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From fully-nonlinear to semilinear evolution equations: two symmetry-integrable examples[1]

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Abstract: In this paper we derive two examples of fully-nonlinear symmetry-integrable evolution equations with algebraic nonlinearities, namely one class of 3rd-order equations and a 5th-order equation. To achieve this we study the equations' Lie-Bäcklund symmetries and apply multipotentialisations, hodograph transformations and generalised hodograph transformations to map the equations to known semilinear integrable evolution equations. As a result of this, we also obtain interesting symmetry-integrable quasilinear equations of order five and order seven, which we display explicitly.

1 Introduction

In [5] we reported a set of 3rd-order fully-nonlinear symmetry-intergable equations that are invariant under a projective transformation in u and in [7] we reported some results on 3rd-order fully-nonlinear symmetry-integrable equations with rational functions in the third derivative. In the current paper we report further results on fully-nonlinear equations whereby we now focus on two special cases, namely a class of 3rd-order evolution equations and a 5th-order evolution equation, both with algebraic nonlinearities in their highest derivative. In particular, we establish the semi-linearisations of the equations by systematically applying the procedure of multipotentialisation, hodograph transformations and, in some cases, generalised hodograph transformations. We refer to [1], [2] and [4] for details on the potentialisation of

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evolution equations by the use of adjoint symmetries, whereas further details on generalised hodograph tranformations can be found in [9] and [12].

The Schwarzian derivative S plays an important role in our discussion and is defined in terms of the dependent variable u by

$$S[u] := \frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2}, \tag{1.1}$$

which is applied throughout this paper. See for example [11] for a discussion on the Schwarzian derivative.

The paper is organised as follows: In Section 2 we consider a class of fully-nonlinear 3rd-order equations, identify those equations that admit local Lie-Bäcklund symmetries and map the obtained equations to known semilinear 3rd-order equations. Recursion operators are found for these equations which provide the associated symmetry-integrable hierarchies. In Section 3 we consider a single 5th-order fully-nonlinear evolution equation and establish its Lie-Bäcklund symmetries as well as its mapping to a known 5th-order semilinear integrable equation. In Section 4 we draw our conclusions of the reported results and mention some related open problems.

2 A class of third-order fully-nonlinear evolution equations

When seeking symmetry-integrable 3rd-order evolution equations the following statement [7] is useful to determine the possible nonlinearities in the highest derivative u_{xxx} [5]:

Lemma 1: *If a 3rd-order evolution equation of the form*

$$u_t = F(x, t, u, u_x, u_{xx}, u_{xxx}) \tag{2.1}$$

is symmetry-integrable for a given function F then this function must satisfy the following condition:

$$9\left(\frac{\partial F}{\partial u_{xxx}}\right)^2 \frac{\partial^4 F}{\partial u_{xxx}^4} - 45\frac{\partial F}{\partial u_{xxx}}\frac{\partial^2 F}{\partial u_{xxx}^2}\frac{\partial^3 F}{\partial u_{xxx}^3} + 40\left(\frac{\partial^2 F}{\partial u_{xxx}^2}\right)^3 = 0. \tag{2.2}$$

In [7] we have reported the general solution of (2.2), namely

$$F(x, t, u, u_x, u_{xx}, u_{xxx}) = \frac{P_3\,(u_{xxx} + P_2)}{\left[(u_{xxx} + P_2)^2 + P_1\right]^{1/2}} + P_4, \tag{2.3}$$

where $P_j = P_j(x, t, u, u_x, u_{xx})$, $j = 1, 2, 3, 4$, are arbitrary and smooth functions of their arguments. We listed several singular solutions of (2.2) that are not included in the general solution (2.3) and used the rational solutions in the classification of fully-nonlinear symmetry-integrable equations with rational nonlinearities of order three [7].

An additional singular solution of (2.2) that has not been reported in [7] is

$$F(x, t, u, u_x, u_{xx}, u_{xxx}) = \frac{P_1(x, t, u, u_x, u_{xx})}{\sqrt{u_{xxx}}} + P_2(x, t, u, u_x, u_{xx}), \tag{2.4}$$

which is the form of F that we will focus on in the current paper. Here P_1 and P_2 are arbitrary and smooth functions. We furthermore restrict ourselves to evolution equations that do not depend explicitly on their independent variables x and t and apply the standard condition for Lie-Bäcklund symmetries for evolution equations, namely

$$L_E[u]Q\Big|_{E=0} = 0, \tag{2.5}$$

where $E := u_t - F(u, u_x, u_{xx}, u_{xxx})$, and $Q = Q(u, u_x, u_{xx}, \ldots, u_{nx})$ is the characteristic of the Lie-Bäcklund symmetry generator

$$Z_{LB} = Q(u, u_x, u_{xx}, \ldots, u_{nx})\frac{\partial}{\partial u}, \quad n > 3. \tag{2.6}$$

