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Potentialisations of a class of fully-nonlinear symmetry-integrable evolution equations

Marianna Euler and Norbert Euler *

International Society of Nonlinear Mathematical Physics, Auf der Hardt 27, 56130 Bad Ems, Germany
and
Centro Internacional de Ciencias, Av. Universidad s/n, Colonia Chamilpa, 62210 Cuernavaca, Morelos, Mexico

* *Dr.Norbert.Euler@gmail.com*

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Abstract: We consider here the class of fully-nonlinear symmetry-integrable third-order evolution equations in 1+1 dimensions that were proposed recently in *Open Communications in Nonlinear Mathematical Physics*, vol. 2, 216–228 (2022). In particular, we report all zero-order and higher-order potentialisations for this class of equations using their integrating factors (or multipliers) up to order four. Chains of connecting evolution equations are also obtained by multi-potentialisations.

1 Introduction

We recently reported a class of fully-nonlinear symmetry-integrable evolution equations in 1+1 dimensions with rational nonlinearities in their highest derivative [4]. This class of equations admit an infinite number of higher-order generalised symmetries, also called Lie-Bäcklund symmetries, and the equations admit recursion operators that generate these symmetries. This class of equations is presented by the following four cases:

Proposition 1. *[4]: The following evolution equations, in the class* $u_t = F(u_x, u_{xx}, u_{xxx})$ *where* F *a rational function in* u_{xxx}*, are symmetry-integrable:*

- **Case I:** *The equation*

$$u_t = \frac{u_{xx}^6}{(\alpha u_x + \beta)^3 u_{xxx}^2} + Q(u_x), \tag{1.1}$$

where $\{\alpha,\ \beta\}$ *are arbitrary constants, not simultaneously zero, and* $Q(u_x)$ *needs to satisfy*

$$(\alpha u_x + \beta)\frac{d^5Q}{du_x^5} + 5\alpha\frac{d^4Q}{du_x^4} = 0, \tag{1.2}$$

which admits for $\alpha \neq 0$ *the general solution*

$$Q(u_x) = c_5\left(u_x + \frac{\beta}{\alpha}\right)^3 + c_4\left(u_x + \frac{\beta}{\alpha}\right)^2 + c_3\left(u_x + \frac{\beta}{\alpha}\right)$$
$$+c_2\left(u_x + \frac{\beta}{\alpha}\right)^{-1} + c_1. \tag{1.3}$$

and for $\alpha = 0$*, the general solution is*

$$Q(u_x) = c_5u_x^4 + c_4u_x^3 + c_3u_x^2 + c_2u_x + c_1. \tag{1.4}$$

Here c_j *are arbitrary constants.*

- **Case II:** *The equation*

$$u_t = \frac{u_{xx}^3\left(\lambda_1 + \lambda_2 u_{xx}\right)^3}{u_{xxx}^2}, \tag{1.5}$$

where $\{\lambda_1,\ \lambda_2\}$ *are arbitrary constants but not simultaneously zero.*

- **Case III:** *The equation*

$$u_t = \frac{(\alpha u_x + \beta)^{11}}{\left[(\alpha u_x + \beta)u_{xxx} - 3\alpha u_{xx}^2\right]^2}, \tag{1.6}$$

where $\{\alpha,\ \beta\}$ *are arbitrary constants but not simultaneously zero.*

- **Case IV:** *The equation*

$$u_t = \frac{4u_x^5}{(2b\,u_x^2 - 2u_xu_{xxx} + 3u_{xx}^2)^2} \equiv \frac{u_x}{(b-S)^2}, \tag{1.7}$$

where b *is an arbitrary constant and* S *is the Schwarzian derivative*

$$S := \frac{u_{xxx}}{u_x} - \frac{3}{2}\left(\frac{u_{xx}}{u_x}\right)^2. \tag{1.8}$$

The recursion operators for each equation listed in Proposition 1 is given in [4] (regarding recursion operators we refer to [5] and [2], and the references therein).

The current article is organised as follows: In Section 2 we discuss the method that is applied here for the potentialisation of the evolution equations. We state a useful Proposition for the calculation of the conserved currents and define the concept of a higher-order potentialisation, whereby the "usual" potentialisation (see for example [2]) is defined as the *zero-order potentialisation* of an evolution equation. In Section 3 we report all potentisalisations of the equations listed in Proposition 1 and also perform multi-potentialisations where possible. In Section 4 we make some concluding remarks.

2 Our method using higher-order potentialisations

In this section we describe the method that is apply to systematically map the equations listed in Proposition 1 by using the equations' conservation laws. The first step in to calculate the equations' integrating factors to obtain, in each case, an appropriate conserved current and flux. We give a short review and introduce our notations.

On the notation: *For a given function $\Psi(x,u,u_x,u_{xx},u_{xxx},\ldots,u_{nx})$, where u_{nx} denotes the nth-derivative of u with respect to x, we use the notation $\Psi[x,u]$ to indicate the dependencies. The order of Ψ is defined by the highest x-derivation in the argument of the function. The same notation is also used to indicate the dependencies of a linear operator.*

It is well known that every integrating factor $\Lambda[x,u]$ (also known as a multiplier) of a given nth-order evolution equation

$$E := u_t - F[x,u] = 0 \tag{2.1}$$

leads to a conserved current Φ^t and a corresponding flux Φ^x for (2.1), such that

$$D_t\Phi^t[x,u] + D_x\Phi^x[x,u]\Big|_{E=0} = 0. \tag{2.2}$$

Now, Λ is an integrating factor of (2.1) if and only if

$$\hat{E}[u]\,(\Lambda[x,u]E) = 0, \tag{2.3}$$

where $\hat{E}[u]$ denotes the Euler operator

$$\hat{E}[u] := \frac{\partial}{\partial u} - D_t\circ\frac{\partial}{\partial u_t} - D_x\circ\frac{\partial}{\partial u_x} + D_x^2\circ\frac{\partial}{\partial u_{xx}} - D_x^3\circ\frac{\partial}{\partial u_{xxx}} + \cdots . \tag{2.4}$$

Note that condition (2.3) is equivalent to

$$L_E^*[u]\Lambda[x,u]\Big|_{E=0} = 0 \tag{2.5a}$$

$$\text{and}\quad L_\Lambda[u]E - L_\Lambda^*[u]E = 0. \tag{2.5b}$$

Here L_E denotes the linear operator

$$L_E[u] = \frac{\partial E}{\partial u} + \frac{\partial E}{\partial u_t}D_t + \frac{\partial E}{\partial u_x}D_x + \frac{\partial E}{\partial u_{xx}}D_x^2 + \frac{\partial E}{\partial u_{xxx}}D_x^3 + \cdots \tag{2.6a}$$

and L_E^* the adjoint of L_E, namely

$$L_E^*[u] := \frac{\partial E}{\partial u} - D_t\circ\frac{\partial E}{\partial u_t} - D_x\circ\frac{\partial E}{\partial u_x} + D_x^2\circ\frac{\partial E}{\partial u_{xx}} - D_x^3\circ\frac{\partial E}{\partial u_{xxx}} + \cdots . \tag{2.6b}$$

For more details, we refer to [1] and [2], and the references therein. Note that the first condition (2.5a) requires Λ to be an adjoint symmetry for (2.1), whereas the second condition (2.5b) requires Λ to be self-adjoint, which means that Λ must necessarily be even-order. If

Λ is an adjoint symmetry for (2.1), then, due to the linear form of the adjoint symmetry condition (2.5a), we know that Λ must be linear in its highest derivative. This observation is essential (see Proposition 2 below).

The relation between a non-zero integrating factors Λ and its corresponding conserved currents Φ^t for (2.1) is given by the relation

$$\Lambda[u] = \hat{E}[u]\Phi^t[x, u], \tag{2.7}$$

whereby the flux Φ^x, say of order m, is related to Λ, F and Φ^t as follows [2]:

$$\Phi^x[x, u] = -D_x^{-1}(\Lambda F) + \sum_{k=1}^{m}\sum_{j=0}^{m-k}(-1)^k (D_x^j F)\, D_x^{k-1}\left(\frac{\partial \Phi^t}{\partial u_{(j+k)}}\right). \tag{2.8}$$

Following the relation (2.7), it is clear that the order of a conserved current Φ^t for (2.1) is closely related to the order of the corresponding integrating factor Λ. To potentialise an evolution equation (2.1), if at all possible, we make use of the equation's integrating factors whereby the corresponding lowest order conserved current Φ^t are of particular interest. In this sense the following statement, which follows directly from (2.5a) and (2.5b), is useful:

Proposition 2: *Assume that a given nth-order evolution equation of the form (2.1), viz.*

$$u_t = F[x, u],$$

admits an integrating factor Λ of order $2m$, where m is a natural number or zero and $n > 1$. Then Λ is of the form

$$\Lambda[x, u] = g_1(x, u, u_x, \dots, u_{mx})u_{(2m)x} + g_0(x, u, u_x, \dots, u_{(2m-1)x}) \tag{2.9}$$

and the lowest order conserved current Φ^t of (2.1) is of order m, i.e.

$$\Phi^t[x, u] = \Phi^t(x, u, u_x, \dots, u_{mx}), \tag{2.10}$$

whereby

$$\frac{\partial^2 \Phi^t}{\partial u_{mx}^2} = (-1)^m g_1(u, u_x, \dots, u_{mx}). \tag{2.11}$$

In the current article we are interested in potentialisations of zero and higher-order, which we define as follows:

Definition: *Consider an nth-order evolution equation of the form (2.1) and assume that it admits an integrating factor Λ with corresponding conserved current Φ^t and flux Φ^x.*

a) Equation (2.1) is said to be **potentialisable of order zero** *if there exist a new dependent variable $v(x, t)$, where*

$$v_x := \Phi^t[x, u], \text{ that is} \tag{2.12a}$$

$$v_t = -\Phi^x[x,u], \tag{2.12b}$$

such that

$$\Phi^x[x,u]\Big|_{v_x=\Phi^t} = G_0(x, v_x, \dots, v_{nx}). \tag{2.12c}$$

The so constructed evolution equation

$$v_t = -G_0(x, v_x, \dots, v_{nx}) \tag{2.12d}$$

is a **zero-order potential equation** *for (2.1), where the explicit form of G_0 depends on the given equation (2.1), on its conserved current Φ^t, and on the corresponding flux Φ^x.*

b) Equation (2.1) is said to be **potentialisable of order one** *if there exist a new dependent variable $w(x,t)$, where*

$$w_x := D_x\Phi^t[x,u], \text{ that is} \tag{2.13a}$$
$$w_t = -D_x\Phi^x[x,u], \tag{2.13b}$$

such that

$$D_x\Phi^x[x,u]\Big|_{w_x=D_x\Phi^t} = G_1(x, w, w_x, \dots, w_{nx}). \tag{2.13c}$$

The so constructed evolution equation

$$w_t = -G_1(x, w, w_x, \dots, w_{nx}) \tag{2.13d}$$

is a **first-order potential equation** *for (2.1), where the explicit form of G_1 depends on the given equation (2.1), on its conserved current Φ^t, and on the corresponding flux Φ^x.*

c) Equation (2.1) is said to be **potentialisable of order** *p if there exist a new dependent variable $w(x,t)$, where $p \geq 1$ and*

$$w_x := D_x^p\Phi^t[x,u], \text{ that is} \tag{2.14a}$$
$$w_t = -D_x^p\Phi^x[x,u], \tag{2.14b}$$

such that

$$D_x^p\Phi^x[x,u]\Big|_{w_x=D_x^p\Phi^t} = G_p(x, w, w_x, \dots, w_{nx}). \tag{2.14c}$$

The so constructed evolution equation

$$w_t = -G_p(x, w, w_x, \dots, w_{nx}) \tag{2.14d}$$

is a **potential equation of order** *p for (2.1), where the explicit form of G_p depends on the given equation (2.1), on its conserved current Φ^t, and on the corresponding flux Φ^x.*

Remarks:

1. From the above Definition it is clear that an evolution equation of the form (2.1), which admits a zero-order potentialisation with $v_x = \Phi^t$, will also admit a first-order potentialisation $w_x = D_x\Phi^t$ for the same conserved current Φ^t, whereby $v_x = w$. However, an equation that does not admit a zero-order potentialisation may, or may not, admit a first-order potentialisation. This is also the case for higher-order potentialisations.

2. Of course, a given equation (2.1) might not admit a potentialisation of any order for a specific integrating factor. As far as we know, this can not be established *a priori* without the knowledge of the integrating factors and the corresponding conservation laws. It is therefore sensible, in our opinion, to do a classification of all integrating factors with all possible corresponding potentialisations that follow, in particular for symmetry-integrable evolution equation.

3. In the current work, we are restricting ourselves to integrating factors that do not depend explicitly on their independent variables.

3 Potentialisations of the equations of Proposition 1

We now consider each equation listed in Proposition 1, namely Case I, Case II, Case III and Case IV, and derive all possible potentialisations of the fully-nonlinear symmetry-integrable equations listed in this proposition and, where possible, we construct further equations by multi-potentialisations. Diagrams are given in some cases to make the connections between the equations more clear.

