On 2nd-order fully-nonlinear equations with links to 3rd-order fully-nonlinear equations

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Abstract: We derive the general conditions for fully-nonlinear symmetry-integrable second-order evolution equations and their first-order recursion operators. We then make use of the established Propositions to find a link between a class of fully-nonlinear third-order symmetry integrable evolution equations and fully-nonlinear second-order symmetry-integrable evolution equations.

1 Introduction

We recently reported a class of third-order fully-nonlinear symmetry-integrable evolution equations in 1+1 dimensions with rational nonlinearities in their highest derivative [2]. For this class of equations we have furthermore reported all the potentialisations and some multipotentialisations in [3]. It is interesting to note that some of these potential equations are in fact members of hierarchies of fully-nonlinear second-order evolution equations. In the current article we identify all those potential equations and give the explicit recursion operators that generate those hierarchies. In particular, the fully-nonlinear third-order symmetry-integrable evolution equations that are relevant here are

\[ u_t = \frac{u_{xx}^3}{u_{xxx}} \left( \lambda_1 + \lambda_2 u_{xx} \right)^3 \]
and
\[ u_t = \frac{1}{u_{xxx}}, \]
where \( \lambda_1 \) and \( \lambda_2 \) are arbitrary constants. For more details on symmetry-integrable hierarchies and recursion operators, we refer to [4] and [2], and the references therein.

This article is organised as follows: In Section 2 we provide two Propositions by which one are able to identify second-order fully nonlinear equations that admit third-order Lie-Bäcklund symmetries (see Proposition 1) and by which we can obtain a recursion operator for those second-order fully-nonlinear equations (see Proposition 2). In Section 3 we apply the mentioned Propositions to establish links to the two fully-nonlinear equations given above. Finally in Section 4 we make our conclusions and propose some further studies that could be of interest.

2 On fully-nonlinear second-order equations

We consider a second-order equation of the form
\[ u_t = F(u, u_x, u_{xx}) \]  
(2.1)
where the non-constant function \( F \) is to be determined such that (2.1) is fully nonlinear in \( u_{xx} \) and admits a third-order Lie-Bäcklund symmetry generated by \( Z = \eta[u] \frac{\partial}{\partial u} \); hence a recursion operator \( R \) that generates this third-order symmetry and consequently an infinite number of higher-order Lie-Bäcklund symmetries. The hierarchy of symmetry-integrable evolution equations is then
\[ u_{tm} = R^m[u] u_t, \quad m = 0, 1, 2, \ldots. \]  
(2.2)

Our first step is to find the general form of \( F \) and \( \eta \) to satisfy this requirement as a necessary condition. This is given by

**Proposition 1.** Consider the class of second-order evolution equations of the form (2.1) viz
\[ u_t = F(u, u_x, u_{xx}). \]

Under the assumption that (2.1) is symmetry-integrable, the following statements must be true:

1. The function \( F \) in (2.1) satisfies the condition
\[ \frac{\partial^3 F}{\partial u_{xx}^3} \left( \frac{\partial F}{\partial u_{xx}} \right)^{-1} - \frac{3}{2} \left( \frac{\partial^2 F}{\partial u_{xx}^2} \right)^2 \left( \frac{\partial F}{\partial u_{xx}} \right)^{-2} = 0 \]  
(2.3)
with general solution
\[ F(u, u_x, u_{xx}) = -\frac{A(u, u_x)}{u_{xx} + B(u, u_x)} + C(u, u_x), \]  
(2.4)
where \( A, B \) and \( C \) are arbitrary functions of their arguments. The quasi-linear and semi-linear form of (2.1) follow from the singular solution of (2.3), namely the solution for which \( \partial^2 F/\partial u_{xx}^2 = 0 \). [Note that (2.3) is the Schwarzian derivative in \( F \).]
2. Any third-order Lie-Bäcklund symmetry generator

\[ Z := \eta(x, u, u_x, u_{xx}, u_{xxx}) \frac{\partial}{\partial u} \]  

for (2.4) is of the general form

\[ \eta(x, u, u_x, u_{xx}, u_{xxx}) = \frac{c_0 A^{3/2} u_{xxx}}{(u_{xx} + B)^3} - \frac{c_0 A^{3/2}}{(u_{xx} + B)^3} \left( B \frac{\partial B}{\partial u_x} - u_x \frac{\partial B}{\partial u} \right) \]