Here $L_E[u]$ denotes the linear operator

$$L_E[u] := \frac{\partial E}{\partial u} + \frac{\partial E}{\partial u_t}D_t + \frac{\partial E}{\partial u_x}D_x + \frac{\partial E}{\partial u_{xx}}D_x^2 + \frac{\partial E}{\partial u_{xxx}}D_x^3. \tag{2.7}$$

This leads to

Proposition 1. *Equation*

$$u_t = \frac{P_1(u, u_x, u_{xx})}{\sqrt{u_{xxx}}} + P_2(u, u_x, u_{xx}) \tag{2.8}$$

is symmetry-integrable if and only if $P_1 = \Psi(u_{xx})$ *and* $P_2 = 0$*, so that (2.8) is of the form*

$$u_t = \frac{\Psi(u_{xx})}{\sqrt{u_{xxx}}}, \tag{2.9a}$$

where $\Psi(u_{xx})$ *must satisfy the following condition:*

$$\Psi\frac{d^2\Psi}{du_{xx}^2} - \frac{1}{3}\left(\frac{d\Psi}{du_{xx}}\right)^2 + \frac{3\beta}{4}\Psi^{-2/3} = 0 \tag{2.9b}$$

with β *an arbitrary constant. A hierarchy of symmetry-integrable equations is generated by*

$$u_{t_j} = R^j[u]u_t, \qquad j = 1, 2, 3, \ldots, \tag{2.10}$$

where u_t *is given by (2.9a) and* $R[u]$ *is the following recursion operator*

$$R[u] = G_2D_x^2 + G_1D_x + G_0 + I_1D_x^{-1}\circ\Lambda_1 + I_2D_x^{-1}\circ\Lambda_2 \tag{2.11}$$

with

$$G_2 = \frac{\Psi^{2/3}}{u_{xxx}} \tag{2.12a}$$

$$G_1 = \frac{\Psi^{2/3}}{2}\frac{u_{4x}}{u_{xxx}^2} - \frac{5}{3\Psi^{1/3}}\frac{d\Psi}{du_{xx}} \tag{2.12b}$$

$$G_0 = -\frac{\Psi^{2/3}}{2}\frac{u_{5x}}{u_{xxx}^2} + \frac{3\Psi^{2/3}}{4}\frac{u_{4x}^2}{u_{xxx}^3} + \frac{1}{3\Psi^{1/3}}\frac{d\Psi}{du_{xx}}\frac{u_{4x}}{u_{xxx}} - \frac{8}{9\Psi^{4/3}}\left(\frac{d\Psi}{du_{xx}}\right)^2 u_{xxx}$$

$$+\frac{5}{3\Psi^{1/3}}\frac{d^2\Psi}{du_{xx}^2}u_{xxx} + k_0 \tag{2.12c}$$

$$I_1 = \beta \tag{2.12d}$$

$$\Lambda_1 = \frac{u_{4x}}{\Psi^2} - \frac{2}{\Psi^3}\frac{d\Psi}{du_{xx}} \tag{2.12e}$$

$$I_2 = \frac{\Psi}{(u_{xxx})^{1/2}} \tag{2.12f}$$

$$\Lambda_2 = \frac{1}{2\Psi^{1/3}}\frac{u_{6x}}{u_{xxx}^{3/2}} - \frac{9}{4\Psi^{1/3}}\frac{u_{4x}u_{5x}}{u_{xxx}^{5/2}} - \frac{1}{2\Psi^{4/3}}\frac{d\Psi}{du_{xx}}\frac{u_{5x}}{u_{xxx}^{1/2}} - \frac{13}{6\Psi^{4/3}}\frac{d^2\Psi}{du_{xx}^2}u_{xxx}^{1/2}u_{4x}$$

$$+\frac{14}{9\Psi^{7/3}}u_{xx}^2 u_{xxx}^{1/2}u_{4x} + \frac{15}{8\Psi^{1/3}}\frac{u_{4x}^3}{u_{xxx}^{7/2}} + \frac{5}{12\psi^{4/3}}\frac{d\Psi}{du_{xx}}\frac{u_{4x}^2}{u_{xxx}^{3/2}} + \frac{76}{27\Psi^{7/3}}\frac{d\Psi}{du_{xx}}\frac{d^2\Psi}{du_{xx}^2}u_{xxx}^{5/2}$$

$$+\frac{41}{27\Psi^{4/3}}\frac{d^3\Psi}{du_{xx}^3}u_{xxx}^{5/2} - \frac{404}{243\Psi^{10/3}}\left(\frac{d\Psi}{du_{xx}}\right)^3 u_{xxx}^{5/2} + \frac{2\beta}{\Psi^4}\frac{d\Psi}{du_{xx}}u_{xxx}^{5/2}$$

$$-\frac{\beta}{\Psi^3}u_{xxx}^{1/2}u_{4x}. \tag{2.12g}$$

Here k_0 is an arbitrary constant, and $\Psi(u_{xx})$ and β must satisfy condition (2.9b).