Case I: We consider equation (1.1), viz.

$$u_t = \frac{u_{xx}^6}{(\alpha u_x + \beta)^3 u_{xxx}^2} + Q(u_x),$$

where $\{\alpha,\ \beta\}$ are arbitrary constants, not simultaneously zero, and where $Q(u_x)$ satisfies (1.2).

In its most general form, equation (1.1) admits the following integrating factor of order six:

$$\begin{aligned}
{}^{I}\Lambda[u] =& \frac{u_{xx}^4 u_{6x}}{(\alpha u_x + \beta)^3\, u_{xxx}^3} + \left(\frac{12u_{xx}^3}{(\alpha u_x + \beta)^3\, u_{xxx}^2} - \frac{9\alpha u_{xx}^5}{(\alpha u_x + \beta)^4\, u_{xxx}^3}\right) u_{5x}\\
&+ \left(\frac{24\alpha u_{xx}^2}{(\alpha u_x + \beta)^3\, u_{xxx}} - \frac{72\alpha u_{xx}^4}{(\alpha u_x + \beta)^4\, u_{xxx}^2} + \frac{36\alpha^2 u_{xx}^6}{(\alpha u_x + \beta)^5\, u_{xxx}^3}\right) u_{4x}\\
&- \frac{9u_{xx}^4 u_{4x} u_{5x}}{(\alpha u_x + \beta)^3\, u_{xxx}^4} + \left(\frac{27\alpha u_{xx}^5}{(\alpha u_x + \beta)^4\, u_{xxx}^4} - \frac{28u_{xx}^3}{(\alpha u_x + \beta)^3\, u_{xxx}^3}\right) u_{4x}^2\\
&+ \frac{12u_{xx}^4 u_{4x}^3}{(\alpha u_x + \beta)^3\, u_{xxx}^5} - \frac{24u_{xx} u_{xxx}}{(\alpha u_x + \beta)^3} + \frac{30\alpha^3 u_{xx}^7}{(\alpha u_x + \beta)^6\, u_{xxx}^2} + \frac{108\alpha u_{xx}^3}{(\alpha u_x + \beta)^4}
\end{aligned}$$

$$-\frac{108\alpha^2 u_{xx}^5}{(\alpha u_x+\beta)^5\, u_{xxx}} - \frac{1}{2}\frac{d^3Q}{du_x^3}\, u_{xx}. \tag{3.1}$$

Here $Q(u_x)$ should satisfy condition (1.2), whereby α and β are arbitrary constant that are not simultaneously zero. Using the integrating factor (3.1) we obtain the following conserved current Φ^t and flux Φ^x for (1.1):

$$^I\Phi^t[u] = -\frac{u_{xx}^4}{2(\alpha u_x+\beta)^3 u_{xxx}} + \frac{u u_{xx}}{2u_x^2}\left(\frac{dQ}{du_x} - u_x\frac{d^2Q}{du_x^2}\right) \tag{3.2a}$$

$$\begin{aligned}
^I\Phi^x[u] =\ & \frac{u_{xx}^{10}u_{5x}}{(\alpha u_x+\beta)^6 u_{xxx}^5} - 2\frac{u_{xx}^{10}u_{4x}^2}{(\alpha u_x+\beta)^6 u_{xxx}^6} + \frac{u u_{xx}^6 u_{4x}}{(\alpha u_x+\beta)^3 u_x^2 u_{xxx}^3}\left(\frac{dQ}{du_x} - u_x\frac{d^2Q}{du_x^2}\right)\\
&+\frac{u_{xx}^9 u_{4x}}{(\alpha u_x+\beta)^6 u_{xxx}^4} - \frac{3\alpha u_{xx}^{11}u_{4x}}{(\alpha u_x+\beta)^7 u_{xxx}^5} - \frac{15\alpha^2 u_{xx}^{12}}{4(\alpha u_x+\beta)^8 u_{xxx}^4} + \frac{15\alpha u_{xx}^{10}}{2(\alpha u_x+\beta)^7 u_{xxx}^3}\\
&+\frac{1}{2}\left(\frac{dQ}{du_x} - u_x\frac{d^2Q}{du_x^2}\right)\left(\frac{3\alpha u u_{xx}^7}{(\alpha u_x+\beta)^4 u_x^2 u_{xxx}^2} + \frac{u_{xx}^6}{(\alpha u_x+\beta)^3 u_x u_{xxx}^2}\right.\\
&\left.-\frac{6u u_{xx}^5}{(\alpha u_x+\beta)^3 u_x^2 u_{xxx}}\right) + \frac{1}{2}\frac{dQ}{du_x}\frac{u_{xx}^4}{(\alpha u_x+\beta)^3 u_{xxx}} - \frac{u u_{xx}}{2u_x^2}\frac{dQ}{du_x}\left(\frac{dQ}{du_x} - u_x\frac{d^2Q}{du_x^2}\right)\\
&-\frac{1}{4u_x}\frac{dQ}{du_x}\left(u_x\frac{dQ}{du_x} - 2Q\right).
\end{aligned} \tag{3.2b}$$

We find that there is no zero-order potentialisation and no higher-order potentialisation related to (3.2a) and (3.2b) for any Q that satisfies (1.2).

We now consider integrating factors of equation (1.1) of order less than six. Depending on the parameters α and β and the form of Q, we obtain the following three distinct cases:

Subcase I.1: Let $\alpha \neq 0$ and $\beta \neq 0$. We find that equation (1.1) viz.

$$u_t = \frac{u_{xx}^6}{(\alpha u_x+\beta)^3 u_{xxx}^2} + Q(u_x),$$

admits two integrating factors of order four that depend on the form of Q, namely for the case $Q^{(2)} = 0$ and $Q^{(3)} = 0$. There exist no integrating factors of order zero or order two.

For $Q^{(2)} = 0$, that is

$$Q(u_x) = c_1 u_x + c_0, \tag{3.3}$$

where c_0 and c_1 are arbitrary constants of integration, we find that

$$u_t = \frac{u_{xx}^6}{(\alpha u_x+\beta)^3 u_{xxx}^2} + c_1 u_x + c_0 \tag{3.4}$$

admits the following two fourth-order integrating factors:

$$^{I}\Lambda_1^1[u] = \frac{(\alpha u_x + \beta)u_{4x}}{u_{xx}^3} - \frac{3(\alpha u_x + \beta)u_{xxx}^2}{u_{xx}^4} + \frac{2\alpha u_{xxx}}{u_{xx}^2} \tag{3.5a}$$

$$^{I}\Lambda_2^1[u] = \frac{(\alpha u_x + \beta)u_{4x}}{u_{xx}^4} - \frac{4(\alpha u_x + \beta)u_{xxx}^2}{u_{xx}^5} + \frac{2\alpha u_{xxx}}{u_{xx}^3}. \tag{3.5b}$$

Integrating factor (3.5a) leads to the following conserved current and flux for (3.4):

$$^{I}\Phi_{1,1}^t[u] = \frac{\alpha u_x + \beta}{u_{xx}} \tag{3.6a}$$

$$\begin{aligned} ^{I}\Phi_{1,1}^x[u] = &-\frac{2u_{xx}^4 u_{4x}}{(\alpha u_x + \beta)^2 u_{xxx}^3} - \frac{5\alpha u_{xx}^5}{(\alpha u_x + \beta)^3 u_{xxx}^2} + \frac{10 u_{xx}^3}{(\alpha u_x + \beta)^2 u_{xxx}} \\ &- \frac{c_1(\alpha u_x + \beta)}{u_{xx}} \end{aligned} \tag{3.6b}$$

and integrating factor (3.5b) gives the following conserved current and flux for (3.4):

$$^{I}\Phi_{1,2}^t[u] = \frac{\alpha u_x + \beta}{u_{xx}^2} \tag{3.7a}$$

$$\begin{aligned} ^{I}\Phi_{1,2}^x[u] = &-\frac{4u_{xx}^3 u_{4x}}{(\alpha u_x + \beta)^2 u_{xxx}^3} - \frac{9\alpha u_{xx}^4}{(\alpha u_x + \beta)^3 u_{xxx}^2} + \frac{24 u_{xx}^2}{(\alpha u_x + \beta)^2 u_{xxx}} \\ &- \frac{c_1(\alpha u_x + \beta)}{u_{xx}^2} - \frac{12}{\alpha(\alpha u_x + \beta)}. \end{aligned} \tag{3.7b}$$

This leads to

Potentialisation I.1: *The only conserved current that leads to a potentialisation of (3.4), viz*

$$\boxed{u_t = \frac{u_{xx}^6}{(\alpha u_x + \beta)^3 u_{xxx}^2} + c_1 u_x + c_0},$$

is $^{I}\Phi_{1,1}^t$ *given by (3.6a). The zero-order potential equation of (3.4) is then*

$$v_t = \frac{2v_{xxx}}{(v_{xx} - \alpha)^3} + \frac{6v_{xx}^2}{(v_{xx} - \alpha)^3 v_x} - \frac{9\alpha v_{xx}}{(v_{xx} - \alpha)^3 v_x} + \frac{3\alpha^2}{(v_{xx} - \alpha)^3 v_x} + c_1 v_x, \tag{3.8a}$$

where

$$v_x = \frac{\alpha u_x + \beta}{u_{xx}} \tag{3.8b}$$

and the first-order potential equation of (3.4), with $w_x = D_x(^{I}\Phi_{1,1}^t)$*, i.e.* $w = v_x$*, is*

$$w_t = \frac{2w_{xxx}}{(w_x - \alpha)^3} - \frac{6w_{xx}^2}{(w_x - \alpha)^4} - \frac{6w_x w_{xx}}{(w_x - \alpha)^3 w} - \frac{3(2w_x - \alpha)w_x}{8(w_x - \alpha)^2 w^2} + c_1 w. \tag{3.8c}$$

The conserved current ${}^I\Phi^t_{1,2}$ given by (3.7a) does not give a potentialisation of (3.4), of any order.

For $Q^{(3)}=0$, that is

$$Q(u_x)=c_2u_x^2+c_1u_x+c_0, \tag{3.9}$$

where c_0, c_1 and $c_2\neq 0$ are arbitrary constants with $c_2\neq 0$, we find that

$$u_t=\frac{u_{xx}^6}{(\alpha u_x+\beta)^3u_{xxx}^2}+c_2u_x^2+c_1u_x+c_0 \tag{3.10a}$$

admits only one integrating factor of order four, namely (3.5a) with the following conserved current and flux:

$${}^I\Phi^t_{1,3}[u]=\frac{\alpha u_x+\beta}{u_{xx}} \tag{3.10b}$$

$$\begin{aligned}{}^I\Phi^x_{1,3}[x,u]=&-\frac{2u_{xx}^4u_{4x}}{(\alpha u_x+\beta)^2u_{xxx}^3}-\frac{5\alpha u_{xx}^5}{(\alpha u_x+\beta)^3u_{xxx}^2}+\frac{10u_{xx}^3}{(\alpha u_x+\beta)^2u_{xxx}}\\&-\frac{(2c_2u_x+c_1)(\alpha u_x+\beta)}{u_{xx}}+4c_2(\alpha u+\beta x),\end{aligned} \tag{3.10c}$$

We find that equation (3.10a) admits no potentialisation of any order related to the integrating factor (3.5a) and its corresponding conserved current (3.10b) for $c_2\neq 0$.

Subcase I.2: Let $\alpha=0$ and $\beta=1$. Equation (1.1) then takes the form

$$u_t=\frac{u_{xx}^6}{u_{xxx}^2}+c_3u_x^3+c_2u_x^2+c_1u_x+c_0, \tag{3.11}$$

where c_j, with $j=0,1,2,3$ are arbitrary constants. We find two distinct cases, namely one case where c_2 and c_3 are both zero, and the case where both c_2 are c_3 are non-zero.