\[ + \frac{3c_0 A^{1/2}}{2(u_{xx} + B)^2} \left( B \frac{\partial A}{\partial u_x} - u_x \frac{\partial A}{\partial u} \right) - \frac{f_1}{u_{xx} + B} + f_2, \]  

whereby \( c_0 \) denotes an arbitrary but non-zero constant, and the functions \( A = A(u, u_x) \), \( B = B(u, u_x) \), \( C = C(u, u_x) \), \( f_1 = f_1(x, u, u_x) \) and \( f_2 = f_2(x, u, u_x) \) need to be determined such that the invariance condition

\[ L_E \eta(x, u, u_x, u_{xx}, u_{xxx}) \bigg|_{E=0} = 0 \]  

is satisfied. Here \( E := u_t - F(u, u_x, u_{xx}) \) and \( L_E \) denotes the linear operator

\[ L_E[u] := \frac{\partial E}{\partial u} + \frac{\partial E}{\partial u_t} D_t + \frac{\partial E}{\partial u_x} D_x + \frac{\partial E}{\partial u_{xx}} D_x^2 + \frac{\partial E}{\partial u_{xxx}} D_x^3 \]  

We now seek 1st-order recursion operators \( R \) for every equation that admits a third-order Lie-Bäcklund symmetry, which, according to Proposition 1, are equations of the general form (2.4). We consider both differential recursion operators

\[ R[u] = G_1(u, u_x, u_{xx}) D_x + G_0(u, u_x) \]  

as well as integro-differential recursion operators

\[ R[u] = G_1(u, u_x, u_{xx}) D_x + G_0(u, u_x) + I(u, u_x, u_{xx}) D_x^{-1} \circ \Lambda(u, u_x, u_{xx}). \]  

Applying the standard condition for \( R \) (see for example [1]), namely

\[ [L_E, R[u]] = D_t R \bigg|_{E=0} \]  

we obtain the following

**Proposition 2.** For finding recursion operators of (2.4) viz.

\[ u_t = -\frac{A(u, u_x)}{u_{xx} + B(u, u_x)} + C(u, u_x). \]

we distinguish between the following four cases:
1. With the assumption \( A = F_1(u_x) \) and \( B(u, u_x) = 0 \) in (2.4), it follows that

\[
\begin{align*}
    u_t = & -\frac{F_1(u_x)}{u_{xx}} + u \left( \frac{a_1}{u_x^2 P(u_x)} - \frac{F_1(u_x)}{u_x P(u_x) du_x} + \frac{1}{2u_x} \frac{dF_1}{du} \right) \\
    & + \frac{a_0 \sqrt{F_1(u_x)}}{P(u_x)} + H(u_x)
\end{align*}
\]

(2.12)

admits a recursion operator of the form (2.10) with

\[
\begin{align*}
    G_1(u_x, u_{xx}) &= \frac{k_1 \sqrt{F_1(u_x)}}{u_{xx}} \quad \text{(2.13a)} \\
    G_0(u_x) &= \frac{k_1 \sqrt{F_1(u_x)}}{2P(u_x)} \frac{dP}{du_x} - \frac{k_1 \sqrt{F_1(u_x)}}{2u_x} - \frac{a_1 k_1}{2u_x P(u_x)} + k_0 \quad \text{(2.13b)} \\
    I(u_x) &= \alpha u_x \quad \text{(2.13c)} \\
    \Lambda(u_x, u_{xx}) &= \frac{P(u_x) u_{xx}}{\sqrt{F_1(u_x)}} \quad \text{(2.13d)}
\end{align*}
\]

where \( a_0, a_1 \) and \( k_0 \) are arbitrary constants, \( k_1 \) and \( \alpha \) are arbitrary non-zero constants, \( H \) is an arbitrary function of \( u_x \) and, furthermore, \( F_1 \) and \( P \) must satisfy the following condition:

\[
\begin{align*}
    0 &= k_1 u_x^2 P^2 \left( u_x \frac{dP}{du_x} + P \right) \frac{d^2 F_1}{du_x^2} - k_1 u_x P \left[ -3u_x^2 P \frac{d^2 P}{du_x^2} + 2u_x \left( \frac{dP}{du_x} \right)^2 - 2u_x P \frac{dP}{du_x} 
    \right. \\
    & + 2P^2 \left. \frac{dF_1}{du_x} + 2k_1 \left( u_x^3 P^2 \frac{dP}{du_x} - 2u_x^2 P \frac{d^2 P}{du_x^2} + u_x^2 P \frac{d^2 P}{du_x^2} + u_x \left( \frac{dP}{du_x} \right)^3 
    \right)
    \right)
\end{align*}
\]

