Open Communications in Nonlinear Mathematical Physics]ocnmp[Vol.5 (2025) pp 81–89 Letter

Letter to the Editors

Two sequences of fully-nonlinear evolution equations and their symmetry properties

*Marianna Euler and Norbert Euler**

International Society of Nonlinear Mathematical Physics (ISNMP)
Auf der Hardt 27, 56130 Bad Ems, Germany
and
Centro Internacional de Ciencias (CIC AC)
Av. Universidad s/n, Colonia Chamilpa, 62210 Cuernavaca, Morelos, Mexico

* *Dr.Norbert.Euler@gmail.com*

Received September 9, 2025; Accepted October 10, 2025

Abstract

We obtain the complete Lie point symmetry algebras of two sequences of odd-order evolution equations. This includes equations that are fully-nonlinear, i.e. nonlinear in the highest derivative. Two of the equations in the sequences have recently been identified as symmetry-integrable, namely a 3rd-order equation and a 5th-order equation [*Open Communications in Nonlinear Mathematical Physics*, *Special Issue in honour of George W Bluman*, ocnmp:15938, 1–15, 2025]. These two examples provided the motivation for the current study. The Lie-Bäcklund symmetries and the consequent symmetry-integrability of the equations in the sequences are also discussed.

1 Introduction

We recently [5] reported two fully-nonlinear symmetry-integrable evolution equations, namely the 3rd-order equation

$$u_t = u_{3x}^{-1/2} \qquad (1.1)$$

and the 5th-order equation

$$u_t = u_{5x}^{-2/3}. \qquad (1.2)$$

Througout this Letter we make use of the notation $u_x = \partial u/\partial x$, $u_{xx} := \partial^2 u/\partial x^2$ and $u_{px} := \partial^p u/\partial x^p$ for $p \geq 3$. Motivated by the symmetry-integrability of equations (1.1) and (1.2), we propose here the following sequence of odd-order evolution equations:

$$u_t = \left(u_{(2k+1)x}\right)^{-n_k}, \quad k = 1, 2, 3, ..., \qquad (1.3)$$

that is

$$u_t = u_{3x}^{-n_1}, \quad u_t = u_{5x}^{-n_2}, \quad u_t = u_{7x}^{-n_3}, \dots \ . \tag{1.4}$$

Here n_k is a number which is, in general, different for every value of k and $n_k \notin \{-1, 0\}$ for any k. It will be shown in Sections 2 and Section 3 that the relation between n_k and k is essential for the symmetry properties and the symmetry-integrability of the equations in this sequence. We note that equations (1.1) and (1.2) are included as the first two members of the sequence (1.3) for the case where $n_k = k/(k+1)$, namely $k = 1$ and $k = 2$, respectively.

Note further that we do not consider here sequences of even order of the form

$$u_t = (u_{2kx})^{n_k}, \quad k = 1, 2, 3, \dots \tag{1.5}$$

since the 2nd-order case (so (1.5) for $k = 1$) is included in our study of general 2nd-order evolution equations [4]) in which we have established that the equation

$$u_t = u_{xx}^{-1}, \tag{1.6}$$

is the only symmetry-integrable equation of the form (1.5). In particular, (1.5) admits the Lie-Bäcklund symmetry generator

$$Z_{LB} = u_{xx}^{-3} u_{3x} \frac{\partial}{\partial u}. \tag{1.7}$$

All other even-order equations of the form (1.5), i.e. for $k \geq 2$, do not satisfy the necessary conditions for symmetry-integrability (see Conjecture 1 in Section 3 below).

In this short Letter we derive the Lie point symmetry algebras of (1.3) in Section 2. The symmetry analysis reveals that (1.3) in fact consists of two sequences, whereby the difference between the two sequences is essentially given by one symmetry. We furthermore discuss the Lie-Bäcklund symmetry structure of the two sequences in Section 3, i.e. the symmetry-integrability of the equations, for which we state a Conjecture with a Corollary. In Section 4 we then sum up our findings and make some concluding remarks.

2 The Lie point symmetries of the sequences

In this section we report the Lie point symmetries of the sequence (1.3). It is well-known how to calculate Lie point symmetries of evolution equations (see for example the books [7] or [1]) so we merely point out our notations here (see [2] for details).

