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Lagrangian multiform structure of discrete and semi-discrete KP systems

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Abstract

A variational structure for the potential AKP system is established using the novel formalism of a Lagrangian multiforms. The structure comprises not only the fully discrete equation on the 3D lattice, but also its semi-discrete variants including several differential-difference equations asssociated with, and compatible with, the partial difference equation. To this end, an overview is given of the various (discrete and semi-discrete) variants of the KP system, and their associated Lax representations, including a novel 'generating PDE' for the KP hierarchy. The exterior derivative of the Lagrangian 3-form for the lattice potential KP equation is shown to exhibit a double-zero structure, which implies the corresponding generalised Euler-Lagrange equations. Alongside the 3-form structures, we develop a variational formulation of the corresponding Lax systems via the square eigenfunction representation arising from the relevant direct linearization scheme.

1 Introduction

The notion of Lagrangian multiforms was introduced in [13] to provide a variational formalism for systems integrable in the sense of multidimensional consistency (MDC). Thus, the multiform theory distinguishes itself from conventional variational approaches by the feature that the corresponding Euler-Lagrange equations produce not just a single equation per component of the field variable, but a compatible system of equations on the same dependent variable in a multidimensional space of independent variables. The corresponding action is a functional of not only the field variables, but also of the d-dimensional hypersurfaces over which a Lagrangian d-form is integrated in an ambient space of arbitrary dimension.

Initially set up for integrable systems in 1+1 dimensions (Lagrangian 2-forms) and 1+0 dimensions (Lagrangian 1-forms, cf. [38]), the first example of a 2+1-dimensional/3-dimensional case (Lagrangian 3-forms) was established in the fully discrete case in [14], and in the fully continuous case in [33], of the KP system. Semi-discrete KP systems were so far not covered, and the fully discrete case was mainly related to the bilinear form of KP due to Hirota, [11], but not the more natural nonlinear (potential) KP form first given in [21] which reads

$$(p-q+\widehat{u}-\widetilde{u})(r+\widehat{\widetilde{u}})+(q-r+\bar{u}-\widehat{u})(p+\widehat{\bar{u}})+(r-p+\widetilde{u}-\bar{u})(q+\widetilde{\bar{u}})=0\ . \quad (1.1)$$

Here the dependent variable $u=u(n,m,h)$ depends on discrete lattice variables n,m,h labelling a three-dimensional lattice, and p,q,r are associated lattice parameters which one can associate with the links on the lattice in n,m, and h-directions respectively. The accents $\widetilde{\ }$, $\widehat{\ }$, $\bar{\ }$ denote elementary lattice shifts in the three lattice directions, i.e. $\widetilde{u}=T_pu=u(n+1,m,h)$, $\widehat{u}=T_qu=u(n,m+1,h)$, $\bar{u}=T_ru=u(n,m,h+1)$ with shift operators T_p,T_q,T_r in the n, m, and h-directions[1]. Eq. (1.1) can be shown to lead to the potential KP equation in a specific full continuum limit, yielding

$$\partial_x\left(u_t-\tfrac{1}{4}u_{xxx}-\tfrac{3}{2}u_x^2\right)=\tfrac{3}{4}u_{yy}\ , \quad (1.2)$$

for the same function u but in terms of appropriate continuous variables x,y,t.

It is well-known, from its construction in [21] as well as through the classification problem addressed in [1], that (1.1) is multidimensionally consistent, which implies that the lattice can be extended to a multidimensional lattice of arbitrary dimension in which on each elementary cube the equation can be imposed and that all equations on all faces of elementary hypercubes are consistent. It is this fundamental integrability property that characterises many integrable lattice systems, and that we aim at capturing in the variational formalism of Lagrangian multiforms.

In this note I will first give a brief review of the discrete, semi-discrete KP systems associated with (1.1), altogether forming a large mixed MDC integrable system, and I will present the corresponding linear systems (i.e., Lax representations). I will also summarise the direct linearising transform approach, cf. [8, 21], as it is linked to the 'square eigenfunction' expansions that is needed for the variational description of the Lax pair, noting that the multiform description as a byproduct also yields a variational formulation of not only the nonlinear equation under consideration, but also of the Lax pairs, cf. [34]. There are several distinct semi- discrete forms appearing, namely a form with two discrete variables and one continuous variable ξ, cf. (2.1) below, and various differential-difference equations with two continuous variables, ξ and σ or τ, and one discrete variable n or m, namely (2.7) below. In addition there are other semi-discrete equations with other combinations of (discrete or continuous) independent variables, such as (2.12) and (2.13), where $v=p-q+\widehat{u}-\widetilde{u}$. Importantly, all these equations are not separate formulae, but can be imposed *simultaneously* on the single dependent variable function $u=u(n,m,h;\xi,\sigma,\tau)$ because of the important property of *multidimensional consistency* that makes all these equations mutually compatible. Thus, one could consider the entire system of all these variant equations as the object that should be considered the 'KP system' , which includes

[1]The notation of the shift operator T_p as acting on the variable n, etc., should not be confused with a shift in the parameter p.

also the standard KP hierarchy. The continuous variables σ, τ can be identified with the so-called Miwa variables, [17], by means of

$$\partial_\tau u = \sum_{j=0}^{\infty} \frac{1}{p^{j+1}} \frac{\partial}{\partial t_j}\ , \quad \partial_\sigma u = \sum_{j=0}^{\infty} \frac{1}{q^{j+1}} \frac{\partial}{\partial t_j}\ , \tag{1.3}$$

where the t_j, $j = 1, 2, \cdots$ are the usual time-variables of the KP hierarchy. As a novel result, a fully continuous coupled PDE system (in terms of u and v), (2.16) together with (2.14), is presented with independent variables ξ, τ, σ which encodes the entire KP hierarchy. This is what we would call a *generating PDE* for the hierarchy. This system allows us to circumvent the rather laborious representation of the KP hierarchy through pseudo-differential operators, since it is closed-form PDE system, and only requires the expansions (1.3) to unpeel the equation in order to obtain the KP hierarchy in terms of the usual time-variables.

The outline of the paper is as follows. In section 2, we summarise the KP system and present the corresponding linear problems in terms of auxiliary functions φ and ψ (the latter solving an adjoint Lax pair). In section 3 we present the direct linearising transform and the corresponding quadratic eigenfunction expansion. In section 4 we give the Lagrangian multiform for the semidiscrete KP, and derive from it the one for the fully discrete case. In particular, we show that the latter possesses the double zero property, which guarantees not only the closure relation, but also the relevant Euler-Lagrange system in terms of the so-called corner equations. Finally, in the Discussion section we mention some future challenges.

2 The discrete, semi-discrete and continuous KP system

Apart from (1.1), we are interested in the Lagrangian structure of the following integrable equation. cf. [23],

$$\partial_\xi \ln(p - q + \widehat{u} - \widetilde{u}) = \widehat{\widetilde{u}} + u - \widehat{u} - \widetilde{u}\ , \tag{2.1}$$

which arises as a so-called straight continuum limit[2] of (1.1). Eq. (2.1) is a semi-discrete version of the KP equation with two discrete independent variables n, m and one continuous independent variable ξ. As mentioned before, the scalar field u can, in principle, be considered to be a function depending on an arbitrary number of discrete variables (with the notation introduced in section 1) and possesses, like (1.1), the MDC property. This equation was used in [24] to derive a discrete-time version of the Calogero-Moser system by pole reduction, and was considered also in [38] as a starting point for the first occurrence of a Lagrangian 1-form structure, namely for both the discrete- and continuous-time Calogero-Moser (CM) systems, but an initial obstacle in that paper was that, while we established a variational description of both the discrete and continuous CM hierarchy, we didn't have a Lagrangian description of the unreduced equation, namely (2.1) itself. This gap in the treatment we want to address here, alongside the establishment of the Lagrange structure for the fully discrete KP system.

[2] For this terminology we refer to [10], Ch. 5.

The system defined by (2.1), extended in all lattice directions, is multidimensionally consistent, subject to (1.1) itself, which for latter convenience we can also cast in the equivalent form[3]:

$$\frac{p-r+\widehat{\bar{u}}-\widehat{\widetilde{u}}}{p-r+\bar{u}-\widetilde{u}}=\frac{q-r+\widetilde{\bar{u}}-\widehat{\widetilde{u}}}{q-r+\bar{u}-\widehat{u}}=\frac{p-q+\widehat{\bar{u}}-\widetilde{\bar{u}}}{p-q+\widehat{u}-\widetilde{u}}\ , \tag{2.2}$$

Identifying (2.2) as a discrete potential KP equation, by virtue of its link to (1.2), a non-potential version of this equation was derived in [8], cf. also [26] for alternative versions of the latter. As mentioned before, (2.2) is itself multidimensionally consistent in terms of consistency on the tessaract, or more precisely on the octahedral lattice (cf. [1] for a detailed discussion).

Both (2.2) and (2.1) form part of a broader structure, which is revealed through the corresponding linear problems and further extensions. First, we note that eq. (2.1) arises as the compatibility condition from the following Lax pair

$$\widetilde{\varphi}=\varphi_\xi+(p+u-\widetilde{u})\varphi \tag{2.3a}$$
$$\widehat{\varphi}=\varphi_\xi+(q+u-\widehat{u})\varphi \tag{2.3b}$$

or alternatively from the adjoint Lax pair

$$\psi=-\widetilde{\psi}_\xi+\widetilde{\psi}(p+u-\widetilde{u}) \tag{2.4a}$$
$$\psi=-\widehat{\psi}_\xi+\widehat{\psi}(q+u-\widehat{u}) \tag{2.4b}$$

whereas the fully discrete equation arises from the (inhomogeneous) Lax triplets

$$\widetilde{\varphi}=(p-\widetilde{u})\varphi+\chi\ , \tag{2.5a}$$
$$\widehat{\varphi}=(q-\widehat{u})\varphi+\chi\ , \tag{2.5b}$$
$$\bar{\varphi}=(r-\bar{u})\varphi+\chi\ , \tag{2.5c}$$

or from the adjoint triplet

$$\psi=\widetilde{\psi}(p+u)-\widetilde{\theta}\ , \tag{2.6a}$$
$$\psi=\widehat{\psi}(q+u)-\widehat{\theta}\ , \tag{2.6b}$$
$$\psi=\bar{\psi}(r+u)-\bar{\theta}\ , \tag{2.6c}$$

where χ and θ are some auxiliary fields that can be pairwise eliminated from (2.5) and (2.6) to yield Lax pairs in a more conventional (homogeneous linear) form.