Proof: Using the Ansatz (2.11) with the standard recursion operator condition (see for example [4])

$$[L_E[u],\, R[u]] = D_t R[u]\Big|_{E=0}, \tag{2.13}$$

where E defines the equation (2.8) and L_E the linear operator (2.7), we find that $P_1 = \Psi(u_{xx})$, $P_2 = 0$ with Ψ and β that satisfy equation (2.9b). This establishes that equation (2.9a) is symmetry-integrable under these conditions. □

Applying Proposition 1 we find that the second member of the hierarchy (2.10) is given by the following 5th-order equation:

$$u_{t_1} = -\frac{\Psi^{5/3}}{2}\frac{u_{5x}}{u_{xxx}^{5/2}} + \frac{5\Psi^{5/3}}{8}\frac{u_{4x}^2}{u_{xxx}^{7/2}} + \frac{5\Psi^{2/3}}{6}\frac{d\Psi}{du_{xx}}\frac{u_{4x}}{u_{xxx}^{3/2}} + \frac{4\Psi^{2/3}}{3}\frac{d^2\Psi}{du_{xx}^2}u_{xxx}^{1/2}$$

$$-\frac{23}{18\Psi^{1/3}}\left(\frac{d\Psi}{du_{xx}}\right)^2 u_{xxx}^{1/2} + \frac{\beta}{\Psi}u_{xxx}^{1/2} + \frac{k_0\Psi}{u_{xxx}^{1/2}} \tag{2.14}$$

For solutions of (2.9b) we state the following

Proposition 2. *Equation (2.9b), viz.*

$$\Psi \frac{d^2\Psi}{du_{xx}^2} - \frac{1}{3}\left(\frac{d\Psi}{du_{xx}}\right)^2 + \frac{3\beta}{4}\Psi^{-2/3} = 0,$$

admits the following three solutions:

1. *For any constant β the general solution for (2.9b) is*

$$\Psi(u_{xx}) = (2c_0)^{-3/8}\left[(u_{xx}+c)^2 - \beta c_0\right]^{3/4}, \tag{2.15}$$

where c and c_0 are arbitrary constants with $c_0 \neq 0$.

2. *For $\beta \neq 0$ an additional solution for (2.9b), besides (2.15), is*

$$\Psi(u_{xx}) = (2\beta)^{3/8}(u_{xx}+c)^{3/4}, \tag{2.16}$$

where c is an arbitrary constant.

3. *For $\beta = 0$ an additional solution for (2.9b), besides (2.15) with $\beta = 0$, is*

$$\Psi(u_{xx}) = c, \tag{2.17}$$

where c is an arbitrary constant.

Remark 1: *Equation (2.9b) can easily be linearised in a 1st-order ordinary differential equation via a Bernoulli equation.*

Applying Proposition 1 and Proposition 2 we identify the following three cases of fully-nonlinear 3rd-order symmetry-integrable equations:

Case 1.1: Consider the solution (2.15) whereby we let $c_0 = 1/2$ without loss of generality. This leads to the symmetry-integrable equation

$$u_t = \frac{\left[(u_{xx}+c)^2 - \dfrac{\beta}{2}\right]^{3/4}}{\sqrt{u_{xxx}}}, \tag{2.18}$$

and by Proposition 1 its recursion operator is given by (2.11) with

$$\Psi(u_{xx}) = \left[(u_{xx}+c)^2 - \frac{\beta}{2}\right]^{3/4} \tag{2.19}$$

for any β and any constant c. By introducing the new variable $v(x,t) = u_{xx}$, equation (2.18) takes the form

$$v_t = -\frac{K^{3/4}}{2v_x^{1/2}}S[v] - \frac{3v_x^{3/2}}{2K^{5/4}}\left(\frac{v^2}{2} + cv + \frac{c^2}{2}\right) + \frac{v_x^{3/2}}{K^{1/4}}, \tag{2.20}$$

where

$$K = (v + c)^2 - \frac{\beta}{2}. \tag{2.21}$$

We note that, for the case $c = 0$ and $\beta = 0$, equation (2.18) becomes

$$u_t = \frac{u_{xx}^{3/2}}{\sqrt{u_{xxx}}}, \tag{2.22}$$

and equation (2.20) becomes

$$v_t = -\frac{1}{2}\left(\frac{v^3}{v_x}\right)^{1/2} S[v] + \frac{3}{4}\left(\frac{v_x^3}{v}\right)^{1/2}. \tag{2.23}$$

