Open Communications in Nonlinear Mathematical Physics]ocnmp[Special Issue 2, 2024 pp 28–45

Proceedings of the OCNMP-2024 Conference:
Bad Ems, 23-29 June 2024

Two-fold degeneracy of a class of rational Painlevé V solutions

H. Aratyn[1], *J.F. Gomes*[2], *G.V. Lobo*[2] *and A.H. Zimerman*[2]

[1] *Department of Physics, University of Illinois at Chicago, 845 W. Taylor St. Chicago, Illinois 60607-7059, USA*

[2] *Instituto de Física Teórica-UNESP, Rua Dr Bento Teobaldo Ferraz 271, Bloco II, 01140-070 São Paulo, Brazil*

Received March 8, 2024; Accepted April 11, 2024

Abstract

We present a construction of a class of rational solutions of the Painlevé V Hamilton equations that exhibit a two-fold degeneracy, meaning that there exist two distinct solutions that share identical parameters.

The fundamental object of our study is the orbit of translation operators of the $A_3^{(1)}$ affine Weyl group acting on the underlying seed solution that only allows action of some symmetry operations. By linking points on this orbit to rational solutions, we establish conditions for such degeneracy to occur after involving in the construction additional Bäcklund transformations that are inexpressible as translation operators. This approach enables us to derive explicit expressions for these degenerate solutions. An advantage of this formalism is that it easily allows generalization to higher Painlevé systems associated with dressing chains of even period $N > 4$.

1 Introduction

Painlevé equations are second order nonlinear differential equations with solutions without any movable critical singularities in the complex plane, a property referred to as Painlevé property (see e.g. [5]). These solutions are generally not solvable in terms of elementary functions however for special values of the underlying parameters the Painlevé equations possess rational and hypergeometric-type of solutions.

Although the discovery of Painlevé equations has its origin in, mathematically motivated, search for equations satisfying the Painlevé property, these equations and their solutions found many practical applications and play an important role in several branches of mathematical physics, algebraic geometry, applied mathematics,fluid dynamics and statistical mechanics. A list of the areas where the Painlevé equations found their applications includes correlation functions of the Ising model, random matrix theory, plasma physics, asymptotics of nonlinear partial differential equations, quantum cohomology, conformal field theory, general relativity, nonlinear and fiber optics, Bose-Einstein condensation [5, 10]. Special solutions, such as rational solutions, turned out to play key role in these applications and various methods were applied in their study.

This project is dedicated to the study of rational solutions of Painlevé V equation by presenting an approach that finds conditions for existence of degeneracy of these solutions, derives systematically their form and also explains in a fundamental way the origin of degenracy in the setting of Painlevé V Hamiltonian formalism.

Painlevé V equation is invariant under the extended affine Weyl group $A_3^{(1)}$ of Bäcklund transformations [7]. A central object of our study is a commutative subgroup of translation operators of $A_3^{(1)}$ and an orbit formed by their actions on two different types of seed solutions, one being invariant under an internal automorphism π of $A_3^{(1)}$.

In a recent paper [1], we have shown how by acting with translation operators on a seed solution, which is invariant under automorphism π, one obtains Umemura polynomials for Painlevé V equation and their relevant recurrence relations [8]. For the other remaining seed solution we have shown that only actions by selected translation operators are allowed while the remaining translation operators produce divergencies.

The presence of degeneracy for a family of rational solutions of Painlevé V equation was recently pointed out in [4], which also presented an explicit construction of special function solutions in terms of the generalized Laguerre polynomials.

The novelty of our contribution is that here we link the origin of degeneracy of rational solutions to existence of divergencies resulting from actions of various translation operators and Bäcklund transformations on the underlying seed solution and use it to explicitly construct the two-fold degenerated solutions of the Painlevé V Hamilton equations (5) and resulting degeneracy of the Painlevé V equation (6) and to find the underlying consistency relations that dictate values of the parameters of degenerated solutions (see also a preprint [2] for initial study of such approach). The degeneracy of rational solutions of Painlevé V equation (6) can be linked to its invariance under the simultaneous $\gamma \to -\gamma, x \to -x$ transformation. However on the level of Hamiltonian formalism with two canonical variables which are not both transfroming trivially under $\gamma \to -\gamma, x \to -x$ we find that the right framework is provided by the method that employs the translation operators and their orbits presented in this paper. In addition this formalism lends itself to be applied to study of degeneracy for higher Painlevé systems with the $A_{2k+1}^{(1)}, k > 1$ affine Weyl symmetry group as understood on basis of their connections with higher dressing chains of even periodicity [1].

