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On 2nd-order fully-nonlinear equations with links to 3rd-order fully-nonlinear equations

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Abstract: We derive the general conditions for fully-nonlinear symmetry-integrable second-order evolution equations and their first-order recursion operators. We then make us of the established Propositions to find a link between a class of fully-nonlinear third-order symmetry integrable evolution equations and fully-nonlinear second-order symmetry-integrable evolution equations.

1 Introduction

We recently reported a class of third-order fully-nonlinear symmetry-integrable evolution equations in 1+1 dimensions with rational nonlinearities in their highest derivative [2]. For this class of equations we have furthermore reported all the potentialisations and some multi-potentialisations in [3]. It is interesting to note that some of these potential equations are in fact members of hierarchies of fully-nonlinear second-order evolution equations. In the current article we identify all those potential equations and give the explicit recursion operators that generate those hierarchies. In particular, the fully-nonlinear third-order symmetry-integrable evolution equations that are relevant here are

$$u_t = \frac{u_{xx}^3}{u_{xxx}^2} \left(\lambda_1 + \lambda_2 u_{xx}\right)^3$$

and

$$u_t = \frac{1}{u_{xxx}^2},$$

where λ_1 and λ_2 are arbitrary constants. For more details on symmetry-integrable hierarchies and recursion operators, we refer to [4] and [2], and the references therein.

This article is organised as follows: In Section 2 we provide two Propositions by which one are able to identify second-order fully nonlinear equations that admit third-order Lie-Bäcklund symmetries (see Proposition 1) and by which we can obtain a recursion operator for those second-order fully-nonlinear equations (see Proposition 2). In Section 3 we apply the mentioned Propositions to establish links to the two fully-nonlinear equations given above. Finally in Section 4 we make our conclusions and propose some further studies that could be of interest.

2 On fully-nonlinear second-order equations

We consider a second-order equation of the form

$$u_t = F(u, u_x, u_{xx}) \tag{2.1}$$

where the non-constant function F is to be determined such that (2.1) is fully nonlinear in u_{xx} and admits a third-order Lie-Bäcklund symmetry generated by $Z = \eta[u]\dfrac{\partial}{\partial u}$; hence a recursion operator R that generates this third-order symmetry and consequently an infinite number of higher-order Lie-Bäcklund symmetries. The hierarchy of symmetry-integrable evolution equations is then

$$u_{t_m} = R^m[u]\, u_t, \qquad m = 0, 1, 2, \dots \ . \tag{2.2}$$

Our first step is to find the general form of F and η to satisfy this requirement as a necessary condition. This is given by

Proposition 1. *Consider the class of second-order evolution equations of the form (2.1) viz*

$$u_t = F(u, u_x, u_{xx}).$$

Under the assumption that (2.1) is symmetry-integrable, the following statements must be true:

1. *The function F in (2.1) satisfies the condition*

$$\frac{\partial^3 F}{\partial u_{xx}^3}\left(\frac{\partial F}{\partial u_{xx}}\right)^{-1} - \frac{3}{2}\left(\frac{\partial^2 F}{\partial u_{xx}^2}\right)^2\left(\frac{\partial F}{\partial u_{xx}}\right)^{-2} = 0 \tag{2.3}$$

with general solution

$$F(u, u_x, u_{xx}) = -\frac{A(u, u_x)}{u_{xx} + B(u, u_x)} + C(u, u_x), \tag{2.4}$$

where A, B and C are arbitrary functions of their arguments. The quasi-linear and semi-linear form of (2.1) follow from the singular solution of (2.3), namely the solution for which $\partial^2 F/\partial u_{xx}^2 = 0$. [Note that (2.3) is the Schwarzian derivative in F.]

2. Any third-order Lie-Bäcklund symmetry generator

$$Z := \eta(x, u, u_x, u_{xx}, u_{xxx})\frac{\partial}{\partial u} \tag{2.5}$$

for (2.4) is of the general form

$$\eta(x, u, u_x, u_{xx}, u_{xxx}) = \frac{c_0 A^{3/2} u_{xxx}}{(u_{xx}+B)^3} - \frac{c_0 A^{3/2}}{(u_{xx}+B)^3}\left(B\frac{\partial B}{\partial u_x} - u_x\frac{\partial B}{\partial u}\right)$$
$$+\frac{3c_0 A^{1/2}}{2(u_{xx}+B)^2}\left(B\frac{\partial A}{\partial u_x} - u_x\frac{\partial A}{\partial u}\right) - \frac{f_1}{u_{xx}+B} + f_2, \tag{2.6}$$