In order to establish the semi-linearisation of equation (2.18) we need to consider the following two subcases, which distinguishes between the cases where β is zero or not:

Subcase 1.1a: Let $\beta = 0$. Equation (2.18) then takes the following form:

$$u_t = \frac{(u_{xx} + c)^{3/2}}{\sqrt{u_{xxx}}}. \tag{2.24}$$

By performing a multipotentialisation of (2.24) we obtain

$$V_t = V^3 V_{xxx} - \frac{3}{2^{4/3}}\frac{V_x}{V}, \tag{2.25}$$

where

$$V(x,t) = -\frac{1}{2^{1/3}}\left(\frac{u_{xx} + c}{u_{xxx}}\right)^{1/2}. \tag{2.26}$$

We prefer not to show here the details of the multipotentialisation that leads to (2.25) but the interested reader can easily verify (2.26). Applying now the generalised hodograph transformation

$$\mathcal{GHT}: \begin{cases} dX = f_1(x,V)dx + f_2(x,V,V_x,V_{xx})dt \\ dT = dt \\ U(X,T) = x \end{cases} \tag{2.27}$$

we obtain the following semilinear equation

$$\boxed{U_T = U_X S[U] + \frac{3}{2^{7/3}}\frac{1}{U_X}}, \tag{2.28}$$

where

$$f_1 = \frac{1}{V} \tag{2.29a}$$

$$f_2 = -VV_{xx} + \frac{1}{2}V_x^3 - \frac{3}{2^{7/3}}\frac{1}{V^2} \tag{2.29b}$$

with V given by (2.26). A 2nd-order recursion operator for equation (2.28) has been reported in [12].

Subcase 1.1b: Let $\beta \neq 0$. Applying the procedure of multipotentialisation on equation (2.18), *viz.*

$$u_t = \frac{\left[(u_{xx}+c)^2 - \dfrac{\beta}{2}\right]^{3/4}}{\sqrt{u_{xxx}}},$$

we obtain

$$Q_t = \frac{1}{Q_x^2}S[Q] - \frac{3}{2^{-7/3}}Q_x^2, \tag{2.30}$$

where

$$Q_x = -\frac{2^{1/3}u_{xxx}}{\left[(u_{xx}+c)^2 + \dfrac{\beta}{2}\right]^{1/4}}. \tag{2.31}$$

Furthermore equation (2.30) maps to the semilinear equation

$$\boxed{U_T = U_X S[U] - \frac{3}{2^{7/3}}\frac{1}{U_X}} \tag{2.32}$$

by the standard hodograph transformation

$$\mathcal{HT}: \begin{cases} X = Q(x,t) \\ T = t \\ U(X,T) = x \end{cases} \tag{2.33}$$

which completes the semi-linearisation of (2.18). A 2nd-order recursion operator for equation (2.32) has been reported in [3]

Case 1.2: Consider the solution (2.16) where we let $\beta = 1/2$, which is for simplicity but without loss of generality. This leads to the symmetry-integrable equation

$$u_t = \frac{(u_{xx}+c)^{3/4}}{\sqrt{u_{xxx}}}, \tag{2.34}$$

whereby its recursion operator is given by (2.11) with $\beta = 1/2$, c an arbitrary constant, and

$$\Psi(u_{xx}) = (u_{xx}+c)^{3/4}. \tag{2.35}$$

By introducing the new variable $v(x,t)=u_{xx}$, equation (2.34) takes the form

$$v_t=-\frac{1}{2}\left[\frac{(v+c)^{3/2}}{v_x}\right]^{1/2}S[v]-\frac{3}{16}\left[\frac{v_x^3}{(v+c)^{5/2}}\right]^{1/2}. \tag{2.36}$$

For a semi-linearisation we apply the procedure of multipotentialisation and find that (2.34) maps to

$$V_t=\frac{S[V]}{V_x^2}, \tag{2.37}$$

where

$$V_x^2=-\frac{u_{xxx}}{2^{2/3}(u_{xx}+c)^{1/2}}\ . \tag{2.38}$$

Furthermore, it is well-known that equation (2.37) maps to the Schwarzian KdV

$$\boxed{U_T=U_XS[U]} \tag{2.39}$$

by the standard hodograph transformation

$$\mathcal{HT}:\begin{cases} X=V(x,t)\\ T=t\\ U(X,T)=x.\end{cases} \tag{2.40}$$

A 2nd-order recursion operator for Schwarzian KdV (2.39) is well known and has for example been reported in [3]