Let $E(x, t, u, u_t, u_x, u_{xx}, \dots, u_{nx}) = 0$ denote a general nth-order partial differential equation. For evolution equations we have

$$E := u_t - F(x, t, u, u_x, u_{xx}, \dots, u_{nx}). \tag{2.1}$$

The invariance condition for the Lie point symmetries of (2.1) with the generator

$$Z = \xi_1(x, t, u) \frac{\partial}{\partial x} + \xi_2(x, t, u) \frac{\partial}{\partial t} + \eta(x, t, u) \frac{\partial}{\partial u} \tag{2.2}$$

is

$$L_E[u]Q\Big|_{E=0} = 0, \tag{2.3}$$

where Q denotes the corresponding invariant surface

$$Q = \eta(x,t,u) - \xi_1(x,t,u)u_x - \xi_2(x,t,u)u_t \tag{2.4}$$

and $L_E[u]$ denotes the linear operator

$$L[u] = \frac{\partial E}{\partial u_t}D_t + \frac{\partial E}{\partial u} + \frac{\partial E}{\partial u_x}D_x + \frac{\partial E}{\partial u_{xx}}D_x^2 + \cdots + \frac{\partial E}{\partial u_{nx}}D_x^n\Big|_{E:=u_t-F} \tag{2.5a}$$

$$= D_t - \frac{\partial F}{\partial u} - \frac{\partial F}{\partial u_x}D_x - \frac{\partial F}{\partial u_{xx}}D_x^2 - \cdots - \frac{\partial F}{\partial u_{nx}}D_x^n. \tag{2.5b}$$

Here D_t and D_x denote the total x-derivative and total t-derivative, respectively.

Applying the invariance condition (2.3) for all equations in the sequence (1.3) results in the following

Proposition 1. *For the general invariant surface (2.4) of the sequence (1.3), viz.*

$$u_t = \left(u_{(2k+1)x}\right)^{-n_k}, \quad k = 1, 2, 3, ...,$$

we distinguish between two cases:

Case 1: *For the sequence*

$$u_t = \left(u_{(2k+1)x}\right)^{-\frac{k}{k+1}}, \quad k = 1, 2, 3, ..., \tag{2.6}$$

the most general invariant surface is

$$Q := \left[k(a_1 + 2a_2x) + \left(\frac{k+1}{2k+1}\right)b_1\right]u - u_x\left(a_0 + a_1x + a_2x^2\right)$$

$$-u_t\left(b_0 + b_1t\right) + f(x), \tag{2.7a}$$

where

$$\frac{d^{2k+1}f(x)}{dx^{2k+1}} = 0 \tag{2.7b}$$

with $k = 1, 2, \ldots$. Here $a_0,\ a_1,\ a_2,\ b_0$ and b_1 are arbitrary constants. This gives the Lie symmetry algebra for (2.6) of dimension $2k+6$ which is spanned by the following set of Lie point symmetry generators:

$$\{Z_1 = \frac{\partial}{\partial x}, \quad Z_2 = \frac{\partial}{\partial t}, \quad Z_3 = t\frac{\partial}{\partial t} + \left(\frac{k+1}{2k+1}\right)u\frac{\partial}{\partial u}, \quad Z_4 = x\frac{\partial}{\partial x} + ku\frac{\partial}{\partial u},$$

$$Z_5 = x^2\frac{\partial}{\partial x} + 2kxu\frac{\partial}{\partial u}, \quad Z_{p+5} = x^{p-1}\frac{\partial}{\partial u}\}, \quad p = 1, 2, \ldots, 2k+1. \tag{2.8}$$

Case 2: *For the sequence*

$$u_t = \left(u_{(2k+1)x}\right)^{-n_k}, \quad n_k \neq \frac{k}{k+1}, \quad k = 1, 2, 3, ..., \tag{2.9}$$

the most general invariant surface is

$$Q := \left(\frac{a_1 n_k(2k+1) - b_1 k}{n_k(k+1) - k}\right) u - u_x \left(\frac{(n_k+1)a_1 - b_1}{n_k(k+1) - k} + a_0\right)$$

$$-u_t\left(b_0 + b_1 t\right) + f(x), \tag{2.10a}$$

where

$$\frac{d^{2k+1} f(x)}{dx^{2k+1}} = 0 \tag{2.10b}$$

with $k = 1, 2, \ldots$. *Here* $a_0,\ a_1,\ b_0$ *and* b_1 *are arbitrary constants. This gives the Lie symmetry algebra for (2.9) of dimension* $2k+5$ *which is spanned by the following set of Lie point symmetry generators:*

$$\{Z_1 = \frac{\partial}{\partial x}, \quad Z_2 = \frac{\partial}{\partial t}, \quad Z_3 = x\frac{\partial}{\partial x} - \left[(k+1)n_k - k\right] t\frac{\partial}{\partial t} + ku\frac{\partial}{\partial u},$$