Furthermore we have the following single-shift semi-continuous KP equations[4]

$$\partial_\xi \ln(1+u_\tau)=\widetilde{u}+\underset{\widetilde{}}{u}-2u\ , \tag{2.7a}$$
$$\partial_\xi \ln(1+u_\sigma)=\widehat{u}+\underset{\widehat{}}{u}-2u\ , \tag{2.7b}$$

[3]Note that (2.2) is just one single equation, even though written in two alternative ways.

[4]Here and in what follows, the under-accents $\underset{\widetilde{}}{\cdot}$ and $\underset{\widehat{}}{\cdot}$ denote the backward shifts to the shifts $\widetilde{\cdot}$ and $\widehat{\cdot}$ respectively, i.e. $\underset{\widetilde{}}{u}=T_p^{-1}u=u(n-1,m,h)$ and $\underset{\widehat{}}{u}=T_q^{-1}u=u(n,m-1,h)$. Similarly, we also have $\underline{u}=T_r^{-1}u=u(n,m,h-1)$.

which involve the continuous Miwa variables τ and σ of (1.3), which are associated with the lattice parameters p and q, and hence with the lattice shifts T_p and T_q respectively (and similar equations associated with any of the lattice directions in the system). Eqs. (2.7), which are closely connected to the 2D Toda equation, can be obtained from a 'skew continuum limit' (again in the terminology of Ch. 5 of [10]) from (2.2), but they arise also as the compatibility conditions of (2.3) together with

$$\varphi_\tau = -(1+u_\tau)\underset{\sim}{\varphi}\ , \tag{2.8a}$$
$$\varphi_\sigma = -(1+u_\sigma)\underset{\frown}{\varphi}\ , \tag{2.8b}$$

or from the adjoint linear equations

$$\psi_\tau = (1+u_\tau)\widetilde{\psi}\ , \tag{2.9a}$$
$$\psi_\sigma = (1+u_\sigma)\widehat{\psi}\ , \tag{2.9b}$$

together with (2.4) respectively. From these linear systems we can derive the following purely continuous Lax pairs:

$$\varphi_{\xi\tau} + (p+u-\widetilde{u})\varphi_\tau + (1+u_\tau)\varphi = 0\ , \tag{2.10a}$$
$$\varphi_{\xi\sigma} + (q+u-\widehat{u})\varphi_\sigma + (1+u_\sigma)\varphi = 0\ , \tag{2.10b}$$
$$v\,\varphi_{\sigma\tau} = (1+\widetilde{u}_\sigma)\varphi_\tau - (1+\widehat{u}_\tau)\varphi_\sigma\ , \tag{2.10c}$$

where we have set $v = p-q+\widehat{u}-\widetilde{u}$. The analogous relations for the adjoint function ψ are

$$\psi_{\xi\tau} - (p+\underset{\sim}{u}-u)\psi_\tau + (1+u_\tau)\psi = 0\ , \tag{2.11a}$$
$$\psi_{\xi\sigma} - (q+\underset{\frown}{u}-u)\psi_\sigma + (1+u_\sigma)\psi = 0\ , \tag{2.11b}$$
$$\underset{\frown}{\underset{\sim}{v}}\,\psi_{\sigma\tau} = (1+\underset{\frown}{u}_\tau)\psi_\sigma - (1+\underset{\sim}{u}_\sigma)\psi_\tau\ . \tag{2.11c}$$

While (2.7) involve the variable ξ, we also have relations that only involve the continuous variables τ and σ, namely

$$v\,u_{\sigma\tau} \;=\; (1+u_\tau)(1+\widetilde{u}_\sigma) - (1+u_\sigma)(1+\widehat{u}_\tau)\ , \tag{2.12a}$$
$$\underset{\frown}{\underset{\sim}{v}}\,u_{\sigma\tau} \;=\; (1+u_\sigma)(1+\underset{\frown}{u}_\tau) - (1+u_\tau)(1+\underset{\sim}{u}_\sigma)\ , \tag{2.12b}$$

which complements (2.7). The following relations on the variable u also hold true

$$\frac{1+\widehat{u}_\tau}{1+u_\tau} = \frac{v}{\underset{\sim}{v}}\ ,\quad \frac{1+\widetilde{u}_\sigma}{1+u_\sigma} = \frac{v}{\underset{\frown}{v}}\ , \tag{2.13a}$$
$$\frac{u_{\sigma\tau}}{(1+u_\sigma)(1+u_\tau)} = \frac{1}{\underset{\frown}{v}} - \frac{1}{\underset{\sim}{v}}\ , \tag{2.13b}$$

and which arise from the Lax relations (2.5) and (2.8), or equivalently (2.6) and (2.9). From the latter relations, we can derive a fully continuous KP system, i.e. a coupled PDE system for u and v in terms of the independent variables ξ, σ and τ, eliminating all the lattice shifts. In fact, from (2.13b) using (2.1) we can derive the relation

$$\frac{1}{\underset{\frown}{v}} + \frac{1}{\underset{\sim}{v}} = \frac{4 + \frac{u_{\sigma\tau}}{(1+u_\sigma)(1+u_\tau)}\partial_\xi \ln\left(\frac{u_{\sigma\tau}^2}{(1+u_\sigma)(1+u_\tau)}\right)}{2v + \partial_\xi \ln\left(\frac{1+u_\tau}{1+u_\sigma}\right)} =: F \tag{2.14}$$

where the expression on the r.h.s., which we call $\mathcal{F}$, only depends on v and derivatives of u, but no longer involving any shifts. Combining (2.14) with (2.13b) we can solve for $\underset{\frown}{v}$ and $\underset{\sim}{v}$ in terms of v and u (and its derivatives) and then use the relation

$$\partial_\xi \ln \underset{\frown}{v} = \underset{\frown}{v} - v + \partial_\xi \ln(1+u_\sigma) \ , \tag{2.15a}$$

or the the alternative relation[5]

$$\partial_\xi \ln \underset{\sim}{v} = -\underset{\sim}{v} + v + \partial_\xi \ln(1+u_\tau) \ , \tag{2.15b}$$

and insert the expression for $\underset{\frown}{v}$ or $\underset{\sim}{v}$ respectively obtained from (2.14) and (2.13b), and obtain a PDE involving v, v_ξ and the derivatives of u up to $u_{\xi\xi\sigma\tau}$, which reads

$$v = \partial_\xi \ln\left((1+u_\sigma)\mathcal{F} + \frac{u_{\sigma\tau}}{1+u_\tau}\right) + \frac{2}{\mathcal{F} + \frac{u_{\sigma\tau}}{(1+u_\tau)(1+u_\sigma)}} \ , \tag{2.16a}$$

$$= -\partial_\xi \ln\left((1+u_\tau)\mathcal{F} - \frac{u_{\sigma\tau}}{1+u_\sigma}\right) + \frac{2}{\mathcal{F} - \frac{u_{\sigma\tau}}{(1+u_\tau)(1+u_\sigma)}} \ , \tag{2.16b}$$

To complement this relation, we also have from (2.13a) the PDE:

$$\begin{aligned} 2v_{\sigma\tau} =& \partial_\sigma \left[(1+u_\tau)v\left(\mathcal{F} - \frac{u_{\sigma\tau}}{(1+u_\tau)(1+u_\sigma)}\right)\right] \\ & - \partial_\tau \left[(1+u_\sigma)v\left(\mathcal{F} + \frac{u_{\sigma\tau}}{(1+u_\tau)(1+u_\sigma)}\right)\right] \ , \end{aligned} \tag{2.16c}$$

which together with (2.16a) or (2.16b), inserting the expression (2.14) for $\mathcal{F}$, forms a coupled system of PDEs for u and v. If one were to eliminate v from this system, and obtain a single PDE in terms of u, the highest derivative would be $u_{\xi\xi\sigma\sigma\tau\tau}$. We will refer to this coupled system (2.16) of PDEs as the *generating PDE of the KP hierarchy*, as by inserting the power series expansions (1.3) it should contain all equations in the KP hierarchy that can be obtained from the Sato scheme, [31].

[5]That these two relations are equivalent follows from the relation

$$\partial_\xi \ln\left(\frac{\underset{\approx}{v}}{\underset{\frown\sim}{v}}\right) = 2v - \underset{\sim}{v} - \underset{\frown}{v} + \partial_\xi \ln\left(\frac{1+u_\tau}{1+u_\sigma}\right) = v + \underset{\frown\sim}{v} - \underset{\frown}{v} - \underset{\sim}{v} \ ,$$

which is a consequence of (2.1) and where is used that

$$\partial_\xi \ln\left(\frac{1+u_\tau}{1+u_\sigma}\right) = \underset{\frown\sim}{v} - v \ ,$$

which in turn is a consequence of (2.7).