whereby c_0 denotes an arbitrary but non-zero constant, and the functions $A = A(u, u_x)$, $B = B(u, u_x)$, $C = C(u, u_x)$, $f_1 = f_1(x, u, u_x)$ and $f_2 = f_2(x, u, u_x)$ need to be determined such that the invariance condition

$$L_E \eta(x, u, u_x, u_{xx}, u_{xxx})\Big|_{E=0} = 0 \tag{2.7}$$

is satisfied. Here $E := u_t - F(u, u_x, u_{xx})$ and L_E denotes the linear operator

$$L_E[u] := \frac{\partial E}{\partial u} + \frac{\partial E}{\partial u_t}D_t + \frac{\partial E}{\partial u_x}D_x + \frac{\partial E}{\partial u_{xx}}D_x^2 + \frac{\partial E}{\partial u_{xxx}}D_x^3. \tag{2.8}$$

We now seek 1st-order recursion operators R for every equation that admits a third-order Lie-Bäcklund symmetry, which, according to Proposition 1, are equations of the general form (2.4) We consider both differential recursion operators

$$R[u] = G_1(u, u_x, u_{xx})D_x + G_0(u, u_x) \tag{2.9}$$

as well as integro-differential recursion operators

$$R[u] = G_1(u, u_x, u_{xx})D_x + G_0(u, u_x) + I(u, u_x, u_{xx})D_x^{-1} \circ \Lambda(u, u_x, u_{xx}). \tag{2.10}$$

Applying the standard condition for R (see for example [1]), namely

$$[L_E,\ R[u]] = D_t R\Big|_{E=0} \tag{2.11}$$

we obtain the following

Proposition 2. *For finding recursion operators of (2.4) viz.*

$$u_t = -\frac{A(u, u_x)}{u_{xx} + B(u, u_x)} + C(u, u_x).$$

we distinguish between the following four cases:

1. *With the assumption* $A = F_1(u_x)$ *and* $B(u, u_x) = 0$ *in (2.4), it follows that*

$$u_t = -\frac{F_1(u_x)}{u_{xx}} + u\left(\frac{a_1\sqrt{F_1(u_x)}}{u_x^2 P(u_x)} - \frac{F_1(u_x)}{u_x P(u_x)}\frac{dP}{du_x} + \frac{F_1(u_x)}{u_x^2} - \frac{1}{2u_x}\frac{dF_1}{du_x}\right)$$
$$+\frac{a_0\sqrt{F_1(u_x)}}{P(u_x)} + H(u_x) \tag{2.12}$$

admits a recursion operator of the form (2.10) with

$$G_1(u_x, u_{xx}) = \frac{k_1\sqrt{F_1(u_x)}}{u_{xx}} \tag{2.13a}$$

$$G_0(u_x) = \frac{k_1\sqrt{F_1(u_x)}}{2P(u_x)}\frac{dP}{du_x} - \frac{k_1\sqrt{F_1(u_x)}}{2u_x} - \frac{a_1 k_1}{2u_x P(u_x)} + k_0 \tag{2.13b}$$

$$I(u_x) = \alpha u_x \tag{2.13c}$$

$$\Lambda(u_x, u_{xx}) = \frac{P(u_x)u_{xx}}{\sqrt{F_1(u_x)}}, \tag{2.13d}$$

where a_0, a_1 *and* k_0 *are arbitrary constants,* k_1 *and* α *are an arbitrary non-zero constants,* H *is an arbitrary function of* u_x *and, furthermore,* F_1 *and* P *must satisfy the following condition:*

$$0 = k_1 u_x^2 P^2\left(u_x\frac{dP}{du_x} + P\right)\frac{d^2F_1}{du_x^2} - k_1 u_x P\left[-3u_x^2 P\frac{d^2P}{du_x^2} + 2u_x^2\left(\frac{dP}{du_x}\right)^2 - 2u_x P\frac{dP}{du_x}\right.$$
$$\left. + 2P^2\right]\frac{dF_1}{du_x} + 2k_1\left[u_x^3 P^2\frac{d^3P}{du_x^3} - 2u_x^3 P\frac{dP}{du_x}\frac{d^2P}{du_x^2} + u_x^2 P^2\frac{d^2P}{du_x^2} + u_x^3\left(\frac{dP}{du_x}\right)^3\right.$$
$$\left. -u_x^2 P\left(\frac{dP}{du_x}\right)^2 - u_x P^2\frac{dP}{du_x} + P^3\right]F_1 + 2\left(4\alpha u_x^3 P^3 - k_1 a_1^2\right)\left(u_x\frac{dP}{du_x} + P\right). \tag{2.14}$$