In section 2, we present the Hamiltonian approach to Painlevé V equation and discuss the construction of rational solutions by actions of translation operators. We describe solutions formed out by actions with $T_2^{-n_2}$ and $T_4^{n_4}$ translation operators, with $n_i, i = 2, 4$ being positive integers, on the seed solution :

$$|q = z,\, p = 0\rangle_{\alpha_{\mathsf{a}}}\,, \tag{1}$$

that describes a solution of Hamilton equations (5) with values of q, p being $q = z,\, p = 0$

and an arbitrary parameter a equal to α_1 and with zero parameters α_2 and α_3. We find the recurrence relation that allows finding explicitly the solutions derived from (1) and obtain a close expression for their parameters in terms of a and integers $n_i, i = 2, 4$.

In section 3, we explain a reason for existence of degenerated solutions in the Hamiltonian formalism due to infinities associated with actions of some Bäcklund transformations on the seed solution (1) and use this observation to find the class of parameters that are being shared by a pair of different in form solutions. We will show that degeneracy occurs for some rational solutions derived from (1) for the parameter a that happens to be an even integer. We propose an explicit construction of such solutions for a Bäcklund transformation M such that infinity is generated if we are to set two sides of inequality

$$M\mathbb{T}(n_2, n_4; \mathsf{a}) \neq \mathbb{T}(m_2, m_4; \mathsf{b}), \; n_i, m_i \in \mathbb{Z}_+, \; i = 2, 4, \tag{2}$$

to be equal. This potential divergence is the cause of degeneracy. In relation (2), the notation is such that $\mathbb{T}(n_2, n_4; \mathsf{a}) = T_2^{-n_2} T_4^{n_4} | q = z, \, p = 0\rangle_{\alpha_\mathsf{a}}$ is a solution linked to the orbit of the seed solution (1) under actions of T_2 and T_4 operators. To be responsible for degeneracy the Bäcklund transformation M must be such that it satisfies two conditions. First that it will cause the divergence, as described in equation (20), and secondly that the equation

$$M\left(\alpha_{n;\mathsf{a}}\right) = \alpha_{m;\mathsf{b}}\,, \tag{3}$$

with the symbol $\alpha_{n;\mathsf{a}} = (\mathsf{a} + 2n_2, \, -2n_2, \, -2n_4, \, 2 - \mathsf{a} + 2n_4)$ (see relation (14)), will have a solution for some values of the parameters $n_i, m_i, i = 2, 4$ and a, b ensuring that both sides of inequality (2) will share the same parameter. These two conditions are shown to be satisfied for M being one of the Bäcklund transformations $M_{12} = s_1 s_2$, $M_{34} = s_3 s_4$, $M_1 = \pi s_1$, $M_4 = \pi^{-1} s_4$ and we call the corresponding set of degenerated solutions an M_i-sequence. One of the main points of this paper is that all these four sequences are equivalent. Specifically, the sequences M_1, M_{12} and M_4 are mapped into each other by Bäcklund transformations, while $M_{3,4}$ happens to be equivalent to M_1 after a simple redefinitions of underlying parameters as discussed in subsections 3.1 - 3.3. The equivalence of these sequences is a new result not contained in unpublished reference [2].

The final section 4, offers conclusions and discussion of the results. This section reviews the results shown in Examples 3.1, 3.2 and 3.4 to obtain an unifying discussion for special values of parameters labeling the degenerated solutions of the Hamilton Painlevé V equations. We find that the condition for a solution constructed in section 2 to be equal to one of the degenerated solutions is that the underlying parameter a of the seed solution is an even integer. We also remark that the fact that the discussion of degeneracy of Painlevé systems is here placed firmly in the setting of the extended affine Weyl group $A^{(1)}_{N-1}, N = 4$ lends itself naturally to being generalized to Painlevé systems associated with higher dressing chains of even period $N > 4$, where more richer degeneracy structure is expected to appear.

2 Background

We will mainly be working with the Hamiltonian approach to Painlevé V equation with the Hamilton:

$$H = -q\,(q - z)\,p\,(p - z) + (1 - \alpha_1 - \alpha_3)\,pq + \alpha_1 zp - \alpha_2 zq\,, \tag{4}$$

where $\alpha_i, i = 1, 2, 3$ are three constant parameters and q, p are two canonical variables that satisfy Hamilton equations: $zq_z = dH/dp$, $zp_z = -dH/dq$:

$$\begin{aligned} zq_z &= -q(q-z)(2p-z) + (1-\alpha_1-\alpha_3)q + \alpha_1 z\,, \\ zp_z &= p(p-z)(2q-z) - (1-\alpha_1-\alpha_3)p + \alpha_2 z\,, \end{aligned} \tag{5}$$