3 τ-function relations

The various relations for the function u can be resolved using the τ-function f. In fact, we have the following relations:

$$u = \partial_\xi \ln f \ , \tag{3.1a}$$

$$p - q + \widehat{u} - \widetilde{u} = (p-q)\frac{f\,\widehat{\widetilde{f}}}{\widehat{f}\,\widetilde{f}} \ , \tag{3.1b}$$

$$1 + u_\tau = \frac{\widetilde{f}\,\underset{\sim}{f}}{f^2} \ , \tag{3.1c}$$

$$1 + u_\sigma = \frac{\widehat{f}\,\underset{\wedge}{f}}{f^2} \ , \tag{3.1d}$$

These identification reduce (2.12), (2.7) and (2.1) to identities The equations for the τ-function comprise the Hirota-Miwa equation

$$.(p-q)\bar{f}\,\widehat{\widetilde{f}} + (q-r)\widetilde{f}\,\widehat{\bar{f}} + (r-p)\widehat{f}\,\widetilde{\bar{f}} = 0 \ , \tag{3.2}$$

as well as the Hirota-type bilinear equations

$$(p-q)\left(f_\sigma \widetilde{f} - f\widetilde{f}_\sigma\right) = f\,\widetilde{f} - \widehat{f}\,\underset{\wedge}{\widetilde{f}} \ , \tag{3.3a}$$

$$(q-p)\left(f_\tau \widehat{f} - f\widehat{f}_\tau\right) = f\,\widehat{f} - \widetilde{f}\,\underset{\sim}{\widehat{f}} \ , \tag{3.3b}$$

which can be ontained from (3.2) by performing a skew continuum limit (in the sense of Ch. 5 of [10]). In fact, this can be viewed as the following remarkable relation connecting the derivatives with regard to the Miwa variables and the lattice shifts on τ-function, namely

$$\partial_\tau f = -\left(T_p^{-1}\frac{\mathrm{d}}{\mathrm{d}p}T_p\right) f \ , \quad \partial_\sigma f = -\left(T_q^{-1}\frac{\mathrm{d}}{\mathrm{d}q}T_q\right) f \ . \tag{3.4}$$

We also have the relation

$$(p-q)u_{\sigma\tau} = \frac{\underset{\wedge}{\widetilde{f}}\,\underset{\sim}{f}\,\widehat{f} - \underset{\sim}{\widehat{f}}\,\widetilde{f}\,\underset{\wedge}{f}}{f^3} \ . \tag{3.5}$$

Finally, there is the Toda type differential-difference equation

$$(p-q)^2\left(f\,f_{\sigma\tau} - f_\sigma f_\tau\right) = \underset{\sim}{\widehat{f}}\,\underset{\wedge}{\widetilde{f}} - f^2 \ . \tag{3.6}$$

Note that a non-autonomous version of (3.6) was presented in [9].

A special solution to the KP system, in fact the pole reduction we studied in [24, 38], is obtained in terms of the τ-function by setting

$$f = \prod_{j=1}^{N}(\xi - x_j) \quad \Rightarrow \quad u = \sum_{j=1}^{N}\frac{1}{\xi - x_j} \ , \tag{3.7}$$

where the zeroes, respectively the poles, x_j are functions of the discrete variables $n, m, \ldots$ as well as of the continuous variables $\sigma, \tau, \ldots$, but not of ξ. By using the Lagrange interpolation formulae

$$\frac{Y(\xi)}{X(\xi)} = 1 + \sum_{j=1}^{N} \frac{1}{\xi - x_j} \frac{Y(x_j)}{X'(x_j)} \ , \tag{3.8a}$$

$$\frac{Y(\xi)Z(\xi)}{X(\xi)^2} = 1 + \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left(\frac{1}{\xi - x_j} \frac{Y(x_j)Z(x_j)}{X'(x_j)^2} \right) \ , \tag{3.8b}$$

$$\frac{Y(\xi)Z(\xi)W(\xi)}{X(\xi)^3} = 1 + \sum_{j=1}^{N} \frac{1}{2} \frac{\partial^2}{\partial x_j^2} \left(\frac{1}{\xi - x_j} \frac{Y(x_j)Z(x_j)W(x_j)}{X'(x_j)^3} \right) \tag{3.8c}$$

where

$$X(\xi) = \prod_{j=1}^{N} (\xi - x_j) \ , \quad Y(\xi) = \prod_{j=1}^{N} (\xi - y_j) \ , \quad Z(\xi) = \prod_{j=1}^{N} (\xi - z_j) \ , \quad W(\xi) = \prod_{j=1}^{N} (\xi - w_j) \ ,$$

where the roots y_j, z_j, w_j do not coincide with any of the roots x_j, and the latter are assumed distinct. Inserting (3.7) into the relations (3.1) and using the expansions (3.8), leads to

$$\sum_{j=1}^{N} \left(\frac{1}{x_i - \widetilde{x}_j} + \frac{1}{x_i - \underset{\sim}{x}_j} \right) - \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{2}{x_i - x_j} = 0 \quad \Leftrightarrow \quad \frac{\partial}{\partial x_i} \left(\frac{\prod_{j=1}^{N} (x_i - \widetilde{x}_j)(x_i - \underset{\sim}{x}_j)}{\prod_{j\neq i} (x_i - x_j)^2} \right) = 0 \ , \tag{3.9a}$$

$$\partial_\tau x_i = \frac{\prod_{j=1}^{N} (x_i - \widetilde{x}_j)(x_i - \underset{\sim}{x}_j)}{\prod_{j\neq i} (x_i - x_j)^2} \ , \tag{3.9b}$$

$$\frac{\prod_{j=1}^{N} (\widehat{x}_i - \widetilde{x}_j) \prod_{j\neq i} (\widehat{x}_i - \widehat{x}_j)}{\prod_{j=1}^{N} (\widehat{x}_i - x_j)(\widehat{x}_i - \widehat{\widehat{x}}_j)} = p - q \ , \tag{3.9c}$$

for $\quad i = 1, \ldots, N,$

(and similar relations with $\widetilde{x}$ and $\widehat{x}$, p and q and τ and σ interchanged), together with Eq. (3.9) are the equations of motion of the discrete-time Calogero-Moser model of [24, 38]. Furthermore, (3.5) inserting (3.7) and using (3.8c) leads to

$$(p - q)\partial_\sigma \partial_\tau x_i = \frac{\partial}{\partial x_i} \left(\frac{\underset{\sim}{\widetilde{X}}(x_i)\widehat{X}(x_i)\underset{\sim}{X}(x_i) - \underset{\sim}{\widehat{X}}(x_i)\widetilde{X}(x_i)\underset{\sim}{X}(x_i)}{X'(x_i)^3} \right) \ , \tag{3.10a}$$

$$0 = \frac{\partial^2}{\partial x_i^2} \left(\frac{\underset{\sim}{\widetilde{X}}(x_i)\widehat{X}(x_i)\underset{\sim}{X}(x_i) - \underset{\sim}{\widehat{X}}(x_i)\widetilde{X}(x_i)\underset{\sim}{X}(x_i)}{X'(x_i)^3} \right) \ . \tag{3.10b}$$

Note that the relation (3.9c) by Lagrange interpolation also leads to

$$1+\sum_{l=1}^{N}\left(\frac{1}{\widehat{x}_j-x_l}\,\frac{\widetilde{X}(x_l)\widehat{X}(x_l)}{X'(x_l)\widehat{\widehat{X}}(x_l)}+\frac{1}{\widehat{x}_j-\widehat{\widehat{x}}_l}\,\frac{\widetilde{X}(\widehat{\widehat{x}}_l)\widehat{X}(\widehat{\widehat{x}}_l)}{X(\widehat{\widehat{x}}_l)\widehat{\widehat{X}}'(\widehat{\widehat{x}}_l)}\right)=0\ , \tag{3.11a}$$

$$p-q+\sum_{l=1}^{N}\left(\frac{1}{(\widehat{x}_j-x_l)^2}\,\frac{\widetilde{X}(x_l)\widehat{X}(x_l)}{X'(x_l)\widehat{\widehat{X}}(x_l)}+\frac{1}{(\widehat{x}_j-\widehat{\widehat{x}}_l)^2}\,\frac{\widetilde{X}(\widehat{\widehat{x}}_l)\widehat{X}(\widehat{\widehat{x}}_l)}{X(\widehat{\widehat{x}}_l)\widehat{\widehat{X}}'(\widehat{\widehat{x}}_l)}\right)=0\ , \tag{3.11b}$$

(and similar relations with $\widetilde{x}$ and $\widehat{x}$, p and q interchanged).

4 Direct Linearising transform

From here on we will assume that the functions φ of the Lax pairs can be characterised by a (spectral type) parameter k, while the adjoint functions can be characterised by a parameter k', the dependence on which we will denote by an index, namely φ_k and $\psi_{k'}$ respectively. These spextal parameters can be identified with the lattice parameters associated with specific directions in the multidimensional lattice, and as such the Lax pair functions can be expressed in terms of the τ-function as

$$\varphi_k=\frac{T_{-k}f}{f}\,\varphi_k^0\ ,\quad \psi_{k'}=\frac{T_{k'}^{-1}f}{f}\,\psi_{k'}^0\ . \tag{4.1}$$

Here the functions φ_k^0 and $\psi_{k'}^0$ are plane-wave factors of the form

$$\varphi_k^0=\left[\prod_\nu(p_\nu+k)^{n_\nu}\right]\exp\left\{k\xi-\sum_\nu\frac{\tau_\nu}{p_\nu+k}\right\}\ , \tag{4.2a}$$

$$\psi_{k'}^0=\left[\prod_\nu(p_\nu-k')^{-n_\nu}\right]\exp\left\{k'\xi+\sum_\nu\frac{\tau_\nu}{p_\nu-k'}\right\}\ . \tag{4.2b}$$

which are the solutions of the Lax pair relations when the potential function $u=u^0=0$. In (4.2) the variables n_ν comprise the discrete variables n and m used before, associated with the corresponding lattice parameters p, q as specific choices for the p_ν, while the corresponding continuous variables τ_ν are the τ and σ variables in the previous sections.

More generally, the direct linearising transform (DLT), [21, 22, 27, 28, 32], cf. also [8], is a very general abstract dressing scheme that builds new solutions from given solutions in terms of linear integral equations of a very general form. I will briefly summarise the scheme here.

It starts with the observation that, as a consequence of the linear Lax equations for φ and ψ, have the following Wronskian type (or closure) relations:

$$\Delta_q\left((T_p\psi_{k'})\varphi_k\right)=\Delta_p\left((T_q\psi_{k'})\varphi_k\right)\ , \tag{4.3a}$$

$$\partial_\xi\left((T_p\psi_{k'})\varphi_k\right)=\Delta_p\left(\psi_{k'}\varphi_k\right)\ , \tag{4.3b}$$

$$\partial_\tau\left((T_p\psi_{k'})\varphi_k\right)=\Delta_p\left((1+u_\tau)(T_p\psi_{k'})(T_p^{-1}\varphi_k)\right)\ . \tag{4.3c}$$

In addition to these there are relations of mixed type, namely

$$\partial_\tau\left((T_q\psi_{k'})\varphi_k\right)=\Delta_q\left((1+u_\tau)(T_p\psi_{k'})(T_p^{-1}\varphi_k)\right)\ , \tag{4.3d}$$

$$\partial_\sigma\left((1+u_\tau)(T_p\psi_{k'})(T_p^{-1}\varphi_k)\right)=\partial_\tau\left((1+u_\sigma)(T_q\psi_{k'})(T_q^{-1}\varphi_k)\right) \tag{4.3e}$$

which are nontrivial, and hold subject to the relations (2.13) on the variable u.