(2.14)

2. With the assumption \( A = F_1(u_x) \) and \( B = B(u_x) \) in (2.4), it follows that

\[
\begin{align*}
    u_t = & -\frac{F_1(u_x)}{u_{xx} + (b_1 + b_2 u_x) \sqrt{F_1(u_x)}} + \frac{\sqrt{F_1(u_x)}}{b_1 + b_2 u_x} 
\end{align*}
\]

(2.15)

admits a recursion operator of the form (2.10) with

\[
\begin{align*}
    G_1(u_x, u_{xx}) &= \frac{k_1 \sqrt{F_1(u_x)}}{u_{xx} + (b_1 + b_2 u_x) \sqrt{F_1(u_x)}} \quad \text{(2.16a)} \\
    G_0(u_x) &= \frac{k_1 b_2 F_1(u_x)}{u_{xx} + (b_1 + b_2 u_x) \sqrt{F_1(u_x)}} - \frac{k_1 b_2 \sqrt{F_1(u_x)}}{b_1 + b_2 u_x} + k_0 \quad \text{(2.16b)}
\end{align*}
\]
\[ I(u_x) = u_x + \frac{b_1}{b_2} \]  
\[ \Lambda(u_x, u_{xx}) = \alpha b_2 \left( \frac{u_{xx} + (b_1 + b_2 u_x) \sqrt{F_1(u_x)}}{(b_1 + b_2 u_x) \sqrt{F_1(u_x)}} \right), \]

where \( b_1 \) and \( \alpha \) are arbitrary constants, whereas \( b_2 \) and \( k_1 \) are arbitrary non-zero constants.

3. With the assumption \( A = F_1(u_x) \), \( B = 0 \) and \( C = C(u_x) \) in (2.14), it follows that

\[ u_t = -\frac{F_1(u_x)}{u_{xx}} + C(u_x). \]  
\[ (2.17) \]

This leads to two subcases:

3.1. Equation (2.17) admits the recursion operator of the form (2.10) with

\[ G_1(u_x, u_{xx}) = \frac{k_1 \sqrt{F_1(u_x)}}{u_{xx}} \]  
\[ (2.18a) \]

\[ G_0(u_x) = -\frac{k_1}{4 \sqrt{F_1(u_x)}} \frac{dF_1}{du_x} + k_0 \]  
\[ (2.18b) \]

\[ I(u_x) = \alpha \]  
\[ (2.18c) \]

\[ \Lambda(u_x, u_{xx}) = \frac{u_{xx}}{F_1(u_x)}, \]  
\[ (2.18d) \]

where \( k_0 \) is an arbitrary constant, \( k_1 \) and \( \alpha \) are arbitrary non-zero constants, \( C \) is an arbitrary function of \( u_x \), and \( F_1 \) must satisfy the following condition:

\[ k_1 \left( 4F_1^4 \frac{d^3 F_1}{du_x^3} - 6F_1^3 \frac{dF_1}{du_x} \frac{d^2 F_1}{du_x^2} + 3F_1^2 \left( \frac{dF_1}{du_x} \right)^3 \right) + 16\alpha F_1^{5/2} \frac{dF_1}{du_x} = 0. \]  
\[ (2.19) \]

3.2. Equation (2.17) admits the recursion operator of the form (2.10) with

\[ G_1(u_x, u_{xx}) = \frac{k_1 \sqrt{F_1(u_x)}}{u_{xx}} \]  
\[ (2.20a) \]

\[ G_0(u_x) = -\frac{k_1}{4 \sqrt{F_1(u_x)}} \frac{dF_1}{du_x} + k_0 \]  
\[ (2.20b) \]

\[ I(u_x) = \alpha u_x \]  
\[ (2.20c) \]

\[ \Lambda(u_x, u_{xx}) = \frac{u_x u_{xx}}{F_1(u_x)}, \]  
\[ (2.20d) \]

where \( k_0 \) is an arbitrary constant, \( k_1 \) and \( \alpha \) are arbitrary non-zero constants, \( C \) is an arbitrary function of \( u_x \), and \( F_1 \) must satisfy the following condition:

\[ k_1 \left( 4F_1^4 \frac{d^3 F_1}{du_x^3} - 6F_1^3 \frac{dF_1}{du_x} \frac{d^2 F_1}{du_x^2} + 3F_1^2 \left( \frac{dF_1}{du_x} \right)^3 \right) \]
$+16\alpha F_1^{5/2}u_x \left( u_x \frac{dF_1}{du_x} - 4F_1 \right) = 0. \quad (2.21)$