$$Z_4 = x\frac{\partial}{\partial x} + \frac{(2k+1)n_k}{n_k+1} u\frac{\partial}{\partial u}, \quad Z_{p+4} = x^{p-1}\frac{\partial}{\partial u}\}, \tag{2.11}$$

$$p = 1, 2, \ldots, 2k+1.$$

3 On the symmetry-integrability of the equations

An evolution equation of order n is said to be symmetry-integrable if it admits an infinite number of local Lie-Bäcklund symmetry generators

$$Z_{LB} = Q(x, t, u, u_x, u_{xx}, \ldots, u_{qx})\frac{\partial}{\partial u}, \tag{3.1}$$

where $q > n$ (for details, see for example [2]). The invariance condition to obtain Lie-Bäcklund symmetry generators is essentially the same as the condition (2.3), albeit Q now depends on derivatives up to some order q. The first step is to establish the possible nonlinearities of the equations in the sequences of Proposition 1. For that we conjecture a necessary condition which is based on many tedious calculations, the details of which we do not present here. To obtain Conjecture 1 (see below) we have considered the evolution equations in their most general form (2.1) with $n \geq 2$ and applied the Lie-Bäcklund symmetry invariance condition to establish a necessary condition for the existence of Lie-Bäcklund symmetry generators of the form (3.1). For this we have sought Lie-Bäcklund symmetries up to order 19. This leads to

Conjecture 1. *The necessary condition for the symmetry-integrability of an evolution equation of the form*

$$u_t = F(x, t, u, u_x, u_{xx}, \ldots, u_{nx}), \quad n \geq 2, \tag{3.2}$$

strictly depends on the order n *as follows:*

1. *For $n = 2$, the necessary condition for the symmetry-integrability of the 2nd-order equation*

$$u_t = F(x,t,u,u_x,u_{xx}) \tag{3.3a}$$

is

$$2\frac{\partial F}{\partial u_{xx}}\frac{\partial^3 F}{\partial u_{xx}^3} - 3\left(\frac{\partial^2 F}{\partial u_{xx}^2}\right)^2 = 0. \tag{3.3b}$$

2. *For $n = 3$,the necessary condition for the symmetry-integrability of the 3rd-order equation*

$$u_t = F(x,t,u,u_x,u_{xx},u_{3x}) \tag{3.4a}$$

is

$$9\left(\frac{\partial F}{\partial u_{3x}}\right)^2\frac{\partial^4 F}{\partial u_{3x}^4} - 45\frac{\partial F}{\partial u_{3x}}\frac{\partial^2 F}{\partial u_{3x}^2}\frac{\partial^3 F}{\partial u_{3x}^3} + 40\left(\frac{\partial^2 F}{\partial u_{3x}^2}\right)^3 = 0. \tag{3.4b}$$

3. *For odd $n \geq 5$, that is $n = (2k+3)$ with $k = 1, 2, 3, ...$, the necessary condition for the symmetry-integrability of*

$$u_t = F(x,t,u,u_x,u_{xx},\dots,u_{2k+3}), \quad k = 1,2,3,\dots \tag{3.5a}$$

is

$$(2k+3)\frac{\partial F}{\partial u_{(2k+3)x}}\frac{\partial^3 F}{\partial u_{(2k+3)x}^3} - (3k+5)\left(\frac{\partial^2 F}{\partial u_{(2k+3)x}^2}\right)^2 = 0 \tag{3.5b}$$

$k = 1, 2, 3, \dots$.

4. *For even $n \geq 4$, that is $n = 2k+2$ with $k = 1, 2, 3, \dots$, the necessary condition for the symmetry-integrability of*

$$u_t = F(x,t,u,u_x,u_{xx},\dots,u_{2k+2}), \quad k = 1,2,3,\dots \tag{3.6a}$$

is

$$\frac{\partial^2 F}{\partial u_{(2k+2)x}^2} = 0, \quad k = 1,2,3,\dots \; . \tag{3.6b}$$

Remarks: *The necessary condition (3.3b) for 2nd-order evolution equations has been established in [4]. Note that (3.3b) is the Schwarzian derivative when we factor out* $2\left(\frac{\partial F}{\partial u_{xx}}\right)^2$. *Condition (3.4b) for 3rd-order equations was derived in [3].*

We now discuss the symmetry-integrability of the two sequences of Proposition 1 in view of Conjecture 1.

Regarding Case 1: We consider the sequence of equations (2.6), i.e.

$$u_t = u_{3x}^{-1/2} \tag{3.7a}$$

$$u_t = u_{5x}^{-2/3} \tag{3.7b}$$

$$u_t = u_{7x}^{-3/4} \tag{3.7c}$$

$$u_t = u_{9x}^{-4/5} \tag{3.7d}$$

etc.