Obviously similar relations to (4.3) also hold for any other pairs of shift variables and parameters and corresponding continuous time-variables in the system. These relations suggest that there is closed 1-form in the space of discrete as well as continuous variables which we symbolically write as

$$\Omega_{k,k'}=\psi_{k'}\varphi_k\,\mathrm{d}\xi+\sum_\nu\left[(T_{p_\nu}\psi_{k'})\varphi_k\,\delta_{p_\nu}+(1+u_{\tau_\nu})(T_{p_\nu}\psi_{k'})(T_{p_\nu}^{-1}\varphi_k)\,\mathrm{d}\tau_\nu\right]\ , \tag{4.4}$$

where ν is a symbol that labels all the lattice directions in the system, and δ_p is a symbol denoting that the finite contribution, given by the corresponding coefficient of the 1-form is in the direction associated with the parameter p. Closedness of this form (4.4) implies that the semisum-integral[6]

$$G_{k,k'}=\sum\!\!\!\!\!\!\int_\Gamma\Omega_{k,k'}\ . \tag{4.5}$$

which combines a sum over discrete path sections as well as a line integral over continuous sections of a mixed discrete/continuous path Γ in the space of discrete and continuous variables n_ν, ξ and τ_ν. As a consequence of the closure relations (4.3) the kernel type quantity $G_{k,k'}$ is independent of the path Γ and only depends on its end-points in the space of these variables. Thus, we have the following relations for its derivatives and differences:

$$\partial_\xi G_{k,k'}=\psi_{k'}\varphi_k\ ,\quad \partial_{\tau_\nu}G_{k,k'}=(1+u_{\tau_\nu})(T_p\psi_{k'})T_p^{-1}\varphi_k\ ,\quad \Delta_pG_{k,k'}=(T_p\psi_{k'})\varphi_k\ .$$

Consider now the integral equations

$$\varphi_k=\varphi_k^0+\iint_D d\zeta(l,l')\varphi_l\,G^0_{k,l'}\ , \tag{4.6a}$$

$$\psi_{k'}=\psi_{k'}^0+\iint_D d\zeta(l,l')G_{l,k'}\psi^0_{l'}\ . \tag{4.6b}$$

where the integral is over a domain D in any appropriate space of the parameters k and k' with an arbitrary measure $\mathrm{d}\zeta(k,k')$ in that space[7], which is independent of the dynamical variables n_ν,τ_ν and ξ. There is no need to specify this integral as long as some assumptions are valid, such as that operations like the lattice shifts T_{p_ν} and derivatives $\partial_\xi,\partial_{\tau_\nu}$ can be moved through the integral without problem.

[6]This partly discrete- partly continuous line integral may be understood analytically in terms of the theory of timescales, cf. e.g. [4, 5].

[7]The integral over k and k' can be over a continuous domain D or over a set of distinct values of these parameters.

We can think of the integral equations (4.6) as an integral *transform* from an initial solution $(u^0,\varphi_k^0,\psi_{k'}^0)$ of the system of equations (comprising the Lax pairs as well as the nonlinear equations for u) to a new 'dressed' solution $(u,\varphi_k,\psi_{k'})$, where

$$u = u^0 - \iint_D d\zeta(l,l')\varphi_l\,\psi_{l'}^0\ . \tag{4.7}$$

It was shown in [8], cf. also [19] for the general matrix case, that this dressing transform has a group property as a consequence of the structure of the kernel function $G_{k,k'}$ as a path independent integrated object, cf. (4.5). Furthermore, the kernel itself obeys the dressing relation, namely

$$G_{k,k'} = G_{k,k'}^0 + \iint_D d\zeta(l,l')G_{l,k'}\,G_{k,l'}^0\ . \tag{4.8}$$

In the case of 'free reference' dressing, i.e. when $u^0=0$ and φ_k^0 and $\psi_{k'}^0$ take the form (4.2) we have the square eigenfunction expansions, cf. [6],

$$\partial u = \iint_D d\zeta(l,l')\varphi_l\psi_{l'}\,\partial\left(\ln(\varphi_k^0\psi_{k'}^0)\right)\ , \tag{4.9a}$$

$$Tu-u = \iint_D d\zeta(l,l')(T\varphi_l)\psi_{l'}\left(\frac{T\psi_{l'}^0}{\psi_{l'}^0}-\frac{\varphi_l^0}{T\varphi_l^0}\right) = \iint_D d\zeta(l,l')\varphi_l(T\psi_{l'})\left(\frac{T\varphi_l^0}{\varphi_l^0}-\frac{\psi_{l'}^0}{T\psi_{l'}^0}\right)\ , \tag{4.9b}$$

where T is any shift operator which commutes with the double integral and ∂ is any first order derivative which commutes with the double integral.

In particular, the following fundamental relation holds

$$\iint_D d\zeta(l,l')\frac{(l+l')\varphi_l^0\psi_{l'}^0}{(a+l')(a'+l)}(T_{a'}^{-1}T_{-l}f)(T_{l'}^{-1}T_{-a}f) = f\,(T_{a'}^{-1}T_{-a}f)-(T_{a'}^{-1}f)(T_{-a}f)\ , \tag{4.10}$$

which is reminiscent of the bilinear identity that plays a central role in the work of the Kyoto school, and from which the hierarchy of Hirota bilinear equations can be derived, cf. [18].

5 Discrete Lagrangian 3-form structure

We will now describe the Lagrangian structure for the KP system comprising both the fully discrete as well as the semi-discrete equations, in terms of Lagrangian difference or differential 3-forms. The most general structure of this form would be

$$\begin{aligned}\mathsf{L} = &\sum_{i<j<k}\mathcal{L}_{p_ip_jp_k}\delta_{p_i}\wedge\delta_{p_j}\wedge\delta_{p_k} + \mathcal{L}_{(\tau_i)(\tau_j)(\tau_k)}\mathrm{d}\tau_i\wedge\mathrm{d}\tau_j\wedge\mathrm{d}\tau_k\\ &+\sum_i\sum_{j<k}\mathcal{L}_{(\tau_i)p_jp_k}\mathrm{d}\tau_i\wedge\delta_{p_j}\wedge\delta_{p_k} + \mathcal{L}_{p_i(\tau_j)(\tau_k)}\delta_{p_i}\wedge\mathrm{d}\tau_j\wedge\mathrm{d}\tau_k\\ &+\sum_{i<j}\mathcal{L}_{(\xi)p_ip_j}\mathrm{d}\xi\wedge\delta_{p_i}\wedge\delta_{p_j} + \mathcal{L}_{(\xi)(\tau_i)(\tau_j)}\mathrm{d}\xi\wedge\mathrm{d}\tau_i\wedge\mathrm{d}\tau_j\\ &+\sum_{i,j}\mathcal{L}_{(\xi)(\tau_i)p_j}\mathrm{d}\xi\wedge\mathrm{d}\tau_i\wedge\delta_{p_j}\ ,\end{aligned} \tag{5.1}$$

which is to be integrated over a 3-dimensional hypersurface, containing smooth as well as semi-discrete and fully discrete coordinate patches. We will not attempt to describe such a weird hypersurface here, along the line as e.g. in [37, 38] for the much simpler situation of semi-discrete 1-curves and 2-surfaces, the main property needed here in the definition of an integral of the type

$$S[u(\xi,\boldsymbol{n},\boldsymbol{\tau});\mathcal{V}] = \int\!\!\!\!\!\!\sum_{\mathcal{V}} \mathsf{L} \tag{5.2}$$

as a functional of both field variables $u(\xi,\boldsymbol{n},\boldsymbol{\tau})$ and semi-discrete hypersurfaces $\mathcal{V}$, is the existence of a generalised Stokes' theorem that applies to closed semi-discrete hypersurfaces. This can be established, but is technically (mostly as a matter of introducing the appropriate notations) quite involved, and we leave the formulation and proof of such a theorem to a future publication. The main point being that, as we want to find the critical point of the action (5.2) for arbitrary hypersurfaces $\mathcal{V}$, it suffices to consider closed hypersurfaces and apply Stokes' theorem to the first order variation of S. Setting the latter to zero as stationarity condition, we arrive at the condition

$$\delta \mathrm{d}\mathsf{L} = 0 \ , \tag{5.3}$$

which form the relevant generalised Euler-Lagrange equations where d denotes the relevant exterior difference/derivative of the Lagrangian 3-form (in the language of a variational bi-complex with horizontal and vertical derivative operators d and δ, the latter comprising discrete exterior 'derivatives' as well as continuous ones). In order to develop the Lagrangian 3-form systematically, we break the process of deriving the EL equations and establishing the corresponding closure relation

$$\mathrm{d}\mathsf{L}\big|_{\mathrm{EL}} = 0 \ , \tag{5.4}$$

up in semi-discrete and fully discrete components, and treat them separately.

In the case we restrict ourself to the fully discrete Lagrangian component we have a discrete Lagrangian 3-form

$$\mathsf{L} = \sum_{i<j<k} \mathcal{L}_{p_ip_jp_k}\delta_{p_i}\wedge\delta_{p_j}\wedge\delta_{p_k} \ , \tag{5.5}$$

where the $\mathcal{L}_{p_ip_jp_k}$ each are the Lagrangian components for any three directions indicated by the lattice parmeters p_i, p_j, p_k (which could comprise the parameters p, q and r of the previous sections as particular choices) each associated with a discrete lattice variable n_{p_i} playing the role of coordinates for the i^{th} direction in that multidimensional lattice. As before, in (4.4) the δ_p denotes a *discrete differential*[8], i.e. the formal symbol indicating that in the restricted action functional

$$S[u(\boldsymbol{n});\nu] = \sum_{\nu}\mathsf{L} = \sum_{\nu_{ijk}\in\nu} \mathcal{L}_{p_ip_jp_k}\delta_{p_i}\wedge\delta_{p_j}\wedge\delta_{p_k} \tag{5.6}$$

the Lagrangian contributions from all elementary rhomboids $\nu_{ijk} = (\boldsymbol{n},\boldsymbol{n}+\boldsymbol{e}_i,\boldsymbol{n}+\boldsymbol{e}_j,\boldsymbol{n}+\boldsymbol{e}_k)$ (with elementary displacement vectors $\boldsymbol{e}_i$ (see Figure 1) along the edges in the 3D

[8]The notation is similar to the one introduced in [16].

lattice associated with the lattice parameter p_i) are simply summed up according to their base point $\boldsymbol{n}$ and their orientation.