Case 1.3: The symmetry-integrable equation

$$u_t=\frac{1}{\sqrt{u_{xxx}}}, \tag{2.41}$$

admits the recursion operator (2.11) with $\Psi=1$ and $\beta=0$. Letting

$$W(x,t)=u_{xxx} \tag{2.42}$$

we obtain

$$W_t=\frac{W_{xxx}}{W^{3/2}}-\frac{9}{2}\frac{W_xW_{xx}}{W^{5/2}}+\frac{15}{4}\frac{W_x^3}{W^{7/2}}. \tag{2.43}$$

We recall [8] that (2.43) is also obtained from

$$\tilde{u}_t=-2\frac{\tilde{u}_x}{\sqrt{S[\tilde{u}]}} \tag{2.44}$$

where

$$W(x,t) = S[\tilde{u}] \tag{2.45}$$

so that the relation between $u(x,t)$ and $\tilde{u}(x,t)$ is

$$u_{xxx} = S[\tilde{u}]. \tag{2.46}$$

In [8] we established that (2.44) maps to the Schwarzian KdV with a hodograph-type transformation. Using this result, we find that equation (2.41) maps to the Schwarzian KdV (2.39), *viz.*

$$\boxed{U_T = U_X S[U]}$$

under the following change of variables:

$$\mathcal{HT}: \begin{cases} X = \displaystyle\int \sqrt{u_{xxx}}\, dx \\ T = t \\ U(X,T) = x. \end{cases} \tag{2.47}$$

For the sake of completeness, we give the recursion operator for the equation

$$U_T = U_X S[U] + \frac{\lambda}{U_X}, \tag{2.48}$$

which includes the semilinear equations (2.28) for $\lambda = 3 \cdot 2^{-7/3}$, (2.32) for $\lambda = -2 \cdot 3^{-7/3}$, and (2.39) for $\lambda = 0$, namely [3]

$$\begin{aligned} R[U] = {} & D_X^2 - \frac{2U_{XX}}{U_X} D_X + \frac{U_{XXX}}{U_X} - \frac{U_{XX}^2}{U_X^2} - \frac{2\lambda}{3U_X^2} + k_0 - \frac{8\lambda}{3} D_X^{-1} \circ \frac{U_{XX}}{U_X^3} \\ & - U_X D_X^{-1} \circ \left(\frac{U_{4X}}{U_X^2} - \frac{4U_{XX}U_{XXX}}{U_X^3} + \frac{3U_{XX}^3}{U_X^4} - \frac{2\lambda U_{XX}}{U_X^4} \right). \end{aligned} \tag{2.49}$$

3 A fifth-order fully-nonlinear evolution equation

When seeking symmetry-integrable 5th-order evolution equations, the following statement [7] is useful to determine the possible nonlinearities in the highest derivative u_{5x} [8]:

Lemma 2: *If a 5th-order evolution equation of the form*

$$u_t = F(x,t,u,u_x,\ldots,u_{5x}) \tag{3.1}$$

is symmetry-integrable for a given function F then this function must satisfy the following condition:

$$5\frac{\partial F}{\partial u_{5x}} \frac{\partial^3 F}{\partial u_{5x}^3} - 8 \left(\frac{\partial^2 F}{\partial u_{5x}^2} \right)^2 = 0. \tag{3.2}$$

The general solution of (3.2) is

$$F(x,t,u,u_x,\ldots,u_{5x}) = \frac{F_1}{(u_{5x}+F_2)^{2/3}} + F_3, \tag{3.3}$$

where $F_j = F_j(x,t,u,u_x,\ldots,u_{4x})$ $(j=1,2,3)$ are arbitrary and smooth functions of their arguments. In the current paper we consider the special case where $F_1 = 1$, $F_2 = F_3 = 0$. Applying now the Lie-Bäcklund symmetry invariance condition (2.5) leads to the following

Proposition 3. *The fully-nonlinear 5th-order equation*

$$u_t = \frac{1}{u_{5x}^{2/3}} \tag{3.4}$$

is symmetry-integrable and admits Lie-Bäcklund symmetries of order $1+6n$ *and order* $5+6n$*, for every natural number* n*, so the symmetries are of order* $\{7, 11, 13, 17, 19, \ldots\}$.

Let us give the first two equations in this hierarchy explicitly: The symmetry-integrable evolution equation associated with the 7th-order Lie-Bäcklund symmetry of equation (3.4) is

$$u_t = \frac{u_{7x}}{u_{5x}^{7/3}} - \frac{7}{6}\frac{u_{6x}^2}{u_{5x}^{10/3}} \tag{3.5}$$

and the associated 11th-order symmetry-integrable equation is

$$u_t = \frac{1}{u_{5x}^{11/3}}\left(u_{11x} - \frac{11u_{6x}u_{10x}}{u_{5x}} - \frac{77u_{7x}u_{9x}}{3u_{5x}} + \frac{682u_{6x}^2u_{9x}}{9u_{5x}^2} - \frac{33u_{8x}^2}{2u_{5x}} - \frac{374u_{6x}^3u_{8x}}{u_{5x}^3}\right.$$
$$\left. + \frac{286u_{6x}u_{7x}u_{8x}}{u_{5x}^2} + \frac{1892u_{7x}^3}{27u_{5x}^2} - \frac{22066u_{6x}^2u_{7x}^2}{27u_{5x}^3} + \frac{107525u_{6x}^4u_{7x}}{81u_{5x}^4} - \frac{752675u_{6x}^6}{1458u_{5x}^5}\right). \tag{3.6}$$