4. The equation

$$u_t = -\frac{F_1(u_x)}{u_{xx} + (b_1 + b_2u_x)\sqrt{F_1(u_x)}} + F_2(u_x) \quad (2.22)$$

admits a recursion operator of the form $(2.9)$ with

$$G_1(u_x) = \frac{k_1\sqrt{F_1}}{u_{xx} + (b_1 + u_xb_2)\sqrt{F_1}} \quad (2.23a)$$

$$G_0(u_x) = \frac{k_1b_2F_1}{u_{xx} + (b_1 + b_2u_x)\sqrt{F_1}} - \frac{k_1b_2}{2}F_2 + k_0, \quad (2.23b)$$

where $b_1$ and $b_2$ are arbitrary constants, $k_1$ is an arbitrary non-zero constant, and $F_1$ and $F_2$ must satisfy the following condition:

$$4F_2^2 \frac{d^3F_1}{du^3_x} - 6F_1 \frac{dF_1}{du_x} \frac{d^2F_1}{du^2_x} + 3 \left( \frac{dF_1}{du_x} \right)^3 - 8F_1^{5/2} \left( (b_1 + b_2u_x) \frac{d^3F_2}{du^3_x} \right)$$

$$+ 3b_2 \frac{d^2F_2}{du^2_x} = 0. \quad (2.24)$$

3 Fully-nonlinear second-order equations related to fully-nonlinear third-order equations

We now apply Proposition 1 and Proposition 2 to identify fully-nonlinear second-order equations that are related to fully-nonlinear third-order equations. For this purpose we consider two main cases:

**Case 1:** Consider the fully-nonlinear symmetry-integrable equation \[ (3.2) \]

$$u_t = \frac{u_{xxx}^3}{u_{xx}} \left( \lambda_1 + \lambda_2u_{xx} \right)^3, \quad (3.1)$$

where $\lambda_1$ and $\lambda_2$ are arbitrary constants. For the potentialisations of this equation we need to distinguish different subcases that depend on the constants $\lambda_1$ and $\lambda_2$.

**Subcase 1.1:** Consider equation $(3.1)$, with $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Then $(3.1)$ admits the zero-order potentialisation \[ (3.2) \]

$$v_t = \frac{\lambda_1^3}{4} \left( 1 + \lambda_1^2\lambda_2v_x^2 \right)^{3/2} \left( \frac{v_{xx}^3v_{xxx}}{v_x^3} - \frac{3v_x^2}{v_{xx}} \right), \quad (3.2)$$
where
\[
v_x = -\frac{2}{\lambda_1^2} \left[ u_{xx} (\lambda_1 + \lambda_2 u_{xx}) \right]^{1/2}.
\] (3.3)

Applying Proposition 1 we find that the fully-nonlinear second-order equation
\[
v_t = - \left( 1 + \lambda_1^2 \lambda_2 v_x^2 \right) \frac{v_x^2}{v_{xx}} - v
\] (3.4)

admits the Lie-Bäcklund symmetry generator \( Z = \eta[v] \frac{\partial}{\partial v} \) for which (3.2) is the third-order flow, i.e.
\[
\eta(x, v, v_x, v_{xx}, v_{xxx}) = \frac{\lambda_1^3}{4} \left( 1 + \lambda_1^2 \lambda_2 v_x^2 \right)^{3/2} \left( \frac{v_x^3 v_{xxx}}{v_{xx}} - \frac{3v_x^2 v_{xx}}{v_{xx}} \right).
\] (3.5)

Applying now Proposition 2 we find that equation (3.4) admits a recursion operator \( R_{11}[v] \) of the form (2.10), where
\[
G_1 = \frac{\lambda_1^3}{4} \left( 1 + \lambda_1^2 \lambda_2 v_x^2 \right)^{1/2} \frac{v_x}{v_{xx}}
\] (3.6a)
\[
G_0 = -\frac{\lambda_1^5 \lambda_2}{4} \left( 1 + \lambda_1^2 \lambda_2 v_x^2 \right)^{-1/2} v_x^2
\] (3.6b)
\[
I = v_x
\] (3.6c)
\[
\Lambda = \frac{\lambda_1^5 \lambda_2}{4} \left( 1 + \lambda_1^2 \lambda_2 v_x^2 \right)^{-3/2} v_{xx}
\] (3.6d)

for which (3.2) is the second member \( v_t = v_t \) in the hierarchy \( v_{tm} = R_{11}^m [v] v_t, m = 0, 1, 2, \ldots \) and \( v_t \) is the equation (3.3).