2. *With the assumption* $A = F_1(u_x)$ *and* $B = B(u_x)$ *in (2.4), it follows that*

$$u_t = -\frac{F_1(u_x)}{u_{xx} + (b_1 + b_2 u_x)\sqrt{F_1(u_x)}} + \frac{\sqrt{F_1(u_x)}}{b_1 + b_2 u_x} \tag{2.15}$$

admits a recursion operator of the form (2.10) with

$$G_1(u_x, u_{xx}) = \frac{k_1\sqrt{F_1(u_x)}}{u_{xx} + (b_1 + b_2 u_x)\sqrt{F_1(u_x)}} \tag{2.16a}$$

$$G_0(u_x) = \frac{k_1 b_2 F_1(u_x)}{u_{xx} + (b_1 + b_2 u_x)\sqrt{F_1(u_x)}} - \frac{k_1 b_2\sqrt{F_1(u_x)}}{b_1 + b_2 u_x} + k_0 \tag{2.16b}$$

$$I(u_x) = u_x + \frac{b_1}{b_2} \tag{2.16c}$$

$$\Lambda(u_x, u_{xx}) = \alpha b_2 \left(\frac{u_{xx} + (b_1 + b_2 u_x)\sqrt{F_1(u_x)}}{(b_1 + b_2 u_x)\sqrt{F_1(u_x)}} \right), \tag{2.16d}$$

where b_1 and α are arbitrary constants, whereas b_2 and k_1 are arbitrary non-zero constants.

3. *With the assumption $A = F_1(u_x)$, $B = 0$ and $C = C(u_x)$ in (2.4), it follows that*

$$u_t = -\frac{F_1(u_x)}{u_{xx}} + C(u_x). \tag{2.17}$$

This leads to two subcases:

3.1. Equation (2.17) admits the recursion operator of the form (2.10) with

$$G_1(u_x, u_{xx}) = \frac{k_1\sqrt{F_1(u_x)}}{u_{xx}} \tag{2.18a}$$

$$G_0(u_x) = -\frac{k_1}{4\sqrt{F_1(u_x)}}\frac{dF_1}{du_x} + k_0 \tag{2.18b}$$

$$I(u_x) = \alpha \tag{2.18c}$$

$$\Lambda(u_x, u_{xx}) = \frac{u_{xx}}{F_1(u_x)}, \tag{2.18d}$$

where k_0 is an arbitrary constant, k_1 and α are arbitrary non-zero constants, C is an arbitrary function of u_x, and F_1 must satisfy the following condition:

$$k_1\left(4F_1^4\frac{d^3F_1}{du_x^3} - 6F_1^3\frac{dF_1}{du_x}\frac{d^2F_1}{du_x^2} + 3F_1^2\left(\frac{dF_1}{du_x}\right)^3\right) + 16\alpha F_1^{5/2}\frac{dF_1}{du_x} = 0. \tag{2.19}$$

3.2. Equation (2.17) admits the recursion operator of the form (2.10) with

$$G_1(u_x, u_{xx}) = \frac{k_1\sqrt{F_1(u_x)}}{u_{xx}} \tag{2.20a}$$

$$G_0(u_x) = -\frac{k_1}{4\sqrt{F_1(u_x)}}\frac{dF_1}{du_x} + k_0 \tag{2.20b}$$

$$I(u_x) = \alpha u_x \tag{2.20c}$$

$$\Lambda(u_x, u_{xx}) = \frac{u_x u_{xx}}{F_1(u_x)}, \tag{2.20d}$$

where k_0 is an arbitrary constant, k_1 and α are arbitrary non-zero constants, C is an arbitrary function of u_x, and F_1 must satisfy the following condition:

$$k_1\left(4F_1^4\frac{d^3F_1}{du_x^3} - 6F_1^3\frac{dF_1}{du_x}\frac{d^2F_1}{du_x^2} + 3F_1^2\left(\frac{dF_1}{du_x}\right)^3\right)$$