We now turn to the task to semi-linearise equation (3.4): We apply the multipotentialisation procedure systematically on equation (3.4), which leads to the quasilinear equation

$$w_t = \frac{w_{5x}}{w_x^5} - \frac{10w_{xx}w_{4x}}{w_x^6} - \frac{10w_{xxx}^2}{w_x^6} + \frac{60w_{xx}^2w_{xxx}}{w_x^7} - \frac{45w_{xx}^4}{w_x^8}, \tag{3.7}$$

where

$$w_x^3 = -\frac{3}{2}u_{5x}. \tag{3.8}$$

Equation (3.7) then takes the quasilinear form

$$V_t = V^5V_{5x} + 5V^4V_xV_{4x} + 10V^4V_{xx}V_{xxx}, \tag{3.9}$$

with the following change of variables:

$$V(x,t) = \frac{1}{w_x}. \tag{3.10}$$

Using generalised hodograph transformations we conclude that the fully-nonlinear 5th-order equation (3.4) maps to the semilinear 5th-order equation

$$\boxed{U_T = U_X\left(\frac{\partial^2 S[U]}{\partial X^2} + 4S^2[U]\right)} \tag{3.11}$$

by the generalised hodograph transformation

$$\mathcal{GHT}: \begin{cases} dX = f_1(x,V)dx + f_2(x,V,V_x,V_{xx},V_{xxx},V_{4x})dt \\ dT = dt \\ U(X,T) = x, \end{cases} \tag{3.12}$$

where

$$f_1 = \frac{1}{V} \tag{3.13a}$$

$$f_2 = -V^3V_{4x} - 2V^2V_xV_{xxx} - 4V^2V_{xx}^2 + 4VV_x^2V_{xx} - V_x^4 \tag{3.13b}$$

and

$$V(x,t) = -\left(\frac{2}{3}\right)^{1/5}\frac{1}{u_{5x}^{1/3}}. \tag{3.13c}$$

Some remarks are in order:

Remark 2: *We find that equation (3.4) does not admit a recursion operator of the form*

$$R[u] = \sum_{j=0}^{6} G_jD_x^j + \sum_{j=1}^{3} I_jD_x^{-1}\circ\Lambda_j, \tag{3.14}$$

whereby we assumed the following dependencies: for the integrating factors $\Lambda_k = \Lambda_k(u,u_x, u_{xx},\ldots,u_{12x})$; for the symmetry coefficients $I_j = I_j(u,u_x,u_{xx},\ldots,u_{5x})$, and for the coefficients of D_x^j we assumed that there are no restrictions in the number of derivatives, i.e. $G_j = G_j(u,u_x, u_{xx},\ldots)$. Since a recursion operator of this form does not exist, we expect that the recursion operator is of a nonlocal type. We will not explore this further here.

Remark 3: *The semilinear equation (3.11) is the third potentialisation (see equation (2.6) [4]) in the chain of multipotentialisations of the Kupershmidt equation*

$$K_t = K_{5x} - 5K_xK_{xxx} - 5K_{xx}^2 - 5K^2K_{xxx} - 20KK_xK_{xx} - 5K_x^3 + 5K^4K_x. \tag{3.15}$$

A recursion operator of order six has been reported for equation (3.11) in [4], where some solution-generating formulas are also given for (3.11).

Remark 4: *Let us furthermore point out that equation (3.9), viz.*

$$V_t = V^5 V_{5x} + 5V^4 V_x V_{4x} + 10V^4 V_{xx} V_{xxx},$$

just like equation (3.4), does not admit a recursion operator of the form (3.18) although (3.9) is of course symmetry-integrable and hence admits Lie-Bäcklund symmetries of the same order as equation (3.7) which generates a hierarchy of symmetry-integrable equations. For example, the 7th-order symmetry-integrable equation in this hierarchy is

$$V_t = V^7 V_{7x} + 14V^6 V_x V_{6x} + 49V^5 V_x^2 V_{5x} + 28V^6 V_{xx} V_{5x} + 35V^4 V_x^3 V_{4x} + 140V^5 V_x V_{xx} V_{4x}$$
$$+35V^6 V_{xxx} V_{4x} + 70V^5 V_x V_{xxx}^2 + 70V^5 V_{xx}^2 V_{xxx} + 70V^4 V_x^2 V_{xx} V_{xxx}. \tag{3.16}$$