A Lagrangian for a 3-dimensional system can be defined on an elementary cube (or rhomboid) $\nu_{ijk}(\boldsymbol{n})$, where $\nu_{ijk}(\boldsymbol{n})$ is specified by the position $\boldsymbol{n} = (\boldsymbol{n}, \boldsymbol{n}+\boldsymbol{e}_i, \boldsymbol{n}+\boldsymbol{e}_j, \boldsymbol{n}+\boldsymbol{e}_k)$ of one of its vertices in the lattice and the lattice directions given by the base vectors $\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k$, as in Figure 1. Thus,

$$\begin{aligned}\mathcal{L}_{ijk}(\boldsymbol{n}) = L(u(\boldsymbol{n}), u(\boldsymbol{n}+\boldsymbol{e}_i), u(\boldsymbol{n}+\boldsymbol{e}_j), u(\boldsymbol{n}+\boldsymbol{e}_k), u(\boldsymbol{n}+\boldsymbol{e}_i+\boldsymbol{e}_j),\\ u(\boldsymbol{n}+\boldsymbol{e}_j+\boldsymbol{e}_k), u(\boldsymbol{n}+\boldsymbol{e}_i+\boldsymbol{e}_k), u(\boldsymbol{n}+\boldsymbol{e}_i+\boldsymbol{e}_j+\boldsymbol{e}_k); p_i, p_j, p_k)\ ,\end{aligned}$$

as well as on lattice parameters p_i, p_j, p_k, (where where for simplicity of notation we have replaced the indices p_i, p_j, p_k simply by i, j, k and where $\Delta_i = T_{p_i} - \mathrm{id}$ is the forward difference operator in the direction labeled by p_i).

According to the derivation proposed in [15], the set of multiform EL equations is obtained by considering the smallest closed 3D hypersurface, which is the boundary of a 4D hypercube (hyper-rhomboid) $\mathcal{R}$, the action of which is given by

$$S[u(\boldsymbol{n}); \partial\mathcal{R}] =: (\square\mathcal{L})_{ijkl} = \Delta_i\mathcal{L}_{jkl} - \Delta_j\mathcal{L}_{kli} + \Delta_k\mathcal{L}_{lij} - \Delta_l\mathcal{L}_{ijk}\ . \tag{5.7}$$

Embedding the system in 4 dimension, the smallest closed 3-dimensional hypersurface is the boundary of the tessaract, consisting of 8 3-dimensional faces, each of which is a rhomboid. Because of the symmetry, we need only take derivatives with respect to u, T_iu, T_iT_ju, $T_iT_jT_ku$ and $T_iT_jT_kT_lu$, and the other equations will follow by cyclic permutation of the lattice directions.

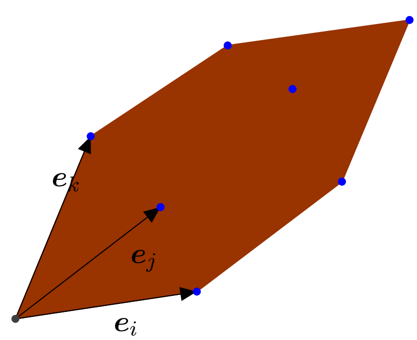

Figure 1. Elementary oriented rhomboid.

The variations with regard to the vertices of the closed hyper-rhomboid lead to the following set of equations, cf. [15],

$$0=\frac{\partial}{\partial u}\Big(-\mathcal{L}_{ijk}+\mathcal{L}_{jkl}-\mathcal{L}_{kli}+\mathcal{L}_{lij}\Big), \tag{5.8a}$$

$$0=\frac{\partial}{\partial T_iu}\Big(-\mathcal{L}_{ijk}-T_i\mathcal{L}_{jkl}-\mathcal{L}_{kli}+\mathcal{L}_{lij}\Big), \tag{5.8b}$$

$$0=\frac{\partial}{\partial T_iT_ju}\Big(-\mathcal{L}_{ijk}-T_i\mathcal{L}_{jkl}+T_j\mathcal{L}_{kli}+\mathcal{L}_{lij}\Big), \tag{5.8c}$$

$$0=\frac{\partial}{\partial T_iT_jT_ku}\Big(-\mathcal{L}_{ijk}-T_i\mathcal{L}_{jkl}+T_j\mathcal{L}_{kli}-T_k\mathcal{L}_{lij}\Big), \tag{5.8d}$$

$$0=\frac{\partial}{\partial T_iT_jT_kT_lu}\Big(T_l\mathcal{L}_{ijk}-T_i\mathcal{L}_{jkl}+T_j\mathcal{L}_{kli}-T_k\mathcal{L}_{lij}\Big), \tag{5.8e}$$

along with the equivalent shifted versions

$$0=\frac{\partial}{\partial u}\Big(-\mathcal{L}_{ijk}+\mathcal{L}_{jkl}-\mathcal{L}_{kli}+\mathcal{L}_{lij}\Big), \tag{5.9a}$$

$$0=\frac{\partial}{\partial u}\Big(-T_i^{-1}\mathcal{L}_{ijk}-\mathcal{L}_{jkl}-T_i^{-1}\mathcal{L}_{kli}+T_i^{-1}\mathcal{L}_{lij}\Big), \tag{5.9b}$$

$$0=\frac{\partial}{\partial u}\Big(-T_i^{-1}T_j^{-1}\mathcal{L}_{ijk}-T_j^{-1}\mathcal{L}_{jkl}+T_i^{-1}\mathcal{L}_{kli}+T_i^{-1}T_j^{-1}\mathcal{L}_{lij}\Big), \tag{5.9c}$$

$$0=\frac{\partial}{\partial u}\Big(-T_i^{-1}T_j^{-1}T_k^{-1}\mathcal{L}_{ijk}-T_j^{-1}T_k^{-1}\mathcal{L}_{jkl}+T_i^{-1}T_k^{-1}\mathcal{L}_{kli}-T_i^{-1}T_j^{-1}\mathcal{L}_{lij}\Big), \tag{5.9d}$$

$$0=\frac{\partial}{\partial u}\Big(T_i^{-1}T_j^{-1}T_k^{-1}\mathcal{L}_{ijk}-T_j^{-1}T_k^{-1}T_l^{-1}\mathcal{L}_{jkl}+T_i^{-1}T_k^{-1}T_l^{-1}\mathcal{L}_{kli}-T_i^{-1}T_j^{-1}T_l^{-1}\mathcal{L}_{lij}\Big). \tag{5.9e}$$

The system of 'corner equations' comprising (5.8), (5.9) supplemented with the 3-form closure relation

$$\Delta_l\mathcal{L}_{ijk}-\Delta_i\mathcal{L}_{jkl}+\Delta_j\mathcal{L}_{kli}-\Delta_k\mathcal{L}_{lij}=0 \tag{5.10}$$

forms the fundamental system of EL equations of a 3D integrable fully discrete system, when we consider it as a system which not only represents the relevant equations of motion, but also as a system of equations for the (possibly unknown) Lagrangian components themselves. However, in applying the multiform scheme to the KP system, we will make a short-cut by regarding the double-zero structure of $\mathsf{d}\mathsf{L}$, following [25, 30, 35], cf. also [36], which provides a short-cut to the relevant EL system in the sense of the variational bi-complex, which implies the closure relation (5.10).

6 Lagrangian 3-forms for (semi-)discrete KP systems

A main aim of this paper is to describe the Lagrangian 3-form structure for the fully discrete KP system (1.1), whereas the only instance so far of a Lagrangian structure for a fully discrete 3-dimensional integrable equation was given in [14], namely for the Hirota bilinear KP equation, cf. also [3] for a geometric interpretation of that result. In order to arrive at that end, we will take an indirect route, namely via a Lagrange structure for the semi-discrete KP stystem (2.1). Furthermore, we will establish the double-zero structure of the fully discrete Lagrangians 3-form.

6.1 Lagrangian structure of the semi-discrete KP

A Lagrangian for eq. (2.1) is given by

$$\mathcal{L}_{(\xi)pq} = v_\xi \ln(p - q + \widehat{u} - \widetilde{u}) + v(\widehat{\widetilde{u}} + u - \widehat{u} - \widetilde{u}) \ , \tag{6.1}$$

where u and v are independently variable fields. The conventional Euler-Lagrange (EL) equations (as a semi-discrete field theory) are given by

$$\frac{\delta \mathcal{L}_{(\xi)pq}}{\delta v} = -\partial_\xi \ln(p - q + \widehat{u} - \widetilde{u}) + \widehat{\widetilde{u}} + u - \widehat{u} - \widetilde{u} = 0 \ , \tag{6.2a}$$

$$\frac{\delta \mathcal{L}_{(\xi)pq}}{\delta u} = \frac{\underset{\wedge}{v}_\xi}{p - q + u - \underset{\wedge}{\widetilde{u}}} - \frac{\underset{\sim}{v}_\xi}{p - q + \underset{\sim}{\widehat{u}} - u} + \underset{\wedge}{\underset{\sim}{v}} + v - \underset{\wedge}{v} - \underset{\sim}{v} = 0 \ , \tag{6.2b}$$

where (6.2a) is (2.1), and (6.2b) a direct consequence provided we have the solution $v = \widehat{u} - \widetilde{u} +$ constant. Note that the latter is only valid 'on-shell' (i.e. on solutions of the EL equations), as posing this off-shell (i.e., as a reduction on the variational system) would trivialize the Lagrangian (6.1). Lagrangians for the DDΔ equations (2.7) are given by

$$\mathcal{L}_{(\xi)(\tau)p} = V_\xi \ln(1 + u_\tau) + V(\widetilde{u} + \underset{\sim}{u} - 2u) \ , \tag{6.3a}$$

$$\mathcal{L}_{(\xi)(\sigma)q} = W_\xi \ln(1 + u_\sigma) + W(\widehat{u} + \underset{\wedge}{u} - 2u) \ , \tag{6.3b}$$

the (conventional) EL equations of which yield

$$\frac{\delta \mathcal{L}_{(\xi)(\tau)p}}{\delta V} = -\partial_\xi \ln(1 + u_\tau) + \widetilde{u} + \underset{\sim}{u} - 2u = 0 \ , \tag{6.4a}$$

$$\frac{\delta \mathcal{L}_{(\xi)(\tau)p}}{\delta u} = -\partial_\tau \left(\frac{V_\xi}{1 + u_\tau} \right) + \widetilde{V} + \underset{\sim}{V} - 2V = 0 \ . \tag{6.4b}$$

(and similar relations arising from (6.3b)). Thus, varying w.r.t. V (resp. W) yield the equations (2.1), while the variations with respect to u yield EL equations that are satisfied by $V = 1 + u_\tau$ and $W = 1 + u_\sigma$ respectively on-shell.