**Subcase 1.2:** Consider equation (3.1), with \( \lambda_1 = 0 \) and \( \lambda_2 = 1 \), i.e.
\[
u_t = \frac{u_{xx}^6}{u_{xxx}^2}
\] (3.7)

Then (3.1) admits the potentialisation [3]
\[
\frac{v_t}{v_x^6} = \frac{v_{xxx}}{v_{xx}^3} - \frac{3v_x^5}{v_{xx}},
\] (3.8)

where
\[
v_x = -\frac{2}{\lambda_1^2} u_{xx}.
\] (3.9)

Applying Proposition 1 we find that the fully-nonlinear second-order equation
\[
v_t = - \frac{v_x^4}{v_{xx}}
\] (3.10)
admits the Lie-Bäcklund symmetry generator $Z = \eta[v] \frac{\partial}{\partial v}$ for which \(3.8\) is the third-order flow, i.e.

$$\eta(x, v, v_x, v_{xx}, v_{xxx}) = \frac{v_x^6 v_{xxx}}{v_{xx}^3} - \frac{3v_x^5}{v_{xx}}$$  \(3.11\)

Applying now Proposition 2 we find that equation \(3.10\) admits a recursion operator $R_{12}[v]$ of the form \(3.9\), where

$$G_1 = \frac{v_x^2}{v_{xx}} \quad (3.12a)$$

$$G_0 = -v_x \quad (3.12b)$$

for which \(3.8\) is the second member $v_{t1}$ in the hierarchy $v_{tm} = R^m_{12}[v] v_t$, $m = 0, 1, 2, \ldots$ and $v_t$ is the equation \(3.10\).

**Subcase 1.3:** Consider equation \(3.1\), with $\lambda_1 = -1$ and $\lambda_2 = 0$, i.e.

$$u_t = -\frac{u_{3x}}{u_{xxx}} \quad (3.13)$$

For this equation we have established several potentialisations in \[3\]. We treat the relevant cases below:

1.3a: Equation \(3.13\) admits the potentialisation \[3\]

$$v_t = \frac{2v_x^3 v_{xxx}}{v_{xx}^3} - \frac{3v_x^2}{v_{xx}}, \quad (3.14)$$

where

$$v_x = -u_{xx}. \quad (3.15)$$

Applying Proposition 1 we find that the fully-nonlinear second-order equation

$$v_t = -\frac{v_x^2}{v_{xx}} \quad (3.16)$$

admits the Lie-Bäcklund symmetry generator $Z = \eta[v] \frac{\partial}{\partial v}$ for which \(3.14\) is the third-order flow, i.e.

$$\eta(x, v, v_x, v_{xx}, v_{xxx}) = \frac{2v_x^3 v_{xxx}}{v_{xx}^3} - \frac{3v_x^2}{v_{xx}} \quad (3.17)$$

Applying now Proposition 2 we find that equation \(3.16\) admits a recursion operator $R_{13}[v]$ of the form \(2.9\), where

$$G_1 = \frac{2v_x}{v_{xx}} \quad (3.18a)$$

$$G_0 = -1 \quad (3.18b)$$

for which \(3.14\) is the second member $v_{t1}$ in the hierarchy $v_{tm} = R^m_{13}[v] v_t$, $m = 0, 1, 2, \ldots$ and $v_t$ is the equation \(3.10\).
1.3b: Furthermore we know from the results reported in [3] that a zero-order potentialisation of equation (3.14) is

\[ V_t = \frac{V_{xxx}}{V_{xx}^3} - 3 \cdot 2^{-2/3} \frac{1}{V_{xx}} - 2^{-1/3} x, \]  

(3.19)

where

\[ V_x = 2^{-1/2} \ln(v_x) \]  

(3.20)

gives the relation to (3.14) and

\[ V_x = 2^{-1/2} \ln| - u_{xx}| \]  

(3.21)

the relation to (3.13). Applying Proposition 1 for (3.19) we find that the fully-nonlinear second-order equation

\[ V_t = -\frac{1}{V_{xx}} \]  