$$+16\alpha F_1^{5/2}u_x\left(u_x\frac{dF_1}{du_x}-4F_1\right)=0. \tag{2.21}$$

4. The equation

$$u_t=-\frac{F_1(u_x)}{u_{xx}+(b_1+b_2u_x)\sqrt{F_1(u_x)}}+F_2(u_x) \tag{2.22}$$

admits a recursion operator of the form (2.9) with

$$G_1(u_x)=\frac{k_1\sqrt{F_1}}{u_{xx}+(b_1+u_xb_2)\sqrt{F_1}} \tag{2.23a}$$

$$G_0(u_x)=\frac{k_1b_2F_1}{u_{xx}+(b_1+b_2u_x)\sqrt{F_1}}-\frac{k_1}{4\sqrt{F_1}}\frac{dF_1}{du_x}+\frac{k_1}{2}(b_1+b_2u_x)\frac{dF_2}{du_x}$$

$$-\frac{k_1b_2}{2}F_2+k_0, \tag{2.23b}$$

where b_1 and b_2 are arbitrary constants, k_1 is an arbitrary non-zero constant, and F_1 and F_2 must satisfy the following condition:

$$4F_1^2\frac{d^3F_1}{du_x^3}-6F_1\frac{dF_1}{du_x}\frac{d^2F_1}{du_x^2}+3\left(\frac{dF_1}{du_x}\right)^3-8F_1^{5/2}\left((b_1+b_2u_x)\frac{d^3F_2}{du_x^3}\right.$$

$$\left.+3b_2\frac{d^2F_2}{du_x^2}\right)=0. \tag{2.24}$$

3 Fully-nonlinear second-order equations related to fully-nonlinear third-order equations

We now apply Proposition 1 and Proposition 2 to identify fully-nonlinear second-order equations that are related to fully-nonlinear third-order equations. For this purpose we consider two main cases:

Case 1: Consider the fully-nonlinear symmetry-integrable equation [2]

$$u_t=\frac{u_{xx}^3}{u_{xxx}^2}\left(\lambda_1+\lambda_2u_{xx}\right)^3, \tag{3.1}$$

where λ_1 and λ_2 are arbitrary constants. For the potentialisations of this equation we need to distinguish different subcases that depend on the constants λ_1 and λ_2.

Subcase 1.1: Consider equation (3.1), with $\lambda_1\neq 0$ and $\lambda_2\neq 0$. Then (3.1) admits the zero-order potentialisation [3]

$$v_t=\frac{\lambda_1^3}{4}\left(1+\lambda_1^2\lambda_2v_x^2\right)^{3/2}\left(\frac{v_x^3v_{xxx}}{v_{xx}^3}-\frac{3v_x^2}{v_{xx}}\right), \tag{3.2}$$

where

$$v_x = -\frac{2}{\lambda_1^2}\left[u_{xx}(\lambda_1+\lambda_2 u_{xx})\right]^{1/2}. \tag{3.3}$$

Applying Proposition 1 we find that the fully-nonlinear second-order equation

$$\boxed{v_t = -\left(1+\lambda_1^2\lambda_2 v_x^2\right)\frac{v_x^2}{v_{xx}} - v} \tag{3.4}$$

admits the Lie-Bäcklund symmetry generator $Z=\eta[v]\dfrac{\partial}{\partial v}$ for which (3.2) is the third-order flow, i.e.

$$\eta(x,v,v_x,v_{xx},v_{xxx}) = \frac{\lambda_1^3}{4}\left(1+\lambda_1^2\lambda_2 v_x^2\right)^{3/2}\left(\frac{v_x^3 v_{xxx}}{v_{xx}^3} - \frac{3v_x^2}{v_{xx}}\right). \tag{3.5}$$

Applying now Proposition 2 we find that equation (3.4) admits a recursion operator $R_{11}[v]$ of the form (2.10), where

$$G_1 = \frac{\lambda_1^3}{4}\left(1+\lambda_1^2\lambda_2 v_x^2\right)^{1/2}\frac{v_x}{v_{xx}} \tag{3.6a}$$

$$G_0 = -\frac{\lambda_1^5\lambda_2}{4}\left(1+\lambda_1^2\lambda_2 v_x^2\right)^{-1/2} v_x^2 \tag{3.6b}$$

$$I = v_x \tag{3.6c}$$

$$\Lambda = \frac{\lambda_1^5\lambda_2}{4}\left(1+\lambda_1^2\lambda_2 v_x^2\right)^{-3/2} v_{xx} \tag{3.6d}$$

for which (3.2) is the second member v_{t_1} in the hierarchy $v_{t_m} = R_{11}^m[v]v_t$, $m=0,1,2,\dots$ and v_t is the equation (3.4) .