Remark: We note that there is an (on-shell) closure relation

$$\left(\partial_\sigma \mathcal{L}_{(\xi)(\tau)p} - \partial_\tau \mathcal{L}_{(\xi)(\sigma)q} \right) \Big|_{\rm EL} = \partial_\xi \left[u_{\tau\sigma} \ln \left(\frac{1 + u_\tau}{1 + u_\sigma} \right) \right] \ , \tag{6.5}$$

where the expression within the straight brackets on the r.h.s. can be considered as a 'null Lagrangian' (one whose conventional EL equations are trivial). The closure relation (6.5) absurdly suggests that there might be a Lagrangian multi-form

$$\mathsf{L} = \mathcal{L}_{(\xi)(\tau)p} \mathrm{d}\xi \wedge \mathrm{d}\tau \wedge \delta_p + \mathcal{L}_{(\xi)(\sigma)q} \mathrm{d}\xi \wedge \mathrm{d}\sigma \wedge \delta_q + \mathcal{L}_{(\tau)(\sigma)} \mathrm{d}\tau \wedge \mathrm{d}\sigma \ , \tag{6.6}$$

which would be a mixed 3-form and 2-form. Whether such an object would make sense should perhaps not be rejected off-hand (the latter term representing perhaps some boundary term) but for now we postpone the investigation of the corresponding multiform structure. Instead, below we will develop the multiform structure associated with (6.1).

6.2 Lagrangians for the Lax representation

Before discussing the multiform structure for (2.1), we make a side-step by showing that there are Lagrangians also for the corresponding Lax represenations of (2.1) and (2.7), namely by using the square eigenfunction representations (4.9). It is here where the multiform structure becomes crucial, as a Lax representation requires the posing of an overdetermined pair of equations, rather than a single equation (the Lax equation). In this spirit a variational description of Lax *pairs* was first given in [34].

The individual components of the Lax representation, namely (2.3a) and (2.8a) and their respective adjoints (2.4a) and (2.9a), can be obtained from the following Lagrangians respectively,

$$\mathsf{L}_p = \tfrac{1}{2}(\widetilde{u}-u)^2 + p(u-\widetilde{u}) + \iint_D d\zeta(l,l')(l+l')\left[\widetilde{\psi}_{l'}\partial_\xi\varphi_l - \psi_{l'}\varphi_l + \widetilde{\psi}_{l'}(p+u-\widetilde{u})\varphi_l\right] \ , \tag{6.7a}$$

$$\mathsf{L}_{(\tau)} = u\partial_\tau\widetilde{u} + \iint_D d\zeta(l,l')(l+l')(2p+l-l')\left[(1+u_\tau)\widetilde{\psi}_{l'}\,\underset{\sim}{\varphi}_l + \widetilde{\psi}_{l'}\partial_\tau\varphi_l\right] \ , \tag{6.7b}$$

and similar Lagrangians associated with the other lattice direction and with the variable σ and parameter q instead of τ and p. The verification that these Lagrangians yield the correct Lax pair equations utilises an application of the formulae (4.9) in the case that $T = T_p$ and $\partial = \partial_\tau$. In fact, for those choices of shift and differential operator, we get by using also (4.2) the square eigenfunction expansion

$$\widetilde{u} - u = \iint_D d\zeta(l,l')(l+l')\widetilde{\psi}_{l'}\varphi_l \ , \quad \widetilde{u} - \underset{\sim}{u} = \iint_D d\zeta(l,l')(l+l')(2p+l-l')\widetilde{\psi}_{l'}\underset{\sim}{\varphi}_l \ .$$

and

$$u_\xi = \iint_D d\zeta(l,l')(l+l')\psi_{l'}\varphi_l \ , \quad u_\tau = \iint_D d\zeta(l,l')\frac{l+l'}{(p+l)(p-l')}\psi_{l'}\varphi_l \ .$$

Thus, variations of (6.7) yields

$$\frac{\delta\mathsf{L}_p}{\delta\psi_{k'}} = \int_{C_{k'}} \mathrm{d}\lambda_{k'}(l)\left[-\varphi_l + \partial_\xi\underset{\sim}{\varphi}_l + (p+\underset{\sim}{u}-u)\underset{\sim}{\varphi}_l\right] = 0 \ , \tag{6.8a}$$

$$\frac{\delta\mathsf{L}_p}{\delta\varphi_k} = \int_{C'_k} \mathrm{d}\lambda'_k(l')\left[-\psi_{l'} - \partial_\xi\widetilde{\psi}_{l'} + (p+u-\widetilde{u})\widetilde{\psi}_{l'}\right] = 0 \ , \tag{6.8b}$$

$$\frac{\delta\mathsf{L}_p}{\delta u} = 2u - \widetilde{u} - \underset{\sim}{u} + \iint_D d\zeta(l,l')(l+l')\left(\widetilde{\psi}_{l'}\varphi_l - \psi_{l'}\underset{\sim}{\varphi}_l\right) = 0 \ , \tag{6.8c}$$

where the latter holds true by virtue of the square eigenfunction expansion, whilst the former two are weak forms of the Lax relations. Note that we have assumed that the functional derivative with regard to the k- and k'-dependent quantity within the double integral reduces the double integral to a single integral over (boundary) curves $C_{k'}$ and C'_k in the space of k respectively k' variables[9].

[9]This would obviously require an analytic justification, for specific integration regions D and measure $\mathrm{d}\zeta$, but that is beyond the scope of this paper. Here we will just assume that this can be done for specific choices of the integrations.

Variations of $\mathsf{L}_{(\tau)}$ yield the following EL equations

$$\frac{\delta \mathsf{L}_{(\tau)}}{\delta \psi_{k'}} = \int_{C_{k'}} \mathrm{d}\lambda_{k'}(l)\,(2p+l-l')\left[(1+\underset{\sim}{u}_\tau)\underset{\approx}{\varphi}_l + \partial_\tau \underset{\sim}{\varphi}_l\right] = 0\ , \tag{6.9a}$$

$$\frac{\delta \mathsf{L}_{(\tau)}}{\delta \varphi_{k}} = \int_{C'_{k}} \mathrm{d}\lambda'_{k}(l')\,(2p+l-l')\left[(1+\widetilde{u}_\tau)\widetilde{\widetilde{\psi}}_{l'} - \partial_\tau \widetilde{\psi}_{l'}\right] = 0\ , \tag{6.9b}$$

$$\frac{\delta \mathsf{L}_{(\tau)}}{\delta u} = \partial_\tau(\widetilde{u}-\underset{\sim}{u}) - \partial_\tau \iint_D d\zeta(l,l')(l+l')(2p+l-l')\widetilde{\psi}_{l'}\underset{\sim}{\varphi}_l = 0\ , \tag{6.9c}$$

which are in some sense weaker forms of the Lax pair (2.9) and where the latter holds true by virtue of the square eigenfunction expansion. The main question is how to fit the Lagrangians L_p and L_q, or alternatively the Lagrangians L_τ and L_σ, into a multiform so that the multiform EL equation produce the respective pairs. This is still an open problem, but the following observation may provide a lead.

Remark1: The eigenfunction Lagrangians (6.7) obey the following closure relation

$$(\Delta_p \mathsf{L}_q - \Delta_q \mathsf{L}_p)\Big|_{\mathrm{EL}} = \partial_\xi(p-q+\widehat{u}-\widetilde{u})\ .$$

This suggests a Lagrangian 2-form

$$\mathbf{L} = \mathsf{L}_p\delta_p\wedge \mathrm{d}\xi + \mathsf{L}_q\delta_q\wedge \mathrm{d}\xi + \mathsf{L}_{pq}\delta_p\wedge\delta_q\ ,$$

where $\mathsf{L}_{pq} = p-q+\widehat{u}-\widetilde{u}$ is a *null Lagrangian* which has trivial EL equations. However, this closure relation does not seem yet either sufficient or appropriate to provide the full variational structure for the Lax representation, and we intend to come back to this matter in a future publication.

Remark2: Although the Lax equations come out in what seems to be a weaker (i.e., integrated) form, they are really are not in any sense weaker as for instance in the case of (6.8) one could introduce the functions

$$\Phi_{k'} = \int_{C_{k'}} \mathrm{d}\lambda_{k'}(l)\varphi_l\ ,\quad \Psi_k = \int_{C'_k} \mathrm{d}\lambda'_k(l')\psi_{l'}\ ,$$

in terms of which we recover the original Lax pairs both the adjoint and the direct forms. In the case of (6.9) it is more subtle, but essentially the variational form of the Lax is as strong as the original form.

The Lax equations for the semi-discrete KP system are thus retrievable from well-chosen Lagrangians, but what we need is a mechanism to obtain these equations simultaneously as Lax pairs from variations of a single structure. For this we need the notion of Lagrangian multiforms, which we will address now in the case of Lagrangains of the form (6.1).