(3.22)

admits the Lie-Bäcklund symmetry generator \( Z = \eta[V] \frac{\partial}{\partial V} \) for which (3.19) is the third-order flow, i.e.

\[ \eta(x, V, V_x, V_{xx}, V_{xxx}) = \frac{V_{xxx}}{V_{xx}^3} - 3 \cdot 2^{-2/3} \frac{1}{V_{xx}} - 2^{-1/3} x. \]  

(3.23)

Applying now Proposition 2 we find that equation (3.22) admits a recursion operator \( R_{131}[V] \) of the form (2.10), where

\[ G_1 = \frac{1}{V_{xx}} \]  

(3.24a)

\[ G_0 = 3 \cdot 2^{-2/3} \]  

(3.24b)

\[ I = 1 \]  

(3.24c)

\[ \Lambda = 2^{-1/3} V_{xx} \]  

(3.24d)

for which (3.19) is the second member \( V_t \) in the hierarchy \( V_{tm} = R_{131}^m[V]V_t \), \( m = 0, 1, 2, \ldots \) and \( V_t \) is the equation (3.22).

1.3c: By a multi-potentialisations of (3.13) we are led to the equation [3]

\[ w_t = \frac{w_{xxx}}{w_{xx}^3} - 2^{-2/3} \frac{3}{w_{xx}}, \]  

(3.25)

where

\[ w_{xx} = 2^{-1/3} \frac{u_{xxx}}{u_{xx}} \]  

(3.26)
gives the relation to (3.13). Applying Proposition 1 for (3.25) we find that the fully-
nonlinear second-order equation

\[ w_t = -\frac{1}{w_{xx}} \]  
(3.27)

admits the Lie-Bäcklund symmetry generator \( Z = \eta[w] \frac{\partial}{\partial w} \) for which (3.25) is the third-
order flow, i.e.,

\[ \eta(x, w, w_x, w_{xx}, w_{xxx}) = \frac{w_{xxx}}{w_{xx}} - 2^{-2/3} \frac{3}{w_{xx}}. \]  
(3.28)

Applying now Proposition 2 we find that equation (3.27) admits a recursion operator
\( R_{132}[w] \) of the form (2.9), where

\[ G_1 = \frac{1}{w_{xx}} \]  
(3.29a)

\[ G_0 = 3 \cdot 2^{-2/3} \]  
(3.29b)

for which (3.25) is the second member \( w_t \) in the hierarchy \( w_{tm} = R_{132}^m[w], m = 0, 1, 2, \ldots \) and \( w_t \) is the equation (3.27).

**Case 2:** Consider the fully-nonlinear symmetry-integrable equation [2]

\[ u_t = \frac{1}{u_{xx}}. \]  
(3.30)

A multi-potentialisation of (3.30) leads the quasi-linear equation [3]

\[ v_t = \frac{v_{xxx}}{v_{xx}^3}, \]  
(3.31)

where

\[ v_x = -2^{-1/3} u_{xx}. \]  
(3.32)

Applying Proposition 1 we find that the fully-nonlinear second-order equation

\[ v_t = -\frac{1}{v_{xx}} \]  
(3.33)

admits the Lie-Bäcklund symmetry generator \( Z = \eta[v] \frac{\partial}{\partial v} \) for which (3.31) is the third-order
flow, i.e.,

\[ \eta(x, v, v_x, v_{xx}, v_{xxx}) = \frac{v_{xxx}}{v_{xx}^3}, \]  
(3.34)
Applying now Proposition 2 we find that equation \( (3.33) \) admits a recursion operator \( R_{14}[v] \) of the form \( (2.9) \), where

\[
G_1 = -\frac{1}{v_{xx}} \quad (3.35a)
\]

\[
G_0 = 0 \quad (3.35b)
\]

for which \( (3.31) \) is the second member \( v_{t_1} \) in the hierarchy \( v_{t_m} = R_{14}^m[v]v_t \), \( m = 0, 1, 2, \ldots \) and \( v_t \) is the equation \( (3.33) \).

4 Concluding remarks

In this article we have provided examples of third-order fully-nonlinear symmetry-integrable equations that are linked to second-order fully nonlinear symmetry-integrable equations. In particular, the examples of Case 1 and Case 2 show that certain potentialisations of considered third-order fully-nonlinear equations are in fact the third-order flows of the Lie-Bäcklund symmetries of particular fully-nonlinear second-order equations. Hence these potential equations, which are quasi-linear third-order equations, are members of hierarchies of symmetry-integrable equations for which the seed equation is a second-order fully-nonlinear equation.