Let $c_2=0$ and $c_3=0$: The equation (3.11) then takes the form

$$u_t=\frac{u_{xx}^6}{u_{xxx}^2}+c_1u_x+c_0. \tag{3.12}$$

Equation (3.12) admits the following two fourth-order integrating factors (no zero-order or second-order integrating factors exist):

$${}^I\Lambda^2_1[u]=k_1\left(\frac{u_{4x}}{u_{xx}^3}-\frac{3u_{xxx}^2}{u_{xx}^4}\right) \tag{3.13a}$$

$${}^I\Lambda^2_2[u]=k_2\left(\frac{u_{4x}}{u_{xx}^4}-\frac{4u_{xxx}^2}{u_{xx}^5}\right). \tag{3.13b}$$

Integrating factor (3.13a) leads to the following conserved current and flux for (3.12):

$$ {}^{I}\Phi^{t}_{2,1}[u] = \frac{k_1}{2u_{xx}} \tag{3.14a} $$

$$ {}^{I}\Phi^{x}_{2,1}[u] = k_1\left(-\frac{u_{xx}^4 u_{4x}}{u_{xxx}^3} + \frac{5u_{xx}^3}{u_{xxx}} - \frac{c_1}{2u_{xx}}\right). \tag{3.14b} $$

The second integrating factor (3.13b) leads to the following conserved current and flux for (3.12):

$$ {}^{I}\Phi^{t}_{2,2}[u] = \frac{k_2}{6u_{xx}^2} \tag{3.15a} $$

$$ {}^{I}\Phi^{x}_{2,2}[u] = k_2\left(-\frac{2u_{xx}^3 u_{4x}}{3u_{xxx}^3} + \frac{4u_{xx}^2}{u_{xxx}} - \frac{c_1}{6u_{xx}^2} + 2u_x\right). \tag{3.15b} $$

This leads to

Potentialisation I.2:

a) Using the conserved current ${}^{I}\Phi^{t}_{2,1}$, given by (3.14a), we find that equation (3.12), viz

$$ \boxed{u_t = \frac{u_{xx}^6}{u_{xxx}^2} + c_1 u_x + c_0}\,, $$

admits the zero-order potentiaisation

$$ v_t = \frac{v_{xxx}}{v_{xx}^3} + \frac{3}{v_x v_{xx}} + c_1 v_x, \tag{3.16} $$

where $k_1 = 2^{2/3}$

$$ v_x = \frac{2^{-1/3}}{u_{xx}} \tag{3.17} $$

The first-order potentialisation of (3.12) with $w_x = D_x({}^{I}\Phi^{t}_{2,1})$ i.e. $w = v_x$, is then

$$ w_t = \frac{w_{xxx}}{w_x^3} - 3\frac{w_{xx}^2}{w_x^4} - 3\frac{w_{xx}}{ww_x^2} - \frac{3}{w^2} + c_1 w_x. \tag{3.18} $$

b) Using the conserved current ${}^{I}\Phi^{t}_{2,2}$, given by (3.15a), we find that equation (3.12), viz

$$ \boxed{u_t = \frac{u_{xx}^6}{u_{xxx}^2} + c_1 u_x + c_0}\,, $$

admits no zero-order potentialisation. However, with

$$ w_x = D_x\left(\frac{k_2}{6}\frac{1}{u_{xx}^2}\right) \tag{3.19} $$

we find the following first-order potential equation for (3.12) in w:

$$w_t = \frac{w^{3/2} w_{xxx}}{w_x^3} - \frac{3 w^{3/2} w_{xx}^2}{w_x^4} + c_1 w_x, \tag{3.20}$$

where $k_2 = 3 \cdot 2^{-5/3}$.

Let $c_3 \neq 0$ and $c_2 \neq 0$: The equation (3.11), viz

$$u_t = \frac{u_{xx}^6}{u_{xxx}^2} + c_3 u_x^3 + c_2 u_x^2 + c_1 u_x + c_0$$

admits one fouth-order integrating factor (none of zero or second-order)

$$^I\Lambda_3^2[u] = \frac{u_{4x}}{u_{xx}^3} - \frac{3u_{xxx}^2}{u_{xx}^4}, \tag{3.21}$$

which leads to the following conserved current and flux for (3.11):

$$^I\Phi_{2,3}^t[u] = \frac{1}{2u_{xx}^2} \tag{3.22a}$$

$$^I\Phi_{2,3}^x[x,u] = -\frac{u_{xx}^4 u_{4x}}{u_{xxx}^3} + \frac{5u_{xx}^3}{u_{xxx}} - \frac{3c_3 u_x^2}{2u_{xx}} - \frac{c_2 u_x}{u_{xx}} - \frac{c_1}{2u_{xx}} + 6c_3 u + 2c_2 x. \tag{3.22b}$$

Using the conserved current $^I\Phi_{2,3}^t$ given by (3.22a), we find that equation (3.11), with $c_3 \neq 0$ and $c_2 \neq 0$, does not lead to any zero-order or higher-order potentialisation.

Subcase I.3: Let $\alpha = 1$ and $\beta = 0$. Equation (1.1) then takes the following form:

$$u_t = \frac{u_{xx}^6}{u_x^3 u_{xxx}^2} + Q(u_x), \ \text{where } u_x Q^{(5)}(u_x) + 5Q^{(4)} = 0. \tag{3.23}$$

We find that (3.23) admits fourth-order integrating factors only in the case where

$$Q(u_x) = c_2 u_x^2 + c_1 u_x + c_0. \tag{3.24}$$

No zero-order or second-order integrating factors exist for (3.23). Two distinct cases must be considered here, namely the case $c_2 = 0$ and $c_2 \neq 0$.

Let $c_2 = 0$: Equation (3.23) then takes the form

$$u_t = \frac{u_{xx}^6}{u_x^3 u_{xxx}^2} + c_1 u_x + c_0. \tag{3.25}$$

Equation (3.25) admits the following two fourth-order integrating factors:

$$^I\Lambda_1^3[u] = 2\left(\frac{u_x u_{4x}}{u_{xx}^3} - \frac{3u_x u_{xxx}^2}{u_{xx}^4} + \frac{2u_{xxx}}{u_{xx}^2}\right) \tag{3.26a}$$

$$^{I}\Lambda_2^3[u] = \frac{u_x u_{4x}}{u_{xx}^4} - \frac{4u_x u_{xxx}^2}{u_{xx}^5} + \frac{2u_{xxx}}{u_{xx}^3}. \tag{3.26b}$$

Integrating factor (3.26a) leads to the following conserved current and flux for (3.25):

$$^{I}\Phi^t_{3,1}[u] = \frac{u_x}{u_{xx}} \tag{3.27a}$$

$$^{I}\Phi^x_{3,1}[u] = -\frac{2u_{xx}^4 u_{4x}}{u_x^2 u_{xxx}^3} - \frac{5u_{xx}^5}{u_x^3 u_{xxx}^2} + \frac{10u_{xx}^3}{u_x^2 u_{xxx}} - \frac{c_1 u_x}{u_{xx}}. \tag{3.27b}$$

and the integrating factor (3.26b) gives the following conserved current and flux for (3.25):

$$^{I}\Phi^t_{3,2}[u] = \frac{u_x}{6u_{xx}^2} \tag{3.28a}$$

$$^{I}\Phi^x_{3,2}[u] = -\frac{2u_{xx}^3 u_{4x}}{3u_x^2 u_{xxx}^3} + \frac{4u_{xx}^2}{u_x^2 u_{xxx}} - \frac{3u_{xx}^4}{2u_x^3 u_{xxx}^2} - \frac{2}{u_x} - \frac{c_1 u_x}{6u_{xx}^2}. \tag{3.28b}$$

This leads to

Potentialisation I.3: *Using the conserved current* $^{I}\Phi^t_{3,1}$*, given by (3.27a), we find that equation (3.25), viz*

$$\boxed{u_t = \frac{u_{xx}^6}{u_x^3 u_{xxx}^2} + c_1 u_x + c_0},$$

admits the zero-order potentiaisation

$$v_t = \frac{2v_{xxx}}{(v_{xx}-1)^3} + \frac{3(2v_{xx}-1)}{2v_x(v_{xx}-1)^2} + c_1 v_x, \tag{3.29}$$

where

$$v_x = \frac{u_x}{u_{xx}}. \tag{3.30}$$

The first-order potentialisation of (3.25) with $w_x = D_x(^{I}\Phi^t_{3,1})$ *i.e.* $w = v_x$*, is then*

$$w_t = \frac{2w_{xxx}}{(w_x-1)^3} - \frac{6w_{xx}^2}{(w_x-1)^4} - \frac{6w_x w_{xx}}{w(w_x-1)^3} - \frac{3w_x(2w_x-1)}{w^2(w_x-1)^2} + c_1 w_x. \tag{3.31}$$

No zero or higher-order potentialisations can be obtained for (3.25) with the conserved current $^{I}\Phi^t_{3,2}$*, given by (3.28a).*

Let $c_2 \neq 0$: The equation

$$u_t = \frac{u_{xx}^6}{u_x^3 u_{xxx}^2} + c_2 u_x^2 + c_1 u_x + c_0 \tag{3.32}$$

admits the same fourth-order integrating factor (3.26a) as equation (3.25) and no zero or second-order integrating factors. This leads to the following conserved current and flux for (3.32):

$$ {}^{I}\Phi^t_{3,3}[u] = \frac{u_x}{u_{xx}} \tag{3.33a}$$

$$ {}^{I}\Phi^x_{3,3}[u] = -\frac{2u_{xx}^4 u_{4x}}{u_x^2 u_{xxx}^3} - \frac{5u_{xx}^5}{u_x^3 u_{xxx}^2} + \frac{10u_{xx}^3}{u_x^2 u_{xxx}} + 2c_2\left(\frac{u_x^2}{u_{xx}} + 2u\right) - \frac{c_1 u_x}{u_{xx}}. \tag{3.33b}$$

We find that (3.32) has no zero-order or higher-order potentialisations using (3.33a).

Case II: We consider (1.5), viz.

$$u_t = \frac{u_{xx}^3\,(\lambda_1 + \lambda_2 u_{xx})^3}{u_{xxx}^2}.$$

Equation (1.5) admits the following four fourth-order integrating factors (no zero-order or second-order integrating factors exist for this equation):

$$ {}^{II}\Lambda_1[u] = \frac{u_{4x}}{u_{xx}^{3/2}(\lambda_1 + \lambda_2 u_{xx})^{3/2}} - \frac{3}{2}\,\frac{(\lambda_1 + 2\lambda_2 u_{xx})u_{xxx}^2}{u_{xx}^{5/2}(\lambda_1 + \lambda_2 u_{xx})^{5/2}} \tag{3.34a}$$

$$ {}^{II}\Lambda_2[u] = \frac{(\lambda_1 + 2\lambda_2 u_{xx})u_{4x}}{u_{xx}^2(\lambda_1 + \lambda_2 u_{xx})^2} - \frac{2\lambda_2(\lambda_1 + 2\lambda_2 u_{xx})u_{xxx}^2}{u_{xx}^2(\lambda_1 + \lambda_2 u_{xx})^3} - \frac{2u_{xxx}^2}{u_{xx}^3(\lambda_1 + \lambda_2 u_{xx})} \tag{3.34b}$$

$$ {}^{II}\Lambda_3[u] = \frac{u_{4x}}{u_{xx}(\lambda_1 + \lambda_2 u_{xx})^2} - \frac{(\lambda_1 + 3\lambda_2 u_{xx})u_{xxx}^2}{u_{xx}^2(\lambda_1 + \lambda_2 u_{xx})^3} \tag{3.34c}$$

$$ {}^{II}\Lambda_4[u] = \frac{u_{4x}}{u_{xx}^2(\lambda_1 + \lambda_2 u_{xx})} - \frac{(2\lambda_1 + 3\lambda_2 u_{xx})u_{xxx}^2}{u_{xx}^3(\lambda_1 + \lambda_2 u_{xx})^2}. \tag{3.34d}$$

The four listed integrating factors, (3.34a) - (3.34d), lead four essentially different subcases, which we now discuss in detail under Subcase II.1 to Subcase II.4 below.

Subcase II.1: We consider the integrating factor ${}^{II}\Lambda_1$ given by (3.34a) for equation (1.5), viz.

$$u_t = \frac{u_{xx}^3\,(\lambda_1 + \lambda_2 u_{xx})^3}{u_{xxx}^2}.$$

We now discuss the case $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, the case $\lambda_1 = 0$ and $\lambda_2 = 1$, and the case $\lambda_1 = 1$ and $\lambda_2 = 0$ below.