Furthermore, the 7th-order equation (3.5) maps to the 7th-order equation (3.16) by the following change of variable:

$$V(x,t) = u_{5x}^{-1/3}. \tag{3.17}$$

Remark 5: *Quite remarkably, equation (3.7), viz.*

$$w_t = \frac{w_{5x}}{w_x^5} - \frac{10 w_{xx} w_{4x}}{w_x^6} - \frac{10 w_{xxx}^2}{w_x^6} + \frac{60 w_{xx}^2 w_{xxx}}{w_x^7} - \frac{45 w_{xx}^4}{w_x^8},$$

admits the following 6th-order recursion operator:

$$R[w] = \sum_{j=0}^{6} G_j D_x^j + 2w_t D_x^{-1} \circ \Lambda_1 + 2D_x^{-1} \circ \Lambda_2, \tag{3.18}$$

where

$$G_6 = \frac{1}{w_x^6} \tag{3.19a}$$

$$G_5 = -\frac{15 w_{xx}}{w_x^7} \tag{3.19b}$$

$$G_4 = -\frac{32 w_{xxx}}{w_x^7} + \frac{123 w_{xx}^2}{w_x^8} \tag{3.19c}$$

$$G_3 = -\frac{27 w_{4x}}{w_x^7} + \frac{354 w_{xx} w_{xxx}}{w_x^8} - \frac{600 w_{xx}^3}{w_x^9} \tag{3.19d}$$

$$G_2 = -\frac{19 w_{5x}}{w_x^7} + \frac{271 w_{xx} w_{4x}}{w_x^8} + \frac{232 w_{xxx}^2}{w_x^8} - \frac{2040 w_{xx}^2 w_{xxx}}{w_x^9} + \frac{1980 w_{xx}^4}{w_x^{10}} \tag{3.19e}$$

$$G_1 = -\frac{4 w_{6x}}{w_x^7} + \frac{79 w_{xx} w_{5x}}{w_x^8} + \frac{149 w_{xxx} w_{4x}}{w_x^8} - \frac{754 w_{xx}^2 w_{4x}}{w_x^9} + \frac{4440 w_{xx}^3 w_{xxx}}{w_x^{10}}$$
$$+\frac{1126 w_{xx} w_{xxx}^2}{w_x^9} - \frac{3060 w_{xx}^5}{w_x^{11}} \tag{3.19f}$$

$$G_0 = -\frac{3w_{7x}}{w_x^7} + \frac{63w_{xx}w_{6x}}{w_x^8} - \frac{690w_{xx}^2 w_{5x}}{w_x^9} + \frac{145w_{xxx}w_{5x}}{w_x^8} - \frac{2494w_{xx}w_{xxx}w_{4x}}{w_x^9}$$

$$+\frac{4794w_{xx}^3 w_{4x}}{w_x^{10}} + \frac{87w_{4x}^2}{w_x^8} - \frac{632w_{xxx}^3}{w_x^9} - \frac{22680w_{xx}^4 w_{xxx}}{w_x^{11}} + \frac{10326w_{xx}^2 w_{xxx}^2}{w_x^{10}}$$

$$+\frac{11340w_{xx}^6}{w_x^{12}} + k_0 \tag{3.19g}$$

$$\Lambda_1 = -\frac{w_{4x}}{w_x^3} + \frac{6w_{xx}w_{xxx}}{w_x^4} - \frac{6w_{xx}^3}{w_x^5} \tag{3.19h}$$

$$\Lambda_2 = \frac{w_{8x}}{w_x^7} - \frac{28w_{xx}w_{7x}}{w_x^8} + \frac{396w_{xx}^2 w_{6x}}{w_x^9} - \frac{68w_{xxx}w_{6x}}{w_x^8} - \frac{106u_{4x}u_{5x}}{w_x^8} - \frac{3636w_{xx}^3 w_{5x}}{w_x^{10}}$$

$$+\frac{1656w_{xx}w_{xxx}w_{5x}}{w_x^9} + \frac{1060w_{xx}w_{4x}^2}{w_x^9} + \frac{23265w_{xx}^4 w_{4x}}{w_x^{11}} + \frac{1420w_{xxx}^2 w_{4x}}{w_x^9}$$

$$-\frac{18900w_{xx}^2 w_{xxx}w_{4x}}{w_x^{10}} - \frac{8520w_{xx}w_{xxx}^3}{w_x^{10}} + \frac{63360w_{xx}^3 w_{xxx}^2}{w_x^{11}} - \frac{103950w_{xx}^5 w_{xxx}}{w_x^{12}}$$