We should point out that not every fully-nonlinear third-order symmetry-integrable equation can be linked in this way to a second-order equation. Moreover, not every potential equation that results from third-order fully-nonlinear symmetry-integrable equation belongs to a symmetry-integrable hierarchy of second-order equations, even if it does so for other potentialisations. For example, the fully-nonlinear third-order symmetry integrable equation \( (3.7) \), viz

\[
u_t = \frac{u_{6 xx}}{u_{2 xxx}}
\]

is related to the fully-nonlinear second-order equation

\[
v_t = -\frac{v_x^4}{v_{xx}}
\]

via the potential equation

\[
v_t = \frac{v_6^6 v_{xxx}}{v_3^3 x xx} - \frac{3 v_5^5}{v_{xx}}
\]

as shown in Subcase 1.2. However, as reported in \( [3] \), equation \( (3.7) \) also admits the zero-order potentialisation

\[
v_t = \frac{v_{xxx}}{v_3^3 x x} + \frac{3}{v_x v_{xxx}},
\]

(4.1)

where

\[v_x = \frac{1}{2^{1/3} u_{xx}}.\]
Using Proposition 1 it is easy to shown that there exists no second-order equation that admits the Lie-Bäcklund symmetry generator

\[ Z = \left( \frac{v_{xxx}}{v_{xx}^2} + \frac{3}{v_x v_{xx}} \right) \frac{\partial}{\partial v}. \]

We conclude that the potential equation (4.1) does not provide a link between a second-order equation and the third-order fully-nonlinear equation (3.7). Furthermore, and example of a fully-nonlinear third-order symmetry-integrable equation that does not admit a potentialisation that links this equation to a second-order symmetry-integrable equation is [3]

\[ u_t = \frac{4u_x^5}{(2bu_x^2 - 2u_x u_{xxx} + 3u_{xx}^2)^2}, \]

for any constant \( b \).

Finally, we give an example of a more general second-order fully-nonlinear symmetry-integrable evolution than those that are known to use to be linked to a fully-nonlinear third-order equation from the results reported in [3]: Applying Proposition 1 and Proposition 2 we find that

\[ v_t = -\frac{v_x^2}{v_{xx}} + c_3 v + c_4 v_x \ln(v_x) + c_4 + c_5 v_x \quad (4.2) \]

admits the recursion operator

\[ R[v] = \frac{27}{4} \left( \frac{v_x}{v_{xx}} \right) D_x - \frac{27}{4} + \frac{27c_3}{4} + \alpha v_x D_x^{-1} v \frac{v_{xx}}{v_x^2}, \quad (4.3) \]

where \( \alpha, c_3, c_4 \) and \( c_5 \) are arbitrary constants. The third-order equation in the hierarchy \( v_{tm} = R^m[v]v_t \) is then

\[ v_{t_1} = -\frac{v_x^3}{v_{xx}} + c_3 v + c_4 v_x \ln(v_x) + c_4 + c_5 v_x \]

\[ + \left( \frac{27c_3c_4}{4} + \alpha c_5 \right) v_x \ln(v_x) + \frac{\alpha c_4}{2} v_x \ln^2(v_x) + c_4 \left( \frac{27}{4}(c_3 - 1) - \alpha \right). \quad (4.4) \]

By letting \( c_3 = c_4 = c_5 = \alpha = 0 \) we obtain

\[ v_{t_1} = \frac{27}{4} \frac{v_x^3}{v_{xx}^3} - \frac{27}{4} \frac{v_x^2}{v_{xx}} \quad (4.5) \]

which is an equation that was obtained in [3] by a multi-potentialisation of (3.13), viz

\[ u_t = -\frac{u_x^3}{u_{xx}^3}. \]

The relation between (3.13) and (4.5) is given my

\[ \frac{v_{xx}}{v_x} = \frac{3}{2} \frac{u_{xxx}}{u_{xx}}. \quad (4.6) \]

In view of this example one may ask whether this method could be used to construct further fully-nonlinear third-order symmetry-integrable evolution equations different from those that have been reported in [2]. Furthermore, the method proposed here could be exploited for the construction of higher-order fully-nonlinear symmetry-integrable equations.
References


