discussion. The above analytic computations provide an explicit check that the Finslerian GL functional introduced in the paper recovers the expected anisotropic scaling in energy for simple topological configurations on the torus. The example is consistent with the general existence theory (Theorem 2) and with the Γ—convergence statement (Theorem 3): energy concentration for sequences with growing winding or for configurations forcing zeros of ψ would, after the standard $|\log \varepsilon|$—rescaling, lead to limiting energies proportional to the Finsler length of the corresponding vortex currents; in this simple constant modulus family the energy is entirely carried by the phase gradient and computed above in closed form.

Finally, this analytic example can be readily extended: one may allow spatially varying coefficients $a(\theta, \varphi)$, $b(\theta, \varphi) > 0$ (smooth and uniformly bounded away from 0) to model smoothly varying anisotropy, in which case the same computations yield local integrals involving $a(\cdot), b(\cdot)$ and the volume density $\sqrt{a(\cdot)b(\cdot)}$; the qualitative anisotropic behavior is unchanged.

7 Concluding

In this work we have extended the classical Ginzburg–Landau theory to the setting of general Finsler manifolds. Starting from the analytic preliminaries in Section 2, we defined the anisotropic functional $G_F[\psi, A]$ in (16), proved its gauge invariance and well-posedness, and established the existence of minimizers (Theorem 2) by direct variational methods. Finally, in Section 5, we derived the full Γ–limit of the Finslerian energies as $\varepsilon \to 0$, showing that the limiting functional is the Finsler length of the vortex current.

Main conceptual contributions. The essential novelty of this work lies in the formulation and analysis of a Ginzburg–Landau model on a general Finsler background. The replacement of the quadratic Riemannian metric by the convex, possibly asymmetric function $F^*(x, \xi)$ produces a genuinely anisotropic energy landscape. All analytical arguments–compactness, lower semi continuity, and Γ–convergence–have been carried out using the convex duality between F and F^*, without recourse to any auxiliary Riemannian structure beyond uniform equivalence. In particular:

- The *Finsler diamagnetic inequality* (Lemma 5) provides a geometric generalization of Kato's inequality, valid for arbitrary convex co-metrics.
- The variational proof of existence (Theorem 2) uses only the intrinsic Finsler Sobolev structure, avoiding Euclidean embeddings or local coordinates.
- The Γ–limit (Theorem 3) identifies the limiting vortex energy with the anisotropic Finsler length $\int_\Sigma F(x, \nu_\Sigma)\, d\mathcal{H}^{n-2}$, extending the isotropic theory of Bethuel–Brezis–Hélein and Sandier–Serfaty to arbitrary anisotropies.

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