As reported in [5], equations (3.7a) and (3.7b) admit Lie-Bäcklund symmetry generators with lowest order 5 and 11, respectively, so that these two equations are symmetry-integrable. Moreover, equation (3.7a) satisfies condition (3.4b) and equation (3.7b) satisfies condition (3.5b) for $k = 1$.

It is easy to show that every equation of order $p \geq 5$ in the sequence (2.6) satisfies the necessary condition (3.5b) for symmetry-integrability as given in Conjecture 1. In particular,

$$u_t = \left(u_{2k+1)x}\right)^{-\frac{k}{k+1}}, \quad k = 2, 3, 4, \ldots \tag{3.8}$$

is equivalent to

$$u_t = \left(u_{(2k+3)x}\right)^{-\frac{k+1}{k+2}}, \quad k = 1, 2, 3, \ldots . \tag{3.9}$$

whereby

$$F = \left(u_{(2k+3)x}\right)^{-\frac{k+1}{k+2}} \tag{3.10}$$

satisfies condition (3.5b) for $k = 1, 2, 3, \ldots$.

Regarding the 7th-order equation (3.7c), we found that this equation does not admit a Lie-Bäcklund symmetry generator up to order 19. Due to the memory restrictions of our computer we are not able to consider Lie-Bäcklund symmetry generators of order higher than 19, so we can therefore not make any statement about the existence of Lie-Bäcklund symmetries for the equations in sequence (2.6) that are of order 7 or higher.

Regarding Case 2: We consider the sequence of equations (2.9), i.e.

$$u_t = u_{3x}^{-n_1}, \quad n_1 \neq \frac{1}{2} \tag{3.11a}$$

$$u_t = u_{5x}^{-n_2}, \quad n_2 \neq \frac{2}{3} \tag{3.11b}$$

$$u_t = u_{7x}^{-n_3}, \quad n_3 \neq \frac{3}{4} \tag{3.11c}$$

$$u_t = u_{9x}^{-n_4}, \quad n_4 \neq \frac{4}{5} \tag{3.11d}$$

etc.

The 3rd-order equation (3.11a) satisfies condition (3.4b) if and only if

$$n_1 \in \{-1, \frac{1}{2}, 2\}. \tag{3.12}$$

The only case of relevance here is $n_1 = 2$, i.e.

$$u_t = u_{3x}^{-2}, \tag{3.13}$$

which has already been established as symmetry-integrable in [6] (see equation (3.7) with $\alpha = 0$ and $\beta = 1$ in [6]).

We now consider the equations of order $p \geq 5$ in the sequence (2.9), namely

$$u_t = \left(u_{(2k+1)x}\right)^{-n_k}, \quad n_k \neq \frac{k}{k+1}, \quad k = 2, 3, 4, \dots\ , \tag{3.14}$$

which is equivalent to

$$u_t = \left(u_{(2k+3)x}\right)^{-n_{k+1}}, \quad n_{k+1} \neq \frac{k+1}{k+2}, \quad k = 1, 2, 3, \dots\ . \tag{3.15}$$

Then

$$F = \left(u_{(2k+3)x}\right)^{-n_{k+1}} \tag{3.16}$$

satisfies condition (3.5b) if and only if

$$n_{k+1} \in \{-1, \frac{k+1}{k+2}\} \tag{3.17}$$

for $k = 1, 2, 3, \dots$. Therefore the sequence (3.14) does not satisfy the necessary condition for symmetry-integrability for all $k = 2, 3, 4, \dots$.

Conjecture 1 now leads to the following

Corollary 1. *The sequence (2.9), viz.*

$$u_t = \left(u_{(2k+1)x}\right)^{-n_k}, \quad n_k \neq \frac{k}{k+1}, \quad k = 1, 2, 3, ...,$$

contains only one symmetry-integrable equation, namely the 3rd-order equation (3.13), viz.

$$u_t = u_{3x}^{-2}.$$

All remaining equations in this sequence are not symmetry-integrable.