Let $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$: The integrating factor (3.34a) then leads to the following conserved current and flux for (1.5):

$$ {}^{II}\Phi^t_{1,1}[u] = -\frac{2}{\lambda_1^2}u_{xx}^{1/2}(\lambda_1 + \lambda_2 u_{xx})^{1/2} \tag{3.35a}$$

$$^{II}\Phi^x_{1,1}[u] = -\frac{2}{\lambda_1^2}(\lambda_1 + 2\lambda_2 u_{xx})(\lambda_1 + \lambda_2 u_{xx})^{5/2}\frac{u_{4x}}{u_{xxx}^3}$$
$$+\frac{4}{\lambda_1^2}(\lambda_1 + \lambda_2 u_{xx})^{3/2}(\lambda_1^2 + 3\lambda_1\lambda_2 u_{xx} + 3\lambda_2^2 u_{xx}^2)\frac{u_{xx}^{3/2}}{u_{xxx}}. \tag{3.35b}$$

Let $\lambda_1 = 0$ and $\lambda_2 = 1$: Equation (1.5) then takes the form

$$u_t = \frac{u_{xx}^6}{u_{xxx}^2} \tag{3.36}$$

and the integrating factor (3.34a) leads to the following conserved current and flux for (3.36):

$$^{II}\Phi^t_{1,2}[u] = \frac{2^{-1/3}}{u_{xx}} \tag{3.37a}$$

$$^{II}\Phi^x_{1,2}[u] = 2^{2/3}\left(-\frac{u_{xx}^4 u_{4x}}{u_{xxx}^3} + \frac{5u_{xx}^3}{u_{xxx}}\right). \tag{3.37b}$$

Let $\lambda_1 \neq 0$ and $\lambda_2 = 0$: Equation (1.5) then takes the form

$$u_t = \lambda_1^3\frac{u_{xx}^3}{u_{xxx}^2} \tag{3.38}$$

and the integrating factor (3.34a) leads to the following conserved current and flux for (3.38):

$$^{II}\Phi^t_{1,3}[u] = \lambda_1^{-3/2}u_{xx}^{1/2} \tag{3.39a}$$

$$^{II}\Phi^x_{1,3}[u] = \lambda^{3/2}\left(\frac{u_{xx}^{5/2}u_{4x}}{u_{xxx}^3} - \frac{2u_{xx}^{3/2}}{u_{xxx}}\right). \tag{3.39b}$$

This leads to

Potentialisation II.1

a) *Using the conserved current* $^{II}\Phi^t_{1,1}$*, given by (3.35a), we find that equation (1.5) viz.*

$$\boxed{u_t = \frac{u_{xx}^3\left(\lambda_1 + \lambda_2 u_{xx}\right)^3}{u_{xxx}^2}}$$

admits the zero-order potentialisation

$$v_t = \frac{\lambda_1^3}{4}\left(1 + \lambda_1^2\lambda_2 v_x^2\right)^{3/2}\left(\frac{v_x^3 v_{xxx}}{v_{xx}^3} - \frac{3v_x^2}{v_{xx}}\right), \tag{3.40}$$

where

$$v_x = -\frac{2}{\lambda_1^2}\left[u_{xx}(\lambda_1 + \lambda_2 u_{xx})\right]^{1/2}.$$

The first-order potentialisation for (1.5) with $w_x = D_x({}^{II}\Phi^t_{1,1})$ *i.e.* $w = v_x$*, is then*

$$w_t = \left(\frac{\lambda_1^3}{4}\right)\frac{(1+\lambda_1^2\lambda_2 w^2)^{3/2}w^3 w_{xxx}}{w_x^3} - \left(\frac{3\lambda_1^3}{4}\right)\frac{(1+\lambda_1^2\lambda_2 w^2)^{3/2}w^3 w_{xx}^2}{w_x^4}$$
$$+\left(\frac{3\lambda_1^3}{4}\right)\frac{(1+\lambda_1^2\lambda_2 w^2)^{1/2}(2+3\lambda_1^2\lambda_2 w^2)w^2 w_{xx}}{w_x^2}$$
$$-\left(\frac{3\lambda_1^3}{4}\right)(1+\lambda_1^2\lambda_2 w^2)^{1/2}(2+5\lambda_1^2\lambda_2 w^2)w. \tag{3.41}$$

b) Using the conserved current ${}^{II}\Phi^t_{1,2}$*, given by (3.37a), we find that equation (3.36) viz.*

$$\boxed{u_t = \frac{u_{xx}^6}{u_{xxx}^2}}$$

admits the zero-order potentialisation

$$v_t = \frac{v_{xxx}}{v_{xx}^3} + \frac{3}{v_x v_{xx}}, \tag{3.42}$$

where

$$v_x = \frac{1}{2^{1/3}u_{xx}}.$$

The first-order potentialisation for (3.36) with $w_x = D_x({}^{II}\Phi^t_{1,2})$ *i.e.* $w = v_x$*, is then*

$$w_t = \frac{w_{xxx}}{w_x^3} - \frac{3w_{xx}^2}{w_x^4} - \frac{3w_{xx}}{w w_x^2} - \frac{3}{w^2}. \tag{3.43}$$

Turning to the multi-potentialisation of (3.36), we find the zero-order potentialisation of (3.42) in

$$\tilde{v}_t = \frac{\tilde{v}_x^6\tilde{v}_{xxx}}{\tilde{v}_{xx}^3} - 3\frac{\tilde{v}_x^5}{\tilde{v}_{xx}}, \tag{3.44}$$

where

$$\tilde{v}_x = -\frac{1}{v_x},$$

as well as the zero-order potentialisation of (3.44) in

$$V_t = \frac{V_x^2 V_{xxx}}{V_{xx}^3} + \frac{V_x}{V_{xx}} - \frac{8}{9}x, \tag{3.45}$$

where

$$V_x = -\frac{1}{27}\frac{1}{\tilde{v}_x^3}.$$

The first-order potentialisation of (3.42), with $v_x = W$, gives

$$W_t = \frac{W_{xxx}}{W_x^3} - 3\frac{W_{xx}^2}{W_x^4} - 3\frac{W_{xx}}{WW_x^2} - \frac{3}{W^2}; \tag{3.46}$$

and the first-order potentialisation of (3.44), with $\tilde{v}_x = \tilde{W}$, gives

$$\tilde{W}_t = \frac{\tilde{W}^6\tilde{W}_{xxx}}{\tilde{W}_x^3} - 3\frac{\tilde{W}^6\tilde{W}_{xx}^2}{\tilde{W}_x^4} + 9\frac{\tilde{W}^5\tilde{W}_{xx}}{\tilde{W}_x^2} - 15\tilde{W}^4. \tag{3.47}$$

Another first-order potentialisation for (3.44) is

$$\tilde{V}_t = -\frac{\tilde{V}^{3/2}\tilde{V}_{xxx}}{\tilde{V}_x^3} + 3\frac{\tilde{V}^{3/2}\tilde{V}_{xx}^2}{\tilde{V}_x^4}, \tag{3.48}$$

where we have applied the following integrating factor of (3.44):

$$\Lambda[\tilde{v}] = -\frac{3}{2}\frac{\tilde{v}_{xx}}{\tilde{v}_x^4} \tag{3.49}$$

and the corresponding conserved current

$$\Phi^t[\tilde{v}] = \frac{1}{4}\frac{1}{\tilde{v}_x^2} \tag{3.50}$$

for the first-order potentialisation $\tilde{V}_x = D_x(\Phi^t)$, so $\tilde{V} = 4^{-1}\tilde{v}_x^{-2}$. Moreover, the first-order potentialisation of (3.45), with $V_x = q$, gives

$$q_t = \frac{q^2 q_{xxx}}{q_x^3} - 3\frac{q^2 q_{xx}^2}{q_x^4} + \frac{qq_{xx}}{q_x^2} + \frac{1}{9}. \tag{3.51}$$

Diagram 1 displays this multi-potentialisation of equation (3.36).

c) *Using the conserved current ${}^{II}\Phi^t_{1,3}$, given by (3.39a), we find that equation (3.38) viz.*

$$\boxed{u_t = \lambda_1^3\frac{u_{xx}^3}{u_{xxx}^2}}$$

admits the zero-order potentialisation

$$v_t = -\frac{\lambda_1^3}{4}\left(\frac{v_x^3 v_{xxx}}{v_{xx}^3} - 3\frac{v_x^2}{v_{xx}}\right), \tag{3.52}$$

where

$$v_x = k\lambda_1^{-3/2}u_{xx}^{1/2}$$

for any non-zero constant k. We now consider $\lambda_1 = -1$, so equation (3.38) takes the form

$$\boxed{u_t = -\frac{u_{xx}^3}{u_{xxx}^2}} \tag{3.53}$$

Turning to the multi-potentialisation of (3.53), we find the zero-order potentialisation of (3.52) in

$$\tilde{v}_t = 2\frac{\tilde{v}_x^3\tilde{v}_{xxx}}{\tilde{v}_{xx}^3} - 3\frac{\tilde{v}_x^2}{\tilde{v}_{xx}} \equiv 2\frac{\tilde{v}_x^4}{\tilde{v}_{xx}^3}\, S[\tilde{v}], \tag{3.54}$$

where

$$\tilde{v}_x = v_x^2 \tag{3.55}$$

and S is the Schwarzian (1.8). Furthermore (3.54) admits the zero-order potentialisation

$$V_t = \frac{V_{xxx}}{V_{xx}^3} - 3\cdot 2^{-2/3}\frac{1}{V_{xx}} - 2^{-1/3}x, \tag{3.56}$$

where

$$V_x = 2^{-1/3}\ln(\tilde{v}_x). \tag{3.57}$$

Equation (3.56) does not admit a zero-order potentialisation.

Turning to the first-order potentialisations of (3.53), we find the following:

- *With $v_x = w$, equation (3.52) takes the form*

$$w_t = \frac{1}{4}\frac{w^3 w_{xxx}}{w_x^3} - \frac{3}{4}\frac{w^3 w_{xx}^2}{w_x^4} + \frac{3}{2}\frac{w^2 w_{xx}}{w_x^2} - \frac{3}{2}w; \tag{3.58}$$

- *With $\tilde{v}_x = \tilde{w}$, equation (3.54) takes the form*

$$\tilde{w}_t = 2\frac{\tilde{w}^3\tilde{w}_{xxx}}{\tilde{w}_x^3} - 6\frac{\tilde{w}^3\tilde{w}_{xx}^2}{\tilde{w}_x^4} + 9\frac{\tilde{w}^2\tilde{w}_{xx}}{\tilde{w}_x^2} - 6\tilde{w}; \tag{3.59}$$

- *With $V_x = W$, equation (3.56) take the form*

$$W_t = \frac{W_{xxx}}{W_x^3} - 3\frac{W_{xx}^2}{W_x^4} + 3\cdot 2^{-2/3}\frac{W_{xx}}{W_x^2} - 2^{-1/3}. \tag{3.60}$$

We find that equation (3.56) admits also a second-order potentialisation which is obtained from equation (3.60) with $W_x = p$, namely

$$p_t = \frac{p_{xxx}}{p^3} - 9\,\frac{p_x p_{xx}}{p^4} + 12\,\frac{p_x^3}{p^5} + 3\,\cdot 2^{-2/3}\,\frac{p_{xx}}{p^2} - 3\,\cdot 2^{1/3}\,\frac{p_x^2}{p^3}. \tag{3.61}$$

Note that (3.61) is a spacial case of one of the integrable equations list in [6], namely equation (4.1.34). Interestingly, equation (3.61) admits only one integrating factor, namely $\Lambda[p] = 1$, which leads to the following zero-order potentialisation of (3.61):

$$q_t = \frac{q_{xxx}}{q_x^3} - 3\frac{q_{xx}^2}{q_x^4} + 3\cdot 2^{-2/3}\,\frac{q_{xx}}{q_x^2}, \tag{3.62}$$

where $q_x = p$. Diagram 2 displays this multi-potentialisation of equation (3.53).

Furthermore we find that (3.62) admits three integrating factors of order zero (none of order two or order four), namely

$$\Lambda_1[q] = 1, \qquad \Lambda_2[q] = q, \quad \textit{and} \quad \Lambda_3[q] = \exp\left(3 \cdot 2^{-2/3} q\right), \tag{3.63}$$

which give the following potentialisations for (3.62):

- *For $\Lambda_1[q] = 1$ we obtain the following zero-order potentialisation of (3.62):*

$$Q_{1,t} = \frac{Q_{1,xxx}}{Q_{1,xx}^3} - 3 \cdot 2^{-2/3} \frac{1}{Q_{1,xx}}, \tag{3.64}$$

  where

$$Q_{1,x} = q. \tag{3.65}$$

  Obviously the first order potentialisation of (3.64) leads back to (3.62).

- *For $\Lambda_1[q] = q$ we obtain the following zero-order potentialisation of (3.62):*

$$Q_{2,t} = \frac{Q_{2,x}^{3/2} Q_{2,xxx}}{Q_{2,xx}^3} - 3 \cdot 2^{-2/3} \frac{Q_{2,x}}{Q_{2,xx}} + 3 \cdot 2^{-5/3}\, x, \tag{3.66}$$

  where

$$Q_{2,x} = \frac{q^2}{4}. \tag{3.67}$$

  The corresponding first-order potentialisation of (3.62) is then

$$\tilde{Q}_{2,t} = \frac{\tilde{Q}_2^{3/2} \tilde{Q}_{2,xxx}}{\tilde{Q}_{2,x}^3} - 3 \frac{\tilde{Q}_2^{3/2} \tilde{Q}_{2,xx}^2}{\tilde{Q}_{2,x}^4} + \frac{3}{2} \frac{\left(\tilde{Q}_2^{1/2} + 2^{1/3} \tilde{Q}_2\right) \tilde{Q}_{2,xx}}{\tilde{Q}_{2,x}^2}$$
$$-3 \cdot 2^{-5/3}, \tag{3.68}$$

  where

$$Q_{2,x} = \tilde{Q}_2. \tag{3.69}$$

- *For $\Lambda_1[q] = \exp\left(3 \cdot 2^{-2/3} q\right)$ we obtain the following zero-order potentialisation of (3.62):*

$$Q_{3,t} = \frac{27}{4} \frac{Q_{3,x}^3 Q_{3,xxx}}{Q_{3,xx}^3} - \frac{27}{4} \frac{Q_{3,x}^2}{Q_{3,xx}}, \tag{3.70}$$

  where

$$Q_{3,x} = \exp\left(3 \cdot 2^{-2/3} q\right). \tag{3.71}$$

The corresponding first-order potentialisation of (3.62) is then

$$\tilde{Q}_{3,t} = \frac{27}{4}\frac{\tilde{Q}_3^3\tilde{Q}_{3,xxx}}{\tilde{Q}_{3,x}^3} - \frac{81}{4}\frac{\tilde{Q}_3^3\tilde{Q}_{3,xx}^2}{\tilde{Q}_{3,x}^4} + 27\frac{\tilde{Q}_3^2\tilde{Q}_{3,xx}}{\tilde{Q}_{3,x}^3}, \tag{3.72}$$

where

$$Q_{3,x} = \tilde{Q}_3. \tag{3.73}$$

Diagram 3 displays the potentialisation of equation (3.62).