Subcase 1.2: Consider equation (3.1), with $\lambda_1=0$ and $\lambda_2=1$, i.e.

$$u_t = \frac{u_{xx}^6}{u_{xxx}^2} \tag{3.7}$$

Then (3.7) admits the potentialisation [3]

$$v_t = \frac{v_x^6 v_{xxx}}{v_{xx}^3} - \frac{3v_x^5}{v_{xx}}, \tag{3.8}$$

where

$$v_x = -2^{1/3}u_{xx}. \tag{3.9}$$

Applying Proposition 1 we find that the fully-nonlinear second-order equation

$$\boxed{v_t = -\frac{v_x^4}{v_{xx}}} \tag{3.10}$$

admits the Lie-Bäcklund symmetry generator $Z = \eta[v]\dfrac{\partial}{\partial v}$ for which (3.8) is the third-order flow, i.e.

$$\eta(x, v, v_x, v_{xx}, v_{xxx}) = \frac{v_x^6 v_{xxx}}{v_{xx}^3} - \frac{3v_x^5}{v_{xx}}. \tag{3.11}$$

Applying now Proposition 2 we find that equation (3.10) admits a recursion operator $R_{12}[v]$ of the form (2.9), where

$$G_1 = \frac{v_x^2}{v_{xx}} \tag{3.12a}$$

$$G_0 = -v_x \tag{3.12b}$$

for which (3.8) is the second member v_{t_1} in the hierarchy $v_{t_m} = R_{12}^m[v]v_t$, $m = 0, 1, 2, \ldots$ and v_t is the equation (3.10) .

Subcase 1.3: Consider equation (3.1), with $\lambda_1 = -1$ and $\lambda_2 = 0$, i.e.

$$u_t = -\frac{u_{xx}^3}{u_{xxx}^2} \tag{3.13}$$

For this equation we have established several potentialisations in [3]. We treat the relevant cases below:

1.3a: Equation (3.13) admits the potentialisation [3]

$$v_t = \frac{2v_x^3 v_{xxx}}{v_{xx}^3} - \frac{3v_x^2}{v_{xx}}, \tag{3.14}$$

where

$$v_x = -u_{xx}. \tag{3.15}$$

Applying Proposition 1 we find that the fully-nonlinear second-order equation

$$\boxed{v_t = -\frac{v_x^2}{v_{xx}}} \tag{3.16}$$

admits the Lie-Bäcklund symmetry generator $Z = \eta[v]\dfrac{\partial}{\partial v}$ for which (3.14) is the third-order flow, i.e.

$$\eta(x, v, v_x, v_{xx}, v_{xxx}) = \frac{2v_x^3 v_{xxx}}{v_{xx}^3} - \frac{3v_x^2}{v_{xx}}. \tag{3.17}$$

Applying now Proposition 2 we find that equation (3.16) admits a recursion operator $R_{13}[v]$ of the form (2.9), where

$$G_1 = \frac{2v_x}{v_{xx}} \tag{3.18a}$$

$$G_0 = -1 \tag{3.18b}$$

for which (3.14) is the second member v_{t_1} in the hierarchy $v_{t_m} = R_{13}^m[v]v_t$, $m = 0, 1, 2, \ldots$ and v_t is the equation (3.16).

1.3b: Furthermore we know from the results reported in [3] that a zero-order potentialisation of equation (3.14) is

$$V_t = \frac{V_{xxx}}{V_{xx}^3} - 3 \cdot 2^{-2/3} \frac{1}{V_{xx}} - 2^{-1/3}\, x, \tag{3.19}$$

where

$$V_x = 2^{-1/2} \ln(v_x) \tag{3.20}$$

gives the relation to (3.14) and

$$V_x = 2^{-1/2} \ln |-u_{xx}| \tag{3.21}$$

the relation to (3.13). Applying Proposition 1 for (3.19) we find that the fully-nonlinear second-order equation

$$\boxed{V_t = -\frac{1}{V_{xx}}} \tag{3.22}$$

admits the Lie-Bäcklund symmetry generator $Z = \eta[V] \dfrac{\partial}{\partial V}$ for which (3.19) is the third-order flow, i.e.