6.3 From semi-discrete to fully discrete KP Lagrangian

We will now first establish the Lagrangian multiform structure for the semi-discrete KP system. Surprisingly this will provide us with a Lagrangian structure for the fully discrete

KP equation (2.2). Considering three copies of the Lagrangian (6.1) in three lattice dimensions (in addition to the continuous one associated with ξ), and observing that *on-shell* the variable v can be identified with $v \doteq p-q+\widehat{u}-\widetilde{u}$, we note that the latter quantity is actually a '2-form valued object'. Thus, for each pair of lattice directions we need to introduce a separate variable v, and consequently in three dimensions the Lagrangian components can be set as:

$$\mathcal{L}_{(\xi)pq} = v_\xi \ln(p - q + \widehat{u} - \widetilde{u}) + v(\widehat{\widetilde{u}} + u - \widehat{u} - \widetilde{u}) \ , \tag{6.10a}$$

$$\mathcal{L}_{(\xi)qr} = w_\xi \ln(q - r + \bar{u} - \widehat{u}) + w(\widehat{\bar{u}} + u - \widehat{u} - \bar{u}) \ , \tag{6.10b}$$

$$\mathcal{L}_{(\xi)rp} = z_\xi \ln(r - p + \widetilde{u} - \bar{u}) + z(\widetilde{\bar{u}} + u - \widetilde{u} - \bar{u}) \ , \tag{6.10c}$$

where on-shell the additional 2-form variables w and z can be identified with

$$w = q - r + \bar{u} - \widehat{u} \ , \quad z = r - p + \widetilde{u} - \bar{u} \ .$$

These are constrained by the conditions

$$v + w + z = 0 \ , \quad \bar{v} + \widetilde{w} + \widehat{z} = 0 \ . \tag{6.11}$$

Furthermore, we note that the potential lattice KP equation (2.2) can be written as

$$\frac{\bar{v}}{v} = \frac{\widetilde{w}}{w} = \frac{\widehat{z}}{z} \ . \tag{6.12}$$

The system comprising (6.11) and (6.12) was considered in the context of the non-Abelian case by [26], while in [8], we showed how by elimination this system gives rise to a scalar *non-potential* lattice KP equation for each of the fields v, w or z. For instance, in terms of v the resulting non-potential lattice KP is the 10-point equation

$$\widehat{\widetilde{v}\widehat{v}}\left[\left(\widehat{\widetilde{\bar{v}}} - \widehat{\widehat{\widetilde{v}}}\right)\widehat{v}\widetilde{\bar{v}} + \widehat{\widehat{\widetilde{v}}}\widetilde{v}\widehat{\bar{v}}\right] + \widehat{\widetilde{\bar{v}}\widehat{\bar{v}}}\left[\widehat{\bar{v}}\widetilde{v}\left(\widehat{\bar{v}} - \widehat{\widehat{v}}\right) + \widehat{\widehat{\bar{v}}}\widehat{v}\widetilde{\bar{v}}\right] \tag{6.13}$$

$$= \widehat{\bar{v}}\widetilde{\bar{v}}\left[\widehat{\widetilde{v}\widehat{\bar{v}}} - \widehat{\widehat{v}\widetilde{\bar{v}}}\right](\bar{v} - \widetilde{v}) + \widehat{\bar{v}}\widehat{v}\widehat{\widetilde{\bar{v}}}\widetilde{\bar{v}}\widehat{v} + \widehat{\widehat{\bar{v}}\widetilde{v}}^{2}\widetilde{v}\widehat{\bar{v}} \ . \tag{6.14}$$

The key indication to establish a multiform structure has been to establish a *closure relation* between the Lagrangian components of a MDC integrable system, [13]. In the present case this is given by the following theorem.

Theorem 6.1. *Between the Lagrangian components* (6.10) *the following 3-form closure relation holds true on-shell (i.e. for solutions of the equations* (6.11) *and* (6.12)*):*

$$\Delta_r \mathcal{L}_{(\xi)pq} + \Delta_p \mathcal{L}_{(\xi)qr} + \Delta_q \mathcal{L}_{(\xi)rp} = \partial_\xi \mathcal{L}_{pqr} \ , \tag{6.15a}$$

where

$$\mathcal{L}_{pqr} = (\bar{v} - v)\ln v + (\widetilde{w} - w)\ln w + (\widehat{z} - z)\ln z \ . \tag{6.15b}$$

Furthermore, the (conventional) Euler-Lagrange equations for the constrained Lagrangian component

$$\mathcal{L}^{\rm c}_{pqr} = \mathcal{L}_{pqr} + \lambda(v + w + z) + \mu(\bar{v} + \widetilde{w} + \widehat{z}) \ , \tag{6.16}$$

where λ and μ are Lagrange multipliers, are satisfied on solutions of the lattice potential KP equation (6.12)*.*

Proof. First we note that on solutions of the EL equations, i.e. on solutions of (2.1) we have

$$\mathcal{L}_{(\xi)pq}\big|_{\rm EL} = \partial_\xi \left(v \ln(p-q+\widehat{u}-\widetilde{u}) \right) \ .$$

Thus by writing out the left-hand side of (6.15a) on-shell we get a total derivative ∂_ξ of a sum of terms that can be rewritten in the form of (6.15b). Furthermore, the EL equations for $\mathcal{L}_{pqr}$, varying with respect to v, w and z independently, are

$$\frac{\delta\mathcal{L}^c}{\delta v} = \ln\left(\underline{v}/v\right) + \frac{\bar{v}}{v} + \lambda - 1 + \underline{\mu} = 0 \ , \tag{6.17a}$$

$$\frac{\delta\mathcal{L}^c}{\delta w} = \ln\left(\underset{\sim}{w}/w\right) + \frac{\widetilde{w}}{w} + \lambda - 1 + \underset{\sim}{\mu} = 0 \ , \tag{6.17b}$$

$$\frac{\delta\mathcal{L}^c}{\delta z} = \ln\left(\underset{\frown}{z}/z\right) + \frac{\widehat{z}}{z} + \lambda - 1 + \underset{\frown}{\mu} = 0 \ , \tag{6.17c}$$

together with the constraints (6.11). A natural solution to these EL equations is

$$\frac{\bar{v}}{v} = \frac{\widetilde{w}}{w} = \frac{\widehat{z}}{z} = e^{-\mu} = 1-\lambda \ ,$$

which supplemented with the constraints (6.11) yield copies of the non-potential lattice KP equation. ■

Theorem 6.1 implies that the semi-discrete Lagrangian 3-form

$$\mathsf{L} = \mathcal{L}_{pqr}\,\delta_p\wedge\delta_q\wedge\delta_r + \mathcal{L}_{(\xi)pq}\,{\rm d}\xi\wedge\delta_p\wedge\delta_q + \mathcal{L}_{(\xi)qr}\,{\rm d}\xi\wedge\delta_q\wedge\delta_r + \mathcal{L}_{(\xi)rp}\,{\rm d}\xi\wedge\delta_r\wedge\delta_p \tag{6.18}$$

is closed on solutions of the system of semi-discrete and fully discrete KP equations.

Remark: A theory of semi-discrete Lagrangian 2-forms was recently proposed in [37], referring to earlier work [16] on a general theory of *difference forms*, cf. also [36]. In [37] a description was given of semi-discrete 2-manifold, while in [38] (in the special case of Calogero-Moser type systems) we gave a description of semi-discrete curves. Although the general case of semi-discrete Lagrangian multiforms was outlined in the Appendix of [37], no examples were given of a system with more than one discrete variable. Thus, the semi- discrete KP case with the semi-discrete 3-form (6.18), seems to be the first concrete example of a higher-dimensional semi-discrete multiform with more than one discrete direction involved in corresponding the semi-discrete multi-time manifold.

However, this is not the end of the story. In fact, the treatment above also gives rise to a fully discrete 3-form structure comprising the Lagrangian components of the form (6.15b) embedded in a multidimensional lattice of at least four lattice directions. This is given by the following:

Theorem 6.2. *The fully discrete Lagrangian 3-form*

$$\mathsf{L} = \mathcal{L}_{pqr}\,\delta_p\wedge\delta_q\wedge\delta_r + \mathcal{L}_{qrs}\,\delta_q\wedge\delta_r\wedge\delta_s + \mathcal{L}_{rsp}\,\delta_r\wedge\delta_s\wedge\delta_p + \mathcal{L}_{spq}\,\delta_s\wedge\delta_p\wedge\delta_q \tag{6.19}$$

where the components are given by (6.15b) *is closed on solutions of the fully discrete lattice KP equation* (2.2)*, i.e. it obeys the following 3-form closure relation on solutions:*

$$(\Box\mathcal{L})_{pqrs} := \Delta_s\mathcal{L}_{pqr} - \Delta_p\mathcal{L}_{qrs} + \Delta_q\mathcal{L}_{rsp} - \Delta_r\mathcal{L}_{spq} = 0 \ . \tag{6.20}$$

Proof. The proof is by direct computation. It takes into account that each component of the 3-form (6.19) is defined on a rhomboid in higher dimension, each face of which is associated with a variable $v_{p_ip_j}$ (where $v_{pq}=v$, $v_{qr}=w$ and $v_{rp}=z$ in the ad-hoc notation used earlier), which on solutions of the EL equations can be identified with $p_i-p_j+T_{p_j}u-T_{p_i}u$, and hence subject to the relations

$$v_{p_ip_j}=-v_{p_jp_i}\ ,\quad v_{p_ip_j}+v_{p_jp_k}+v_{p_kp_i}=0\ ,\quad \frac{T_{p_k}v_{p_ip_j}}{v_{p_ip_j}}=\frac{T_{p_i}v_{p_kp_j}}{v_{p_kp_j}}\ ,$$

the latter relation being the lattice potential KP equation on each rhomboid. Expanding the left-hand side of (6.20) there are various types of terms and their permutations of indices. The terms of the type $T_{p_i}\left[\left(T_{p_j}v_{p_kp_l}\right)\ln(v_{p_kp_l})\right]$ all cancel against each other using the above identification for the $v_{p_kp_l}$ and the lattice potential KP equation. The terms of the type $v_{p_kp_l}\ln(v_{p_kp_l})$ cancel out by the skew symmetry among the cyclic permutations of the indices. The mixed shifted terms of the types $T_{p_i}\left[v_{p_kp_l}\ln(v_{p_kp_l})\right]$ and $\left(T_{p_j}v_{p_kp_l}\right)\ln(v_{p_kp_l})$ conspire together to cancel, using the lattice potential KP and the form of the quantities $v_{p_ip_j}$, where there are two types of cyclic permutations, even and odd, that provide separate cancellations of the contributions arising in the closure. ■

There is an even stronger result that pertains to the double-zero phenomenon of the exterior derivative of the Lagrange multiform, following the ideas in [25, 30, 35], which is stated as follows.