Diagram 1

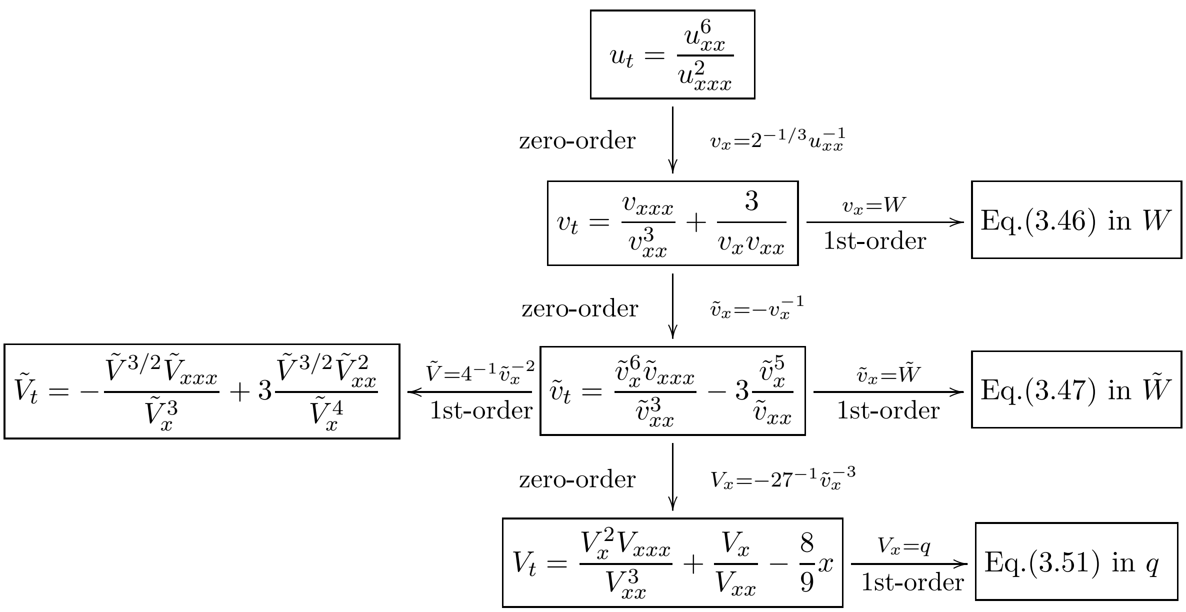


Diagram 2

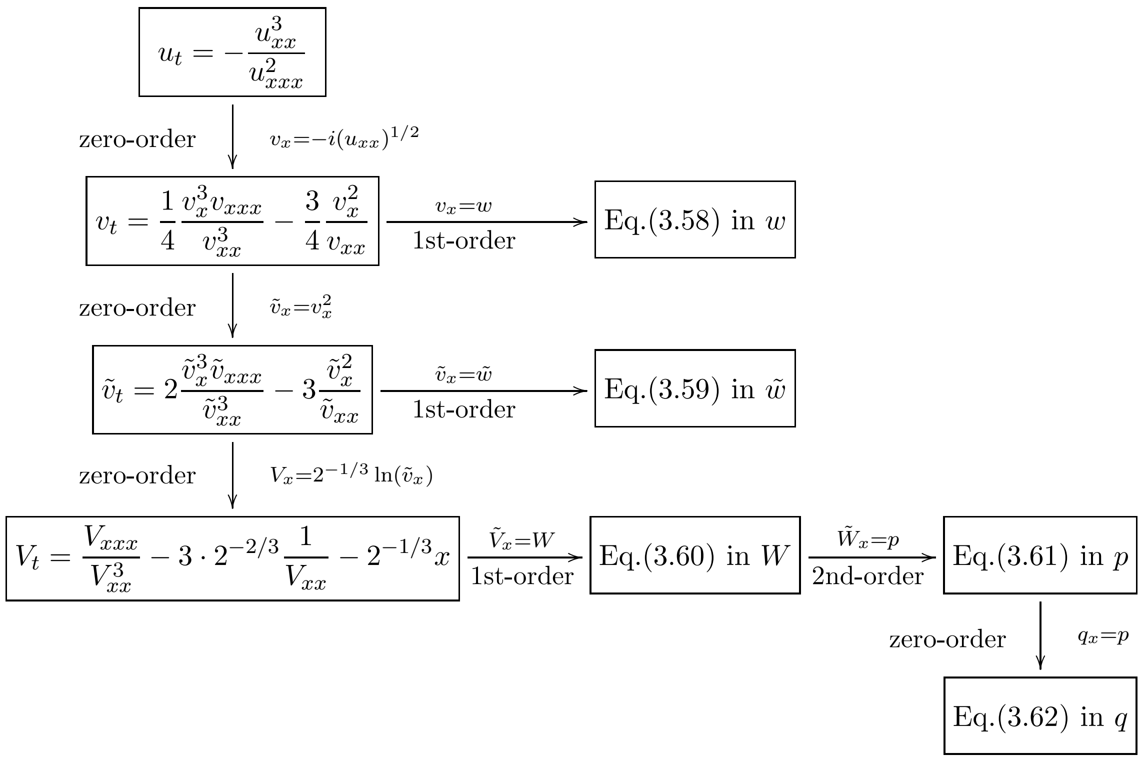
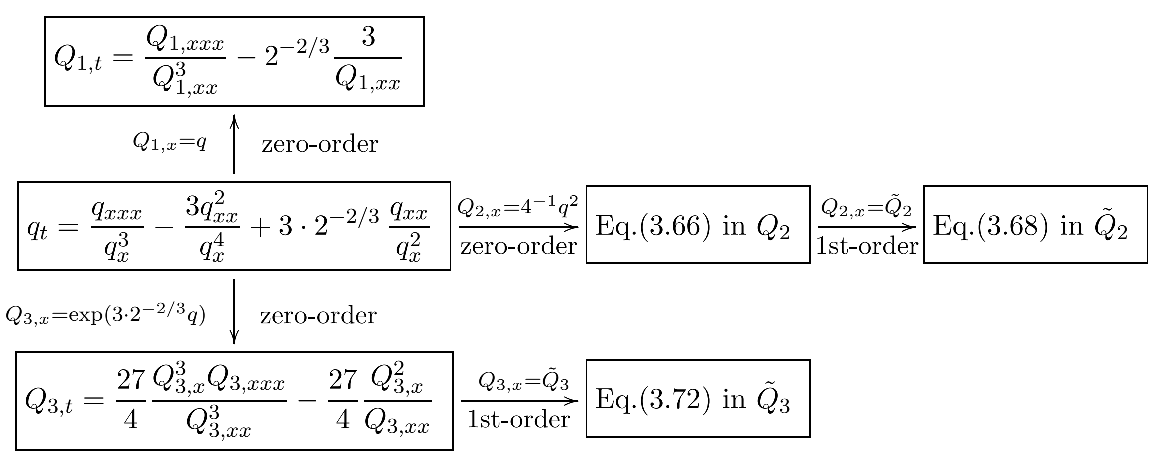


Diagram 3



Subcase II.2: We consider the integrating factor (3.34b) for equation (1.5), viz.

$$u_t = \frac{u_{xx}^3 \left(\lambda_1 + \lambda_2 u_{xx}\right)^3}{u_{xxx}^2}.$$

Let $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$: The integrating factor (3.34b) then leads to the following conserved current and flux for (1.5):

$$^{II}\Phi^t_{2,1}[u] = \frac{2^{-1/3}}{\lambda_1} \ln\left(\lambda_2 + \frac{\lambda_1}{u_{xx}}\right) \tag{3.74a}$$

$$^{II}\Phi^x_{2,1}[x,u] = -2^{2/3}\frac{(\lambda_1+\lambda_2 u_{xx})^2 u_{xx}^2 u_{4x}}{u_{xxx}^3} + 5\cdot 2^{-1/3}\frac{(\lambda_1+\lambda_2 u_{xx})(\lambda_1+2\lambda_2 u_{xx})u_{xx}}{u_{xxx}} + 2^{-1/3}\lambda_1^2\, x. \tag{3.74b}$$

This leads to

Potentialisation II.2

Using the conserved current $^{II}\Phi^t_{2,1}$*, given by (3.74a), we find that equation (1.5) viz.*

$$\boxed{u_t = \frac{u_{xx}^3 \left(\lambda_1 + \lambda_2 u_{xx}\right)^3}{u_{xxx}^2}}$$

admits the zero-order potentialisation

$$v_t = \frac{v_{xxx}}{v_{xx}^3} + \lambda_1 3\cdot 2^{-2/3}\frac{e^{2^{1/3}\lambda_1 v_x} + \lambda_2}{\left(e^{2^{1/3}\lambda_1 v_x} - \lambda_2\right) v_{xx}} - 2^{-1/3}\lambda_1^2\, x, \tag{3.75a}$$

where

$$v_x = \frac{2^{-1/3}}{\lambda_1}\ln\left(\frac{\lambda_1+\lambda_2 u_{xx}}{u_{xx}}\right). \tag{3.75b}$$

The first-order potentialisation of (1.5) with $w_x = D_x(^{II}\Phi^t_{2,1})$ *i.e.* $w = v_x$*, is then*

$$w_t = \frac{w_{xxx}}{w_x^3} - 3\frac{w_{xx}^2}{w_x^4} - 3\cdot 2^{-2/3}\lambda_1\frac{\left(e^{2^{1/3}\lambda_1 w} + \lambda_2\right) w_{xx}}{\left(e^{2^{1/3}\lambda_1 w} - \lambda_2\right) w_x^2} - 3\cdot 2^{2/3}\lambda_1^2\lambda_2\frac{e^{2^{1/3}\lambda_1 w}}{(e^{2^{1/3}\lambda_1 w} - \lambda_2)^2} - 2^{-1/3}\lambda_1^2. \tag{3.75c}$$

We remark that the case $\lambda_1 = 1$ and $\lambda_2 = 0$, as well as the case $\lambda_1 = 0$ and $\lambda_2 = 1$, do not lead to different equations than those already listed in Potentialisation II.1.

Subcase II.3: We consider the integrating factor (3.34c) for equation (1.5), viz.

$$u_t = \frac{u_{xx}^3 \left(\lambda_1 + \lambda_2 u_{xx}\right)^3}{u_{xxx}^2}.$$

Let $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$: The integrating factor (3.34c) then leads to the following conserved current and flux for (1.5):

$$ {}^{II}\Phi^t_{3,1}[u] = \frac{1}{\lambda_1^2}\, u_{xx} \ln\left(\frac{u_{xx}}{\lambda_1 + \lambda_2 u_{xx}}\right) + \frac{1}{\lambda_1\lambda_2} \tag{3.76a}$$

$$\begin{aligned} {}^{II}\Phi^x_{2,1}[u] = {} & \frac{2}{\lambda_1^2}\frac{(\lambda_1+\lambda_2 u_{xx})^2 u_{xx}^3 u_{4x}}{u_{xxx}^3}\left[\lambda_1 + (\lambda_1+\lambda_2 u_{xx})\ln\left(\frac{u_{xx}}{\lambda_1+\lambda_2 u_{xx}}\right)\right] \\ & -\frac{3}{\lambda_1^2}\frac{u_{xx}^2}{u_{xxx}}(\lambda_1 + 2\lambda_2 u_{xx})(\lambda_1+\lambda_2 u_{xx})^2 \ln\left(\frac{u_{xx}}{\lambda_1+\lambda_2 u_{xx}}\right) \\ & -\frac{1}{\lambda_1}\frac{u_{xx}^2}{u_{xxx}}(\lambda_1 + 6\lambda_2 u_{xx})(\lambda_1+\lambda_2 u_{xx}) - \lambda_1 u_x. \end{aligned} \tag{3.76b}$$

Let $\lambda_1 = 0$ and $\lambda_2 = 1$: In this case the integrating factor (3.34c) is identical to the integrating factor (3.34b), which has already been described in Subcase II.2.