4 Concluding remarks

In this short Letter we have introduced two sequences of fully-nonlinear evolution equations, namely

$$u_t = \left(u_{(2k+1)x}\right)^{-\frac{k}{k+1}}, \quad k = 1, 2, 3, ... \tag{4.1}$$

and

$$u_t = \left(u_{(2k+1)x}\right)^{-n_k}, \quad n_k \neq \frac{k}{k+1}, \quad k = 1, 2, 3, ... \tag{4.2}$$

We report the Lie point symmetries of both sequences, where (4.1) admits a Lie point symmetry algebra of dimension $2k + 6$ spanned by the generators (2.8) and the sequence (4.2) admits a Lie point symmetry algebra of dimension $2k + 5$ spanned by the generators (2.11). The difference in dimensions of the two symmetry algebras is due to the Lie point symmetry generator

$$Z_5 = x^2 \frac{\partial}{\partial x} + 2kxu \frac{\partial}{\partial u}, \tag{4.3}$$

which is admitted by the sequence (4.1) but not by the sequence (4.2). Note that (4.3) generates the one-parameter local Lie transformation group

$$\tilde{x} = \frac{x}{1 - \epsilon x}, \quad \tilde{t} = t, \quad \tilde{u} = \frac{u}{(1 - \epsilon x)^{2k}}, \tag{4.4a}$$

where ϵ is the group parameter and $\tilde{x}(\epsilon = 0) = x$, $\tilde{t}(\epsilon = 0) = t$, $\tilde{u}(\epsilon = 0) = u$.

Regarding the symmetry-integrability of the two sequences: We have established that every equation in the sequence (4.1) satisfies the necessary condition for symmetry-integrability. That is, condition (3.4b) for the 3rd-order equation and condition (3.5b) for the remaining equations in sequence (4.1) which is based on Conjecture 1. Furthermore, sequence (4.1) contains at least two symmetry integrable equations, namely [5]

$$u_t = u_{3x}^{-1/2} \quad \text{and} \quad u_t = u_{5x}^{-2/3}. \tag{4.5}$$

As mentioned in Section 3, we are not able to find further symmetry-integrable equations of sequence (4.1) besides the two equations (4.5). It is therefore an open problem to find further symmetry-integrable equations or to prove that these two equations are the only symmetry-integrable equations in this sequence.

Regarding the sequence (4.2): Based on Conjecture 1 we conclude in Corollary 1 that this series contains only one nonlinear symmetry-integrable equation, namely the 3rd-order equation (3.13), *viz*

$$u_t = u_{3x}^{-2}.$$

Furthermore, by solving condition (3.5b), Conjecture 1 leads to

Corollary 2. *Any fully-nonlinear evolution equations of dimension* $1+1$ *of order* $n \geq 5$ *that is symmetry-integrable has to be of the form*

$$u_t = f_1 \left(u_{(2k+3)x} + f_2 \right)^{\frac{k+1}{k+2}} + f_3, \quad k = 1, 2, 3, \dots \tag{4.6}$$

for some functions $f_j = f_j(x, t, u, u_x, u_{xx}, \dots, u_{(2k+2)x})$, $j = 1, 2, 3.$

We hope that this preliminary results will pave the way for a deeper study of the sequences introduced here and possibly other sequences of (fully-)nonlinear evolution equations, in particular the sequence (4.6).

References

[1] Bluman, G W and Anco, S C, *Symmetry and Integration Methods for Differential Equations*, Springer, New York, 2002.

[2] Euler M and Euler N, Nonlocal invariance of the multipotentialisations of the Kupershmidt equation and its higher-order hierarchies In: *Nonlinear Systems and Their Remarkable Mathematical Structures*, N Euler (ed), CRC Press, Boca Raton, 317-351, 2018 or arXiv:2506.07780 [nlin.SI].

[3] Euler M and Euler N, On Möbius-invariant and symmetry-integrable evolution equations and the Schwarzian derivative, Studies in Applied Mathematics, 143(2), 139–156, 2019.

[4] Euler M and Euler N, On 2nd-order fully-nonlinear equations with links to 3rd-order fully-nonlinear equations, *Open Communications in Nonlinear Mathematical Physics*, **vol. 4** (2), Proceedings of the OCNMP-2024 Conference, **ocnmp:13765**, 158–170, 2024. https://doi.org/10.46298/ocnmp.13765

[5] Euler M and Euler N, From fully-nonlinear to semilinear evolution equations: two symmetry-integrable examples, *Open Communications in Nonlinear Mathematical Physics*, **vol. 5**, Special Issue in honour of George W Bluman, **ocnmp:15938**, 1–15, 2025. https://doi.org/10.46298/ocnmp.15938

[6] Euler M and Euler N, On fully-nonlinear symmetry-integrable equations with rational functions in their highest derivative: Recursion operators, *Open Communications in Nonlinear Mathematical Physics*, **2**, **ocnmp:10306**, 216–228, 2022. https://doi.org/10.46298/ocnmp.10306

[7] Olver PJ, *Applications of Lie Groups to Differential Equations*, Springer, New York, 1986.