Theorem 6.3. *The discrete exterior derivative of the Lagrangian 3-form* (6.19) *possesses the double-zero property, meaning that off-shell it can be written as a sum of factors, each of which vanish on the solutions of the EL equations.*

Proof. The proof is by direct computation. Writing the Lagrangian components as

$$\mathcal{L}_{pqr}=(T_rv_{pq}-v_{pq})\ln(v_{pq})+(T_pv_{qr}-v_{qr})\ln(v_{qr})+(T_qv_{rp}-v_{rp})\ln(v_{rp})\ ,\qquad (6.21)$$

and only using the property

$$T_rv_{pq}=T_qv_{pr}+T_pv_{rq}\ ,\qquad (6.22)$$

by direct computation we find that *off-shell*, ie. as an identity in the u-variables,

$$
\begin{aligned}
(\Box \mathsf{L})_{pqrs} =& (T_qT_sv_{pr})T_s\ln\left(\frac{v_{pq}\,T_qv_{rp}}{v_{rp}\,T_rv_{pq}}\right) + (T_pT_sv_{rq})T_s\ln\left(\frac{v_{pq}\,T_pv_{qr}}{v_{qr}\,T_rv_{pq}}\right)\\
&+ (T_pT_rv_{qs})T_r\ln\left(\frac{v_{pq}\,T_pv_{qs}}{v_{qs}\,T_sv_{pq}}\right) + (T_qT_rv_{sp})T_r\ln\left(\frac{v_{pq}\,T_qv_{sp}}{v_{sp}\,T_sv_{pq}}\right)\\
&+ (T_pT_qv_{rs})T_p\ln\left(\frac{v_{qr}\,T_qv_{rs}}{v_{rs}\,T_sv_{qr}}\right) + (T_pT_rv_{sq})T_p\ln\left(\frac{v_{qr}\,T_rv_{sq}}{v_{sq}\,T_sv_{qr}}\right)\\
&+ (T_qT_rv_{qs})T_q\ln\left(\frac{v_{pr}\,T_rv_{sp}}{v_{sp}\,T_sv_{pr}}\right) + (T_pT_qv_{sr})T_q\ln\left(\frac{v_{pr}\,T_pv_{rs}}{v_{rs}\,T_sv_{pr}}\right)\\
&+ (T_qv_{ps})\ln\left(\frac{v_{pq}\,T_qv_{sp}}{v_{sp}\,T_sv_{pq}}\right) + (T_pv_{sq})\ln\left(\frac{v_{pq}\,T_pv_{sq}}{v_{qs}\,T_sv_{pq}}\right)\\
&+ (T_qv_{rs})\ln\left(\frac{v_{rs}\,T_sv_{qr}}{v_{qr}\,T_qv_{rs}}\right) + (T_rv_{sq})\ln\left(\frac{v_{sq}\,T_sv_{qr}}{v_{qr}\,T_rv_{sq}}\right)\\
&+ (T_rv_{qp})\ln\left(\frac{v_{pq}\,T_pv_{qr}}{v_{qr}\,T_rv_{pq}}\right) + (T_qv_{pr})\ln\left(\frac{v_{rp}\,T_pv_{qr}}{v_{qr}\,T_qv_{pr}}\right)\\
&+ (T_sv_{rp})\ln\left(\frac{v_{pr}\,T_pv_{rs}}{v_{rs}\,T_sv_{rp}}\right) + (T_rv_{ps})\ln\left(\frac{v_{sp}\,T_pv_{rs}}{v_{rs}\,T_rv_{sp}}\right)\ ,
\end{aligned}
$$

where we can rewrite these terms using the KP-octahedron quantity[10]

$$
\Omega_{pqr} := (T_rv_{pq})v_{qr} - v_{pq}T_pv_{qr} = (p-q+T_rT_qu - T_rT_pu)(T_ru - r) + \text{cycl.}\ , \tag{6.23}
$$

(which obviously vanishes on solutions of (1.1), but here we define it as a polynomial in the u-variables and its shifts) by writing the expressions within the logarithms as

$$
\frac{(T_rv_{pq})v_{qr}}{v_{pq}T_pv_{qr}} = 1 + \frac{\Omega_{pqr}}{v_{pq}T_pv_{qr}}\ ,
$$

and expanding the logarithms, we find the double-zero expansion

$$
\begin{aligned}
(\Box \mathsf{L})_{pqrs} =& (T_s\Omega_{rpq})T_s\left(\frac{T_qv_{pr}}{v_{rp}T_rv_{pq}} - \frac{T_pv_{rq}}{v_{rq}T_rv_{qp}} + \mathcal{O}(T_s\Omega_{rpq})\right)\\
&+ (T_r\Omega_{sqp})T_r\left(\frac{T_pv_{qs}}{v_{sq}T_sv_{qp}} - \frac{T_qv_{sp}}{v_{sp}T_sv_{pq}} + \mathcal{O}(T_r\Omega_{sqp})\right)\\
&+ (T_p\Omega_{srq})T_p\left(\frac{T_qv_{rs}}{v_{sr}T_sv_{rq}} - \frac{T_rv_{sq}}{v_{sq}T_sv_{qr}} + \mathcal{O}(T_p\Omega_{srq})\right)\\
&+ (T_q\Omega_{spr})T_q\left(\frac{T_rv_{ps}}{v_{sp}T_sv_{pr}} - \frac{T_pv_{sr}}{v_{sr}T_sv_{rp}} + \mathcal{O}(T_q\Omega_{spr})\right)\ ,
\end{aligned} \tag{6.24}
$$

where, using (6.22) all the terms of lower order in the shifts cancel. The factors in each term of (6.24) vanish on solutions of the lattice potential KP equation $\Omega_{\cdot,\cdot,\cdot} = 0$. In fact

[10] Octahedral objects like Ω_{pqr} were introduced in connection with the quadrilateral lattice equations in the ABS list, cf. [2], cf. also the more recent treatment [30].

the second factor in each term the contributions of first and higher order in Ω vanish on solutions of the potential KP, while the terms of the form

$$\frac{T_q v_{pr}}{v_{rp} T_r v_{pq}} - \frac{T_p v_{rq}}{v_{rq} T_r v_{qp}}$$

in the second factors equally vanish on solutions of the lattice KP equation. ■

When considering the generalised EL equations $\delta(\Box \mathsf{L})_{pqrs} = 0$ one has to take into account the identity

$$T_s\Omega_{pqr} - T_p\Omega_{qrs} + T_q\Omega_{rsp} - T_r\Omega_{spq} = 0 \ ,$$

as an identity in the variables u and its shifts, and thus the different prefactors of the form $T\Omega$ in the double-zero expansion are not all independent. This leads to a slightly weaker form of the EL equations coming from the computation

$$\delta\Box\mathsf{L} = \sum \left[(\delta T\Omega)\left(\frac{Tv}{vTv} + \cdots\right) + (T\Omega)\delta\left(\frac{Tv}{vTv} + \cdots\right)\right] = 0 \ ,$$

while splitting the contributions up into functionally independent components. In conclusion, the double-zero structure provides the relevant EL equations for the potential KP system, and thus establishes the multiform structure for the fully discrete KP system.

7 Discussion

One of the initial motivations of this paper was to complete the known link between the Calogero- Moser systems and the KP system, a link first discovered by Krichever for the conventional CM system and the standard KP equation, [12], and later extended to the discrete-time case in [24] based on a pole-reduction of the semi-discrete KP equation (2.1). The corresponding Lagrangian 1-form structure was explored in [38], using the Calogero-Moser system as a laboratory to establish the EL structure of the corresponding Lagrangian 1-forms, but this was partly hampered by the absence of a Lagrangian structure for (2.1), which disallowed us to perform the pole-reduction on the Lagrangian level. That lacuna is hopefully now repaired in the present paper, but the actual pole-reduction from 3-forms to 1-forms remains to be achieved.

More generally, the reduction problem from 3-forms to, say, 2-forms is something that needs to be explored, not only for reductions from the KP system to the corresponding 2D quad equation (H1 of the ABS list), but the more subtle and highly nontrivial reduction to higher-order 2D lattice systems like the lattice Boussinesq system (cf. e.g. [25]) or the lattice Gel'fand-Dikii hierarchy.

Another issue that emerges is the further study of the generating PDE system (2.16) for the KP hierarchy, in particular finding the Lagrange structure. In the linearised case the corresponding PDE can be written entirely in terms of u alone, and reads

$$(p-q)^2 u_{\xi\xi\sigma\sigma\tau\tau} - (p-q)^4 u_{\sigma\sigma\tau\tau} + 2(p-q)^2\left(u_{\sigma\tau\tau\xi} + u_{\sigma\sigma\tau\xi}\right) = u_{\sigma\sigma\xi\xi} + u_{\tau\tau\xi\xi} - 2u_{\sigma\tau\xi\xi} \ , \tag{7.1}$$

which possesses a Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(p-q)^2 u_{\sigma\tau\xi}^2 + \tfrac{1}{2}(p-q)^4 u_{\sigma\tau}^2 - (p-q)^2 u_{\sigma\tau}\left(u_{\tau\xi} + u_{\sigma\xi}\right) + \tfrac{1}{2}\left(u_{\sigma\xi}^2 + u_{\tau\xi}^2 - 2u_{\sigma\xi}u_{\tau\xi}\right) \ . \tag{7.2}$$

How the latter Lagrangian fits into a 3-form structure remains for now an open problem. Ideally, one would like to find another generating PDE system in which the special variable ξ is replaced by yet another Miwa variable, say ρ, which would complete the picture. Such a PDE system, in terms only of Miwa variables, does exist, but not for the dependent variable u; it is the Darboux-KP system the Lagrangian 3-form structure of which was found in [20]. Intriguingly that Lagrangian structure is in fact a curious kind of infinite-dimensional Chern-Simons theory, as was found in [7]. The latter finding begs for a further exploration of the mysterious connection between topological field theories and Lagrangian multiform systems.

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