$$\eta(x, V, V_x, V_{xx}, V_{xxx}) = \frac{V_{xxx}}{V_{xx}^3} - 3 \cdot 2^{-2/3} \frac{1}{V_{xx}} - 2^{-1/3}\, x. \tag{3.23}$$

Applying now Proposition 2 we find that equation (3.22) admits a recursion operator $R_{131}[V]$ of the form (2.10), where

$$G_1 = \frac{1}{V_{xx}} \tag{3.24a}$$

$$G_0 = 3 \cdot 2^{-2/3} \tag{3.24b}$$

$$I = 1 \tag{3.24c}$$

$$\Lambda = 2^{-1/3} V_{xx} \tag{3.24d}$$

for which (3.19) is the second member V_{t_1} in the hierarchy $V_{t_m} = R_{131}^m[V] V_t$, $m = 0, 1, 2, \ldots$ and V_t is the equation (3.22).

1.3c: By a multi-potentialisations of (3.13) we are led to the equation [3]

$$w_t = \frac{w_{xxx}}{w_{xx}^3} - 2^{-2/3} \frac{3}{w_{xx}}, \tag{3.25}$$

where

$$w_{xx} = 2^{-1/3} \frac{u_{xxx}}{u_{xx}} \tag{3.26}$$

gives the relation to (3.13). Applying Proposition 1 for (3.25) we find that the fully-nonlinear second-order equation

$$\boxed{w_t = -\frac{1}{w_{xx}}} \tag{3.27}$$

admits the Lie-Bäcklund symmetry generator $Z = \eta[w]\dfrac{\partial}{\partial w}$ for which (3.25) is the third-order flow, i.e.

$$\eta(x, w, w_x, w_{xx}, w_{xxx}) = \frac{w_{xxx}}{w_{xx}^3} - 2^{-2/3}\frac{3}{w_{xx}}. \tag{3.28}$$

Applying now Proposition 2 we find that equation (3.27) admits a recursion operator $R_{132}[w]$ of the form (2.9), where

$$G_1 = \frac{1}{w_{xx}} \tag{3.29a}$$

$$G_0 = 3 \cdot 2^{-2/3} \tag{3.29b}$$

for which (3.25) is the second member w_{t_1} in the hierarchy $w_{t_m} = R_{132}^m[w]w_t$, $m = 0, 1, 2, \ldots$ and w_t is the equation (3.27).

Case 2: Consider the fully-nonlinear symmetry-integrable equation [2]

$$u_t = \frac{1}{u_{xxx}^2}. \tag{3.30}$$

A multi-potentialisation of (3.30) leads the quasi-linear equation [3]

$$v_t = \frac{v_{xxx}}{v_{xx}^3}, \tag{3.31}$$

where

$$v_x = -2^{-1/3}u_{xx}. \tag{3.32}$$

Applying Proposition 1 we find that the fully-nonlinear second-order equation

$$\boxed{v_t = -\frac{1}{v_{xx}}} \tag{3.33}$$

admits the Lie-Bäcklund symmetry generator $Z = \eta[v]\dfrac{\partial}{\partial v}$ for which (3.31) is the third-order flow, i.e.

$$\eta(x, v, v_x, v_{xx}, v_{xxx}) = \frac{v_{xxx}}{v_{xx}^3}. \tag{3.34}$$

Applying now Proposition 2 we find that equation (3.33) admits a recursion operator $R_{14}[v]$ of the form (2.9), where

$$G_1 = -\frac{1}{v_{xx}} \tag{3.35a}$$

$$G_0 = 0 \tag{3.35b}$$

for which (3.31) is the second member v_{t_1} in the hierarchy $v_{t_m} = R_{14}^m[v]v_t$, $m = 0, 1, 2, \ldots$ and v_t is the equation (3.33).

4 Concluding remarks

In this article we have provided examples of third-order fully-nonlinear symmetry-integrable equations that are linked to second-order fully nonlinear symmetrey-integrable equations. In particular, the examples of Case 1 and Case 2 show that certain potentialisations of considered third-order fully-nonlinear equations are in fact the third-order flows of the Lie-Bäcklund symmetries of particular fully-nonlinear second-order equations. Hence these potential equations, which are quasi-linear third-order equations, are members of hierarchies of symmetry-integrable equations for which the seed equation is a second-order fully-nonlinear equation.