$$+\frac{44550w_{xx}^7}{w_x^{13}}. \tag{3.19i}$$

Here k_0 is an arbitrary constant. It is interesting to note that acting $R[w]$ on w_x does not result in a 7th-order symmetry for (3.7). Instead, $R[w]$ maps the x-translation symmetry to itself, i.e.,

$$R[w]\, w_x = k_0 w_x. \tag{3.20}$$

Nevertheless, we find that equation (3.7) does admit a 7th-order Lie-Bäcklund symmetry and hence a corresponding 7th-order symmetry-integrable equation, namely

$$w_t = \frac{w_{7x}}{w_x^8} - \frac{21w_{xx}w_{6x}}{w_x^8} - \frac{49w_{xxx}w_{5x}}{w_x^8} + \frac{231w_{xx}^2 w_{5x}}{w_x^9} - \frac{28w_{4x}^2}{w_x^8} + \frac{826w_{xx}w_{xxx}w_{4x}}{w_x^9}$$

$$-\frac{1596w_{xx}^3 w_{4x}}{w_x^{10}} + \frac{644w_{xxx}^3}{3w_x^9} - \frac{3444w_{xx}^2 w_{xxx}^2}{w_x^{10}} + \frac{7560w_{xx}^4 w_{xxx}}{w_x^{11}} - \frac{3780w_{xx}^6}{w_x^{12}}. \tag{3.21}$$

With the change of variable

$$W(x,t) = \frac{1}{w_x} \tag{3.22}$$

equation (3.21) takes the form

$$W_t = W^8 W_{7x} - 6W^7 W_x W_{6x} + 21W^6 W_x W_{6x} - 42W^7 W_{xx} W_{5x} + 70W^6 W_{xx} W_{5x}$$

$$+7W^5 W_x^2 W_{5x} + 30W^6 W_x^2 W_{5x} - 70W^7 W_{xxx} W_{4x} + 105W^6 W_{xxx} W_{4x}$$

$$+390W^6W_xW_{xx}W_{4x} - 280W^5W_xW_{xx}W_{4x} - 120W^5W_x^3W_{4x} + 245W^4W_x^3W_{4x}$$
$$+260W^6W_xW_{xxx}^2 - 210W^5W_xW_{xxx}^2 + 630W^6W_{xx}^2W_{xxx} - 540W^5W_{xx}^2W_{xxx}$$
$$-2160W^5W_x^2W_{xx}W_{xxx} + 2510W^4W_x^2W_{xx}W_{xxx} + 360W^4W_x^4W_{xxx}$$
$$-760W^3W_x^4W_{xxx} - 1800W^5W_xW_{xx}^3 + 1830W^4W_xW_{xx}^3 + 3960W^4W_x^3W_{xx}^2$$
$$-4800W^3W_x^3W_{xx}^2 - 720W^3W_x^5W_{xx} + 2280W^2W_x^5W_{xx} - 720W^2W_x^7. \qquad (3.23)$$

4 Concluding remarks

In this paper we have introduced the following set of fully-nonlinear evolution equations:

$$u_t = \frac{\left[(u_{xx}+c)^2 - \dfrac{\beta}{2}\right]^{3/4}}{\sqrt{u_{xxx}}}, \quad u_t = \frac{(u_{xx}+c)^{3/4}}{\sqrt{u_{xxx}}}, \quad u_t = \frac{1}{\sqrt{u_{xxx}}}, \quad u_t = \frac{1}{u_{5x}^{2/3}}. \qquad (4.1)$$

We have established that this set of equations is symmetry-integrable in the sense that the equations admit local Lie-Bäcklund symmetries and that the equations can be mapped to known semilinear integrable equations using a combination of multipotentialisations, hodograph transformations and generalised hodegraph transformations. The 3rd-order fully-nonlinear equations in the set (4.1) admit recursion operators of the standard 2nd-order form which can be applied to generated hierarchies of symmetry-integrable quasilinear equations. On the other hand, according to our calculations, we find that the fully-nonlinear 5th-order equation listed in (4.1) does not admit a standard recursion operator of order six or less. It is therefore an open problem to find a recursion operator for this fully-nonlinear 5th-order equation, which we expect to be nonlocal.

Besides the fully-nonlinear equations listed in (4.1) we have obtained a set of quasilinear 5th-order symmetry-integrable equations, namely the equation (2.14) under the condition (2.9b), as well as the equations (3.7) and (3.9). Furthermore, we have obtained a set of 7th-order quasilinear symmetry-integrable equations, namely (3.5), (3.16), (3.21) and (3.23). These equations result naturally from the multipotentialisations of the fully-nonlinear equations (4.1) and as members of the symmetry-integrable hierarchies.

Finally we should mention that the complete classification of fully-nonlinear symmetry-integrable evolution equations of 3rd and higher order is an ongoing project and the examples studied here is an addition to the results that have already been reported in [5], [7] and [8].

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