Let $\lambda_1 = 1$ and $\lambda_2 = 0$: Equation (1.5) then takes the form (3.38) with $\lambda_1 = 1$, viz.

$$u_t = \frac{u_{xx}^3}{u_{xxx}^2}$$

and the integrating factor (3.34c) leads to the following conserved current and flux for (3.38) with $\lambda_1 = 1$:

$$ {}^{II}\Phi^t_{3,3}[u] = u_{xx}\ln(u_{xx}) - u_{xx} \tag{3.77a}$$

$$ {}^{II}\Phi^x_{3,3}[u] = \frac{2u_{xx}^3 u_{4x}}{u_{xxx}^3}\ln(u_{xx}) - \frac{3u_{xx}^2}{u_{xxx}}\ln(u_{xx}) + \frac{2u_{xx}^2}{u_{xxx}} - u_x. \tag{3.77b}$$

This leads to

Potentialisation II.3

a) Using the conserved current ${}^{II}\Phi^t_{3,1}$, given by (3.76a), we find that equation (1.5) viz.

$$\boxed{u_t = \frac{u_{xx}^3 \left(\lambda_1 + \lambda_2 u_{xx}\right)^3}{u_{xxx}^2}}$$

admits no zero-order potentialisation. Equation (1.5) does however admit a first-order potentialisation with $w_x = D_x({}^{II}\Phi^t_{3,1})$, *i.e.*

$$w(x,t) = \frac{u_{xx}}{\lambda_1^2} \ln\left(\frac{u_{xx}}{\lambda_1 + \lambda_2 u_{xx}}\right) + \frac{1}{\lambda_1\lambda_2}, \tag{3.78}$$

namely

$$\begin{aligned} w_t = &-\frac{2}{\lambda_2}^{3} \frac{w_{xxx}}{w_x^3}(\lambda_2^2 u_{xx} w + \lambda_1\lambda_2 w - 1)^3 + u_{xx}^3 \left(30\lambda_2^3 w + 6\lambda_2^3 \frac{w^3 w_{xx}^2}{w_x^4} - 18\lambda_2^3 \frac{w^2 w_{xx}}{w_x^2}\right)\\ &+u_{xx}^2\left(60\lambda_1\lambda_2^2 w - 18\lambda_2 - 15\lambda_2 \frac{w w_{xx}}{w_x^2}(3\lambda_1\lambda_2 w - 2) + 18\lambda_2 \frac{w^2 w_{xx}^2}{w_x^4}(\lambda_1\lambda_2 w - 1)\right)\\ &+\frac{6}{\lambda_2^3}\frac{w_{xx}^2}{w_x^4}(\lambda_1\lambda_2 w - 1)^3 - \frac{9\lambda_1}{\lambda_2^2}\frac{w_{xx}}{w_x^2}(\lambda_1\lambda_2 w - 1)^2 + 6\lambda_1^3 w - \frac{6\lambda_1^2}{\lambda_2}\\ &+u_{xx}\left(36\lambda_1^2\lambda_2 w - 23\lambda_1 + \frac{18}{\lambda_2}\frac{w w_{xx}^2}{w_x^4}(\lambda_1\lambda_2 w - 1)^2\right.\\ &\left.-\frac{12}{\lambda_2}\frac{w_{xx}}{w_x^2}(\lambda_1\lambda_2 w - 1)(3\lambda_1\lambda_2 w - 1)\right), \end{aligned} \tag{3.79}$$

where u_{xx} *needs to be solved algebraically in terms of* w *from (3.78), which is possible by the Lambert function.*

b) *Using the conserved current* ${}^{II}\Phi^t_{3,3}$ *given by (3.77a), we find that equation (3.38) with* $\lambda_1 = 1$, *viz.*

$$\boxed{u_t = \frac{u_{xx}^3}{u_{xxx}^2}}$$

does not admit a zero-order potentialisation but it does admit a first-order potentialisation with $w_x = D_x({}^{II}\Phi^t_{3,3})$, *i.e.*

$$w = u_{xx}\ln(u_{xx}) - u_{xx}, \tag{3.80}$$

namely

$$\begin{aligned} w_t = &-2\frac{w_{xxx}}{w_x^3}(u_{xx} + w)^3 + 6\frac{w_{xx}^2}{w_x^4}(u_{xx} + w)^3 - 3\frac{w_{xx}}{w_x^2}(u_{xx} + w)(5u_{xx} + 3w)\\ &+6w + 13u_{xx}, \end{aligned} \tag{3.81}$$

where u_{xx} *needs to be solved algebraically from the relation (3.80) in terms of* w *by the Lambert function.*

Subcase II.4: We consider the integrating factor (3.34d) for equation (1.5), viz.

$$u_t = \frac{u_{xx}^3 \left(\lambda_1 + \lambda_2 u_{xx}\right)^3}{u_{xxx}^2}.$$

Let $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$: The integrating factor (3.34d) then leads to the following conserved current and flux for (1.5):

$$^{II}\Phi^t_{4,1}[u] = \frac{2^{-1/3}}{\lambda_1^2}(\lambda_1 + \lambda_2 u_{xx}) \ln\left(\frac{u_{xx}}{\lambda_1 + \lambda_2 u_{xx}}\right) + \frac{2^{-1/3}}{\lambda_1} \tag{3.82a}$$

$$\begin{aligned} ^{II}\Phi^x_{4,1}[x,u] &= 2^{-2/3}\frac{1}{\lambda_1^2}\frac{u_{xx}^2 u_{4x}}{u_{xxx}^2}(\lambda_1 + \lambda_2 u_{xx})^3 \left[\lambda_2 u_{xx} \ln\left(\frac{u_{xx}}{\lambda_1 + \lambda_2 u_{xx}}\right) + \lambda_1\right] \\ &\quad -3\cdot 2^{-1/3}\frac{\lambda_2}{\lambda_1^2}\frac{u_{xx}^2}{u_{xxx}}(\lambda_1 + 2\lambda_2 u_{xx})(\lambda_1 + \lambda_2 u_{xx})^2 \ln\left(\frac{u_{xx}}{\lambda_1 + \lambda_2 u_{xx}}\right) \\ &\quad -2^{-1/3}\frac{1}{\lambda_1}\frac{u_{xx}}{u_{xxx}}(5\lambda_1 + 6\lambda_2 u_{xx})(\lambda_1 + \lambda_2 u_{xx})^2 - 2^{-1/3}\lambda_1\lambda_2 u_x - 2^{-1/3}\lambda_1^2 x. \end{aligned} \tag{3.82b}$$

Let $\lambda_1 = 0$ and $\lambda_2 = 1$: The integrating factor (3.34d) is then identical to integrating factor (3.34a) described in Subcase II.1.

Let $\lambda_1 = 1$ and $\lambda_2 = 0$: The integrating factor (3.34d) then leads to the following conserved current and flux for (3.38) with $\lambda_1 = 1$, viz

$$u_t = \frac{u_{xx}^3}{u_{xxx}^2},$$

namely

$$^{II}\Phi^t_{4,3}[u] = -2^{-1/3}\ln(u_{xx}) \tag{3.83a}$$

$$^{II}\Phi^x_{4,3}[x,u] = -2^{2/3}\left(\frac{u_{xx}^2 u_{4x}}{u_{xxx}^3} - \frac{5}{2}\frac{u_{xx}}{u_{xxx}} - \frac{1}{2}x\right). \tag{3.83b}$$

This leads to

Potentialisation II.4

a) *Using the conserved current* $^{II}\Phi^t_{4,1}$*, given by (3.82a), we find that equation (1.5) viz.*

$$\boxed{u_t = \frac{u_{xx}^3 \left(\lambda_1 + \lambda_2 u_{xx}\right)^3}{u_{xxx}^2}}$$

admits no zero-order potentialisation. Equation (1.5) does however admit a first-order potentialisation with $w_x = D_x({}^{II}\Phi^t_{4,1})$, *i.e.*

$$w(x,t) = \frac{2^{-1/3}}{\lambda_1^2}(\lambda_1 + \lambda_2 u_{xx}) \ln\left(\frac{u_{xx}}{\lambda_1 + \lambda_2 u_{xx}}\right) + \frac{2^{-1/3}}{\lambda_1} \tag{3.84}$$

namely

$$\begin{aligned} w_t &= -\frac{w_{xxx}}{w_x^3}\frac{(2^{1/3}\lambda_2 w u_{xx}+1)^3}{(\lambda_2 u_{xx}+\lambda_1)^2} + 3\frac{w_{xx}^2}{w_x^4}\frac{(2^{1/3}\lambda_2 w u_{xx}+1)^3}{(\lambda_2 u_{xx}+\lambda_1)^2} \\ &-\frac{9\lambda_2^2 w_{xx} w^2}{w_x^2}\frac{(2\lambda_2 u_{xx}+\lambda_1)u_{xx}^2}{(\lambda_2 u_{xx}+\lambda_1)^2} + \frac{3\cdot 2^{2/3}\lambda_2 w_{xx} w}{\lambda_1 w_x^2}\frac{(\lambda_2^2 u_{xx}^2 - 4\lambda_1\lambda_2 u_{xx} - 2\lambda_1^2)u_{xx}}{(\lambda_2 u_{xx}+\lambda_1)^2} \\ &+\frac{3\cdot 2^{-2/3}\lambda_2 w_{xx}}{\lambda_1^2 w_x^2}\frac{(4\lambda_2^2 u_{xx}^2 + 11\lambda_1\lambda_2 u_{xx} + 4\lambda_1^2)u_{xx}}{(\lambda_2 u_{xx}+\lambda_1)^2} - \frac{3\cdot 2^{-2/3} w_{xx}}{\lambda_1^2 w_x^2}(4\lambda_2 u_{xx}+\lambda_1) \\ &-\frac{3\lambda_2(2^{2/3}-2\lambda_1 w)}{\lambda_1}\frac{(5\lambda_2^2 u_{xx}^2 + 5\lambda_1\lambda_2 u_{xx} + \lambda_1^2)u_{xx}}{(\lambda_2 u_{xx}+\lambda_1)^2} \\ &+\frac{2^{-1/3}}{\lambda_1}\frac{(16\lambda_2^2 u_{xx}^2 + 11\lambda_1\lambda_2 u_{xx} + \lambda_1^2)u_{xx}}{(\lambda_2 u_{xx}+\lambda_1)^2}, \end{aligned} \tag{3.85}$$

where u_{xx} *needs to be solved algebraically in* w *from (3.84) in terms of the Lambert function.*

b) *Using the conserved current* ${}^{II}\Phi^t_{4,3}$, *given by (3.83a), we find that equation (3.38) with* $\lambda_1 = 1$, *viz.*

$$\boxed{u_t = \frac{u_{xx}^3}{u_{xxx}^2}}$$

admits the zero-order potentialisation

$$v_t = \frac{v_{xxx}}{v_{xx}^3} + 3\cdot 2^{-2/3}\frac{1}{v_{xx}} - 2^{-1/3}x, \tag{3.86}$$

where

$$v_x = -2^{-1/3}\ln(u_{xx}). \tag{3.87}$$

Equation (3.38) also admits a first-order potentialisation with $w_x = D_x({}^{II}\Phi^t_{4,3})$, *i.e.* $v_x = w$, *namely*

$$w_t = \frac{w_{xxx}}{w_x^3} - 3\frac{w_{xx}^2}{w_x^4} - 3\cdot 2^{-2/3}\frac{w_{xx}}{w_x^2} - 2^{-1/3}, \tag{3.88}$$

and a second-order potentialisation with $W_x = D_x^2({}^{II}\Phi^t_{4,3})$, *i.e.* $W = w_x$, *namely*

$$W_t = \frac{W_{xxx}}{W^3} - 9\frac{W_x W_{xx}}{W^4} - 3\cdot 2^{-2/3}\frac{W_{xx}}{W^2} + 12\frac{W_x^3}{W^5} + 3\cdot 2^{1/3}\frac{W_x^2}{W^3}. \tag{3.89}$$

Case III: We consider (1.6), viz.

$$u_t = \frac{(\alpha u_x + \beta)^{11}}{\left[(\alpha u_x + \beta)u_{xxx} - 3\alpha u_{xx}^2\right]^2},$$

where α and β are arbitrary constants, not simultaneously zero. Equation (1.6) does not admit zero-order or second-order integrating factors, but the equation does admit the following three fourth-order integrating factors:

$$^{III}\Lambda_1^1[u] = \frac{u_{xx}^2 u_{4x}}{(\alpha u_x + \beta)^{11}} + \frac{2u_{xx}u_{xxx}^2}{(\alpha u_x + \beta)^{11}} - \frac{22\alpha u_{xx}^3 u_{xxx}}{(\alpha u_x + \beta)^{12}} + \frac{33\alpha^2 u_{xx}^5}{(\alpha u_x + \beta)^{13}} \tag{3.90a}$$

$$^{III}\Lambda_2^1[u] = \frac{u_{xx}u_{4x}}{(\alpha u_x + \beta)^8} + \frac{u_{xxx}^2}{(\alpha u_x + \beta)^8} - \frac{16\alpha u_{xx}^2 u_{xxx}}{(\alpha u_x + \beta)^9} + \frac{24\alpha^2 u_{xx}^4}{(\alpha u_x + \beta)^{10}} \tag{3.90b}$$

$$^{III}\Lambda_3^1[u] = \frac{u_{4x}}{(\alpha u_x + \beta)^5} - \frac{10\alpha u_{xx}u_{xxx}}{(\alpha u_x + \beta)^6} + \frac{15\alpha^2 u_{xx}^3}{(\alpha u_x + \beta)^7}. \tag{3.90c}$$

Using (3.90a), (3.90b) and (3.90c), we obtain the following respective conserved currents for (1.6):

$$^{III}\Phi_{1,1}^t[u] = \frac{u_{xx}^4}{(\alpha u_x + \beta)^{11}}, \quad {}^{III}\Phi_{1,2}^t[u] = \frac{u_{xx}^3}{(\alpha u_x + \beta)^8},$$

$$\text{and} \quad {}^{III}\Phi_{1,3}^t[u] = \frac{u_{xx}^2}{(\alpha u_x + \beta)^5}. \tag{3.91}$$

We find that all three conserved currents (3.91) do not lead to a potentialisation of (1.6), of any order.