We should point out that not every fully-nonlinear third-order symmetry-integrable equation can be linked in this way to a second-order equation. Moreover, not every potential equation that results from third-order fully-nonlinear symmetry-integrable equation belongs to a symmetry-integrable hierarchy of second-order equations, even if it does so for other potentialisations. For example, the fully-nonlinear third-order symmetry integrable equation (3.7), viz

$$u_t = \frac{u_{xx}^6}{u_{xxx}^2}$$

is related to the fully-nonlinear second-order equation

$$v_t = -\frac{v_x^4}{v_{xx}}$$

via the potential equation

$$v_t = \frac{v_x^6 v_{xxx}}{v_{xx}^3} - \frac{3v_x^5}{v_{xx}}$$

as shown in Subcase 1.2. However, as reported in [3], equation (3.7) also admits the zero-order potentialisation

$$v_t = \frac{v_{xxx}}{v_{xx}^3} + \frac{3}{v_x v_{xx}}, \tag{4.1}$$

where

$$v_x = \frac{1}{2^{1/3} u_{xx}}.$$

Using Proposition 1 it is easy to shown that there exists no second-order equation that admits the Lie-Bäcklund symmetry generator

$$Z = \left(\frac{v_{xxx}}{v_{xx}^3} + \frac{3}{v_x v_{xx}}\right)\frac{\partial}{\partial v}.$$

We conclude that the potential equation (4.1) does not provide a link between a second-order equation and the third-order fully-nonlinear equation (3.7). Furthermore, and example of a fully-nonlinear third-order symmetry-integrable equation that does not admit a potentialisation that links this equation to a second-order symmetry-integrable equation is [3]

$$u_t = \frac{4u_x^5}{(2bu_x^2 - 2u_x u_{xxx} + 3u_{xx}^2)^2},$$

for any constant b.

Finally, we give an example of a more general second-order fully-nonlinear symmetry-integrable evolution than those that are known to use to be linked to a fully-nonlinear third-order equation from the results reported in [3]: Applying Proposition 1 and Proposition 2 we find that

$$v_t = -\frac{v_x^2}{v_{xx}} + c_3 v + c_4 v_x \ln(v_x) + c_4 + c_5 v_x \tag{4.2}$$

admits the recursion operator

$$R[v] = \frac{27}{4}\left(\frac{v_x}{v_{xx}}\right) D_x - \frac{27}{4} + \frac{27c_3}{4} + \alpha v_x D_x^{-1} \circ \frac{v_{xx}}{v_x^2}, \tag{4.3}$$

where α, c_3, c_4 and c_5 are arbitrary constants. The third-order equation in the hierarchy $v_{t_m} = R^m[v]v_t$ is then

$$\begin{aligned} v_{t_1} = {} & \frac{27}{4}\frac{v_x^3 v_{xxx}}{v_{xx}^3} - \frac{27}{4}\frac{v_x^2}{v_{xx}} + \alpha(c_3-1)x v_x + c_3\left(\frac{27c_3}{4} - \frac{27}{4} - \alpha\right) v + \frac{27}{4}(c_4 + c_3 c_5) v_x \\ & + \left(\frac{27c_3c_4}{4} + \alpha c_5\right) v_x \ln(v_x) + \frac{\alpha c_4}{2} v_x \ln^2(v_x) + c_4\left(\frac{27}{4}(c_3-1) - \alpha\right). \end{aligned} \tag{4.4}$$

By letting $c_3 = c_4 = c_5 = \alpha = 0$ we obtain

$$v_{t_1} = \frac{27}{4}\frac{v_x^3 v_{xxx}}{v_{xx}^3} - \frac{27}{4}\frac{v_x^2}{v_{xx}} \tag{4.5}$$

which is an equation that was obtained in [3] by a multi-potentialisation of (3.13), viz

$$u_t = -\frac{u_{xx}^3}{u_{xxx}^2}.$$

The relation between (3.13) and (4.5) is given my

$$\frac{v_{xx}}{v_x} = \frac{3}{2}\frac{u_{xxx}}{u_{xx}}. \tag{4.6}$$

In view of this example one may ask whether this method could be used to construct further fully-nonlinear third-order symmetry-integrable evolution equations different from those that have been reported in [2]. Furthermore, the method proposed here could be exploited for the construction of higher-order fully-nonlinear symmetry-integrable equations.

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