Let $\alpha = 1$ and $\beta = 0$: Equation (1.6) then takes the form

$$u_t = \frac{u_x^{11}}{\left(u_x u_{xxx} - 3u_{xx}^2\right)^2}. \tag{3.92}$$

In addition to the integrating factors (3.90a), (3.90b) and (3.90c) with $\alpha = 1$ and $\beta = 0$, equation (3.92) also admits the following fourth-order integrating factors:

$$^{III}\Lambda_1^2[u] = \left(\frac{uu_{xx}^2}{u_x^{11}} + \frac{u_{xx}}{u_x^9}\right)u_{4x} + \left(\frac{1}{u_x^9} + \frac{2uu_{xx}}{u_x^{11}}\right)u_{xxx}^2 - \left(\frac{22uu_{xx}^3}{u_x^{12}} + \frac{16u_{xx}^2}{u_x^{10}}\right)u_{xxx} + \frac{33uu_{xx}^5}{u_x^{13}} + \frac{24u_{xx}^4}{u_x^{11}} \tag{3.93a}$$

$$^{III}\Lambda_2^2[u] = \left(\frac{uu_{xx}}{u_x^8} + \frac{1}{u_x^6}\right)u_{4x} - \left(\frac{16uu_{xx}^2}{u_x^9} + \frac{10u_{xx}}{u_x^7}\right)u_{xxx} + \frac{uu_{xxx}^2}{u_x^8} + \frac{24uu_{xx}^4}{u_x^{10}} + \frac{15u_{xx}^3}{u_x^8}. \tag{3.93b}$$

Using (3.93a) and (3.93b), we obtain the following respective conserved currents for (3.92):

$$^{III}\Phi^t_{2,1}[u] = \frac{1}{12}\frac{uu_{xx}^4}{u_x^{11}} + \frac{1}{6}\frac{u_{xx}^3}{u_x^9} \qquad \text{and} \qquad ^{III}\Phi^t_{2,2}[u] = \frac{1}{6}\frac{uu_{xx}^3}{u_x^8} + \frac{1}{2}\frac{u_{xx}^2}{u_x^6}. \tag{3.94}$$

We find that both conserved currents (3.94) do not lead to a potentialisation of (3.92), of any order.

Let $\alpha = 0$ and $\beta = 1$: Equation (1.6) then takes the form

$$u_t = \frac{1}{u_{xxx}^2}, \tag{3.95}$$

which admits the following three fourth-order integrating factors (no zero-order or second-order exist):

$$^{III}\Lambda^3_1[u] = u_{xx}^2 u_{4x} + 2u_{xx}u_{xxx}^2 \tag{3.96a}$$

$$^{III}\Lambda^3_2[u] = u_{xx}u_{4x} + u_{xxx}^2 \tag{3.96b}$$

$$^{III}\Lambda^3_3[u] = u_{4x}. \tag{3.96c}$$

Note that the integrating factors (3.96a), (3.96b) and (3.96c) are just the integrating factors (3.90a), (3.90b) and (3.90c), respectively, with $\alpha = 0$ and $\beta = 1$. No additional integrating factors up to order four than those listed here were obtained for equation (3.95). The corresponding conserved currents and fluxes are as follows:

$$^{III}\Phi^t_{3,1}[u] = 2^{-1/3}\frac{1}{512}u_{xx}^4 \tag{3.97a}$$

$$^{III}\Phi^x_{3,1}[u] = 2^{2/3}\frac{1}{64}\left(\frac{u_{xx}^3 u_{4x}}{u_{xxx}^3} + \frac{3u_{xx}^2}{u_{xxx}} - 3u_x\right) \tag{3.97b}$$

$$^{III}\Phi^t_{3,2}[u] = -\frac{1}{54}u_{xx}^2 \tag{3.97c}$$

$$^{III}\Phi^x_{3,2}[x,u] = -\frac{1}{9}\left(\frac{u_{xx}^2 u_{4x}}{u_{xxx}^3} + \frac{2u_{xx}}{u_{xxx}} - 2x\right) \tag{3.97d}$$

$$^{III}\Phi^t_{3,3}[u] = 2^{-8/3}u_{xx}^2 \tag{3.97e}$$

$$^{III}\Phi^x_{3,3}[u] = 2^{-2/3}\left(\frac{u_{xx}u_{4x}}{u_{xxx}^3} + \frac{1}{u_{xxx}}\right). \tag{3.97f}$$

This leads to

Potentialisation III.1

a) *Using the conserved current* $^{III}\Phi^t_{3,1}$*, given by (3.97a), we find that equation (3.95) viz.*

$$\boxed{u_t = \frac{1}{u_{xxx}^2}}$$

admits no zero-order potentialisation. Equation (3.95) does however admit a first-order potentialisation with $w_x = D_x({}^{III}\Phi^t_{3,1})$*, i.e.*

$$w(x,t) = 2^{-1/3}\frac{1}{512}u_{xx}^4, \tag{3.98}$$

namely

$$w_t = -\frac{w^{9/4}w_{xxx}}{w_x^3} + 3\frac{w^{9/4}w_{xx}^2}{w_x^4} - \frac{9}{4}\frac{w^{5/4}w_{xx}}{w_x^2} + \frac{3}{8}w_x^{1/4}. \tag{3.99}$$

b) Using the conserved current ${}^{III}\Phi^t_{3,2}$*, given by (3.97c), we find that equation (3.95) viz.*

$$\boxed{u_t = \frac{1}{u_{xxx}^2}}$$

admits the zero-order potentialisation

$$v_t = \frac{v_x^2 v_{xxx}}{v_{xx}^3} - \frac{2}{9}x, \tag{3.100}$$

where

$$v_x = -\frac{1}{54}u_{xx}^3. \tag{3.101}$$

Equation (3.95) also admits a first-order potentialisation with $w_x = D_x({}^{III}\Phi^t_{3,2})$*, i.e.* $v_x = w$*, namely*

$$w_t = \frac{w^2 w_{xxx}}{w_x^3} - 3\frac{w^2 w_{xx}^2}{w_x^4} + 2\frac{w w_{xx}}{w_x^2} - \frac{2}{9}. \tag{3.102}$$

c) Using the conserved current ${}^{III}\Phi^t_{3,3}$*, given by (3.97e), we find that equation (3.95) viz.*

$$\boxed{u_t = \frac{1}{u_{xxx}^2}}$$

admits the zero-order potentialisation

$$v_t = -\frac{v_x^{3/2}v_{xxx}}{v_{xx}^3}, \tag{3.103}$$

where

$$v_x = 2^{-8/3}u_{xx}^2. \tag{3.104}$$

Equation (3.95) also admits a first-order potentialisation with $w_x = D_x({}^{III}\Phi^t_{3,3})$*, i.e.* $v_x = w$*, namely*

$$w_t = -\frac{w^{3/2}w_{xxx}}{w_x^3} + 3\frac{w^{3/2}w_{xx}^2}{w_x^4} - \frac{3}{2}\frac{w^{1/2}w_{xx}}{w_x^2}. \tag{3.105}$$

We obtained one multipotentialisation for equation (3.95) which is a result of the potentialisation of equation (3.100) viz.

$$v_t = \frac{v_x^2 v_{xxx}}{v_{xx}^3} - \frac{2}{9}x,$$

which we now discuss in detail.

Equation (3.100) admits the following four integrating factors (no zero-order integrating factor exists):

$$^{III}\Lambda_1^4[v] = \frac{v_x^{2/3} v_{4x}}{v_{xx}^3} - 3\frac{v_x^{2/3} v_{xxx}^2}{v_{xx}^4} + \frac{4}{3}\frac{v_{xxx}}{v_{xx}^2 v_x^{1/3}} + \frac{2}{9}\frac{1}{v_x^{4/3}} \tag{3.106a}$$

$$^{III}\Lambda_2^4[v] = \frac{v_{xx}}{v_x^{4/3}} \tag{3.106b}$$

$$^{III}\Lambda_3^4[v] = \frac{v_{xx}}{v_x^{5/3}} \tag{3.106c}$$

$$^{III}\Lambda_4^4[v] = v_{xx} \tag{3.106d}$$

The corresponding conserved currents and fluxes for (3.100) are as follows:

$$^{III}\Phi_{4,1}^t[v] = \frac{v_x^{2/3}}{v_{xx}} \tag{3.107a}$$

$$^{III}\Phi_{4,1}^x[v] = \frac{v_x^{8/3} v_{4x}}{v_{xx}^5} + \frac{2}{3}\frac{v_x^{5/3} v_{xxx}}{v_{xx}^4} - 2\frac{v_x^{8/3} v_{xxx}^2}{v_{xx}^6} + \frac{2}{9}\frac{v_x^{2/3}}{v_{xx}^2} \tag{3.107b}$$

$$^{III}\Phi_{4,2}^t[v] = \frac{9}{4} v_x^{2/3} \tag{3.107c}$$

$$^{III}\Phi_{4,2}^x[v] = -\frac{3}{2}\frac{v_x^{5/3} v_{xxx}}{v_{xx}^3} + \frac{1}{2}\frac{v_x^{2/3}}{v_{xx}} \tag{3.107d}$$

$$^{III}\Phi_{4,3}^t[v] = 3 v_x^{1/3} \tag{3.107e}$$

$$^{III}\Phi_{4,3}^x[v] = -\frac{v_x^{4/3} v_{xxx}}{v_{xx}^3} + \frac{2}{3}\frac{v_x^{1/3}}{v_{xx}} \tag{3.107f}$$

$$^{III}\Phi_{4,4}^t[v] = \frac{1}{64} v_x^2 \tag{3.107g}$$

$$^{III}\Phi_{4,4}^x[v] = -\frac{1}{32}\frac{v_x^3 v_{xxx}}{v_{xx}^3} - \frac{1}{32}\frac{v_x^2}{v_{xx}} + \frac{5}{72} v. \tag{3.107h}$$

This leads to

Potentialisation III.2

a) Using the conserved current ${}^{III}\Phi^t_{4,1}$, given by (3.107a), we find that equation (3.100) viz.

$$\boxed{v_t = \frac{v_x^2 v_{xxx}}{v_{xx}^3} - \frac{2}{9}x},$$

admits the zero-order potentialisation

$$\tilde{v}_{1,t} = \tilde{v}_{1,x}^3 \tilde{v}_{1,xxx}, \tag{3.108}$$

where

$$\tilde{v}_{1,x} = \frac{v_x^{2/3}}{v_{xx}}, \tag{3.109}$$

and the first-order potentialisation with $\tilde{w}_{1,x} = D_x({}^{III}\Phi^t_{4,1})$, i.e. $\tilde{v}_{1,x} = \tilde{w}_1$, namely

$$\tilde{w}_{1,t} = \tilde{w}_1^3 \tilde{w}_{1,xxx} + 3\tilde{w}_1^2 \tilde{w}_{1,x}\tilde{w}_{1,xx}. \tag{3.110}$$

Equation (3.108) also admits the zero-order potentialisation

$$\tilde{w}_{3,t} = \frac{\tilde{w}_{3,xxx}}{\tilde{w}_{3,x}^3} - 3\frac{\tilde{w}_{3,xx}^2}{\tilde{w}_{3,x}^4}, \tag{3.111}$$

where

$$\tilde{w}_{3,x} = \frac{1}{\tilde{v}_{1,x}} \tag{3.112}$$

which corresponds to the integrating factor $\Lambda[\tilde{v}_1] = -2\tilde{v}_{1,x}^{-3} v_{1,xx}$ admitted by equation (3.108).

b) Using the conserved current ${}^{III}\Phi^t_{4,2}$, given by (3.107c), we find that equation (3.100) viz.

$$\boxed{v_t = \frac{v_x^2 v_{xxx}}{v_{xx}^3} - \frac{2}{9}x},$$

admits the zero-order potentialisation

$$\tilde{v}_{2,t} = \frac{\tilde{v}_{2,x}^{3/2}\tilde{v}_{2,xxx}}{\tilde{v}_{2,xx}^3} \tag{3.113}$$

where

$$\tilde{v}_{2,x} = \frac{9}{4} v_x^{2/3}, \tag{3.114}$$

and the first-order potentialisation with $\tilde{w}_{2,x} = D_x({}^{III}\Phi^t_{4,2})$, i.e. $\tilde{v}_{2,x} = \tilde{w}_2$, namely

$$\tilde{w}_{2,t} = \frac{\tilde{w}_2^{3/2}\tilde{w}_{2,xxx}}{\tilde{w}_{2,x}^3} - 3\frac{\tilde{w}_2^{3/2}\tilde{w}_{2,xx}^2}{\tilde{w}_{2,x}^4} + \frac{3}{2}\frac{\tilde{w}_2^{1/2}\tilde{w}_{2,xx}}{\tilde{w}_{2,x}^2}. \tag{3.115}$$

c) Using the conserved current ${}^{III}\Phi^t_{4,3}$*, given by (3.107e), we find that equation (3.100) viz.*

$$\boxed{v_t = \frac{v_x^2 v_{xxx}}{v_{xx}^3} - \frac{2}{9}x},$$

admits the zero-order potentialisation

$$\tilde{v}_{3,t} = \frac{\tilde{v}_{3,xxx}}{\tilde{v}^3_{3,xx}}, \tag{3.116}$$

where

$$\tilde{v}_{3,x} = 3v_x^{1/3} \tag{3.117}$$

and the first-order potentialisation with $\tilde{w}_{3,x} = D_x({}^{III}\Phi^t_{4,3})$*, i.e.* $\tilde{v}_{3,x} = \tilde{w}_3$*, namely*

$$\tilde{w}_{3,t} = \frac{\tilde{w}_{3,xxx}}{\tilde{w}^3_{3,x}} - 3\frac{\tilde{w}^2_{3,xx}}{\tilde{w}^4_{3,x}},$$

which is equivalent to (3.111). Equation (3.100) also admits the second-order potentialisation with $W_{3,x} = D_x^2({}^{III}\Phi^t_{4,3})$*, i.e.* $\tilde{w}_{3,x} = W_3$*, namely*

$$W_{3,t} = \frac{W_{3,xxx}}{W_3^3} - 9\frac{W_{3,x}W_{3,xx}}{W_3^4} + 12\frac{W^3_{3,x}}{W_3^5}. \tag{3.118}$$

d) Using the conserved current ${}^{III}\Phi^t_{4,4}$*, given by (3.107g), we find that equation (3.100) viz.*

$$\boxed{v_t = \frac{v_x^2 v_{xxx}}{v_{xx}^3} - \frac{2}{9}x},$$

admits no zero-order potentialisation. Instead (3.100) admits the first-order potentialisation with $\tilde{w}_{4,x} = D_x({}^{III}\Phi^t_{4,4})$*, i.e.*

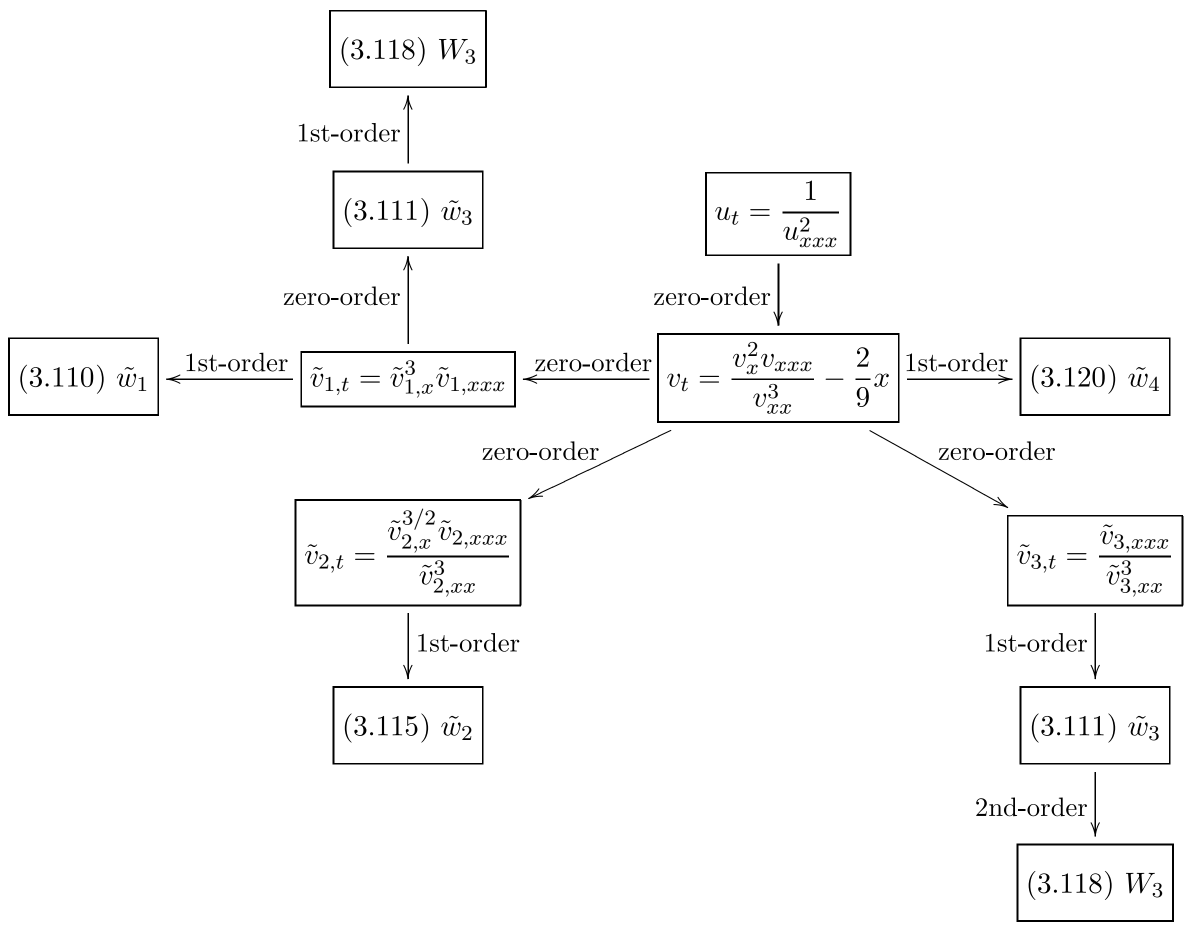
$$\tilde{w}_4 = \frac{1}{64}v_x^2 \tag{3.119}$$

namely

$$\tilde{w}_{4,t} = -\frac{\tilde{w}_4^{5/2}\tilde{w}_{4,xxx}}{\tilde{w}^3_{4,x}} + 3\frac{\tilde{w}_4^{5/2}\tilde{w}^2_{4,xx}}{\tilde{w}^4_{4,x}} - \frac{5}{2}\frac{\tilde{w}_4^{3/2}\tilde{w}_{4,xx}}{\tilde{w}^2_{4,x}} + \frac{5}{9}w_4^{1/2}. \tag{3.120}$$

A graphical description of the above is given in Diagram 4.

Diagram 4



Case IV: We consider (1.7), viz.

$$u_t = \frac{4u_x^5}{(2b\,u_x^2 - 2u_x u_{xxx} + 3u_{xx}^2)^2},$$

where b is an arbitrary constant. Equation (1.7) is Möbius-invariant [3] and can therefore conveniently be expressed in terms of the Schwarzian derivative S, namely

$$u_t = \frac{u_x}{(b-S)^2}, \tag{3.121}$$

where S is given by (1.8). Equation (3.121) admits the following fourth-order integrating factors for arbitrary constant b (no zero-order or second-order integrating factors exist):

$${}^{IV}\Lambda_1[u] = \frac{S_x}{u_x} \tag{3.122a}$$

$$ {}^{IV}\Lambda_2[u] = \left(\frac{u_{xx}^2}{u_x^4} + \frac{2b}{u_x^2}\right) S_x + \frac{2u_{xx}}{u_x^3} S^2 - \frac{4bu_{xx}}{u_x^3} S + \frac{2b^2 u_{xx}}{u_x^3}. \tag{3.122b} $$

For the case where $b = 0$, equation

$$ u_t = \frac{u_x}{S^2}, \tag{3.123} $$

admits, in addition to ${}^{IV}\Lambda_1[u]_{b=0}$ and ${}^{IV}\Lambda_2[u]_{b=0}$ given (3.122a) and (3.122b) respectively, also the integrating factor

$$ {}^{IV}\Lambda_3[u] = \left(\frac{uu_{xx}^2}{u_x^4} - \frac{2u_{xx}}{u_x^2}\right) S_x + \left(\frac{2uu_{xx}}{u_x^3} - \frac{2}{u_x}\right) S^2. \tag{3.124} $$

Corresponding to (3.122a) and (3.122b), the respective conserved current and flux for (3.121) are as follows:

$$ {}^{IV}\Phi_1^t[u] = \frac{1}{4}\frac{u_{xx}^2}{u_x^2} \tag{3.125a} $$

$$ {}^{IV}\Phi_1^x[u] = -\frac{1}{4}\frac{u_{xx}^2}{(S-b)^2 u_x^2} + \frac{u_{xx} S_x}{(S-b)^3 u_x} + \frac{1}{2}\frac{2S-b}{(S-b)^2} \tag{3.125b} $$

$$ {}^{IV}\Phi_2^t[u] = \frac{bu_{xx}^2}{u_x^3} + \frac{1}{12}\frac{u_{xx}^4}{u_x^5} - \frac{b^2}{u_x} \tag{3.125c} $$

$$ {}^{IV}\Phi_2^x[u] = -\frac{1}{12}\frac{u_{xx}^4}{u_x^5(S-b)^2} + \frac{2}{3}\frac{u_{xx}^3 S_x}{u_x^4 (S-b)^3} + \frac{u_{xx}^2(2S-3b)}{u_x^3(S-b)^2} + \frac{4bu_{xx}S_x}{u_x^2(S-b)^3} + \frac{(2S-b)^2}{u_x(S-b)^2}. \tag{3.125d} $$

For the case $b = 0$ corresponding to the integrating factor (3.124), we obtain an additional conserved current and flux for equation (3.123), namely

$$ {}^{IV}\Phi_3^t[u] = \frac{1}{12}\frac{uu_{xx}^4}{u_x^5} - \frac{1}{3}\frac{u_{xx}^3}{u_x^3} \tag{3.126a} $$

$$ {}^{IV}\Phi_3^x[u] = -\frac{1}{12}\frac{uu_{xx}^4}{u_x^5}\frac{1}{S^2} + \frac{1}{3}\frac{u_{xx}^3}{u_x^4}\frac{2uS_x + u_x S}{S^3} - \frac{2u_{xx}^2}{u_x^3}\frac{u_x S_x - uS^2}{S^3} - \frac{4u_{xx}}{u_x}\frac{1}{S} + \frac{4u}{u_x}. \tag{3.126b} $$

This leads to

Potentialisation IV *Using the conserved current* ${}^{IV}\Phi_1^t$*, given by (3.125a), we find that equation (3.121) viz.*

$$ \boxed{u_t = \frac{u_x}{(b-S)^2}}, $$

admits, for arbitrary constant b, the zero-order potentialisation

$$\tilde{v}_t = \frac{1}{\left(-v_{xx} + b v_x^{1/2} + v_x^{3/2}\right)^3} \left[2 v_x^{3/2} v_{xxx} - \frac{3}{2} v_x (6v_x + b) v_{xx} + \frac{1}{2}(6v_x + b)(2v_x + b) v_x^{3/2}\right], \quad (3.127)$$

where

$$v_x = \frac{1}{4} \frac{u_{xx}^2}{u_x^2} \quad (3.128)$$

and the first-order potentialisation with $w_x = D_x({}^{IV}\Phi_1^t)$, i.e. $v_x = w$, namely

$$w_t = \frac{1}{\left(-w_x + b w^{1/2} + 2w^{3/2}\right)^4} \Big[-2w^{3/2} w_x w_{xxx} + 2w^2(2w + b) w_{xxx} + 6w^{3/2} w_{xx}^2 - 3w^{1/2} w_x^2 w_{xx} - 3w(10w + b) w_x w_{xx} + \frac{3}{2}(12w + b) w_x^3 + 4w^{3/2}(6w - b) w_x^2 - w^2(2w + b)(6w - b) w_x \Big]. \quad (3.129)$$

No further potentialisations of any order are possible for equation (3.121) or equation (3.123) using the conserved currents ${}^{IV}\Phi_2^t$ and ${}^{IV}\Phi_3^t$ given by (3.125c) and (3.126a), respectively.

4 Concluding remarks

In this article we are reporting all potentialisations of the class of fully-nonlinear symmetry-integrable equatons listed in Proposition 1 using their integrating factors up to order four, whereby the integrating factors do not depend explicitly on their independent variables. Several mappings to equations using the multi-potentialisations process are also given where possible, although we do not claim to have obtained here a complete list of all possible multi-potentialisations connected to the equations in Proposition 1.

Our results show that the class of equations in Proposition 1 have a rich structure and the systematic use of the zero and higher-order potentialisations, as introduced here, leads to interesting quasi-linear equations, some of which certainly deserve further attention.

We should point out that in all our previous classifications of evolution equations of order three and order five where we have introduced potentialisations were based only on zero-order potentialisations. However, in the current work we have shown that there are many cases where an equation does not admit a zero-order potentialisation but that it can instead admit first and higher-order potentialisations. This means that some of those earlier classifications could possibly be extended by considering higher-order potentialisations. We will address this in the near future.

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