from which one derives Painlevé V equation

$$y_{xx} = -\frac{y_x}{x} + \left(\frac{1}{2y} + \frac{1}{y-1}\right) y_x^2 + \frac{(y-1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma}{x} y + \delta \frac{y(y+1)}{y-1}\,, \tag{6}$$

by eliminating one of the canonical variables and defining $y = (q/z)(q/z-1)^{-1}$, as well as redefining the variable $z \to x$ with $x = \epsilon z^2/2$. The coefficients α, β, γ of the Painlevé V equation are given by:

$$\alpha = \frac{1}{8}\alpha_3^2, \quad \beta = -\frac{1}{8}\alpha_1^2, \quad \gamma = \frac{\alpha_2 - \alpha_4}{2\epsilon}, \quad \delta = -\frac{1}{2}\frac{1}{\epsilon^2}\,, \tag{7}$$

in terms of components $\alpha_i = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ with $\alpha_4 = 2 - \sum_{i=1}^{3} \alpha_i$.

For δ to take a conventional value of $-\frac{1}{2}$ we need $\epsilon^2 = 1$.

The Hamilton equations are directly connected to symmetric Painlevé V equations:

$$z\frac{df_i}{dz} = f_i f_{i+2}\left(f_{i+1} - f_{i-1}\right) + \left(1 - \alpha_{i+2}\right) f_i + \alpha_i f_i\,, \quad f_{i+4} = f_i, \quad i = 1, 2, 3, 4\,,$$

via relations $f_1 = q$, $f_2 = p$, $f_3 = z - q$, $f_4 = z - p$. Since our formalism will be shown to describe degeneracy of Painlevé V Hamilton equations (5) it will also automatically provide such description for the symmetric Painlevé V equations as well as equation (6).

The Hamilton equations are invariant under Bäcklund transformations, $\pi, s_i, i = 1, \ldots, 4$ that satisfy the $A_3^{(1)}$ extended affine Weyl group relations:

$$\begin{aligned} &s_i^2 = 1, \qquad s_i s_j = s_j s_i \ (j \neq i, i \pm 1), \qquad s_i s_j s_i = s_j s_i s_j \ (j = i \pm 1), \\ &\pi^4 = 1, \qquad \pi s_j = s_{j+1}\pi, \quad s_{i+4} = s_i. \end{aligned} \tag{8}$$

An explicit form of these transformations on canonical variables p and q is shown in Table 1, Imposing the periodicity condition $\alpha_{i+4} = \alpha_i$ we can compactly describe the action of the Bäcklund transformations on the constant parameters α_i from equations (5) as :

$$s_i(\alpha_i) = -\alpha_i, \quad s_i(\alpha_{i\pm1}) = \alpha_i + \alpha_{i\pm1}, \quad s_i(\alpha_{i+2}) = \alpha_{i+2}, \quad i = 1, 2, 3, 4\,. \tag{9}$$

Furthermore the automorphism π acts according to

$$\pi(\alpha_i) = \alpha_{i-1}\,. \tag{10}$$

Within the $A_3^{(1)}$ extended affine Weyl group one defines an abelian subgroup of translation operators defined as $T_i = r_{i+3} r_{i+2} r_{i+1} r_i, i = 1, 2, 3, 4$, where $r_i = r_{4+i} = s_i$ for $i = 1, 2, 3$ and $r_4 = \pi$. The translation operators commute among themselves, $T_i T_j = T_j T_i$, and as follows from relations (9) and (10) generate the following translations when acting on the α_i parameters:

$$T_i(\alpha_i) = \alpha_i + 2,\ T_i(\alpha_{i-1}) = \alpha_{i-1} - 2,\ T_i(\alpha_j) = \alpha_j,\ j = i+1, j = i+2\,.$$

	q	p	α_1	α_2	α_3	α_4
s_1	q	$p+\frac{\alpha_1}{q}$	$-\alpha_1$	$\alpha_1+\alpha_2$	α_3	$\alpha_1+\alpha_4$
s_2	$q-\frac{\alpha_2}{p}$	p	$\alpha_1+\alpha_2$	$-\alpha_2$	$\alpha_2+\alpha_3$	α_4
s_3	q	$p-\frac{\alpha_3}{z-q}$	α_1	$\alpha_2+\alpha_3$	$-\alpha_3$	$\alpha_3+\alpha_4$
s_4	$q+\frac{\alpha_4}{z-p}$	p	$\alpha_1+\alpha_4$	α_2	$\alpha_3+\alpha_4$	$-\alpha_4$
π	$z-p$	q	α_4	α_1	α_2	α_3

Table 1. $A_3^{(1)}$ Bäcklund transformations

The translation operators satisfy the following commutation relations

$$s_i T_i s_i = T_{i+1},\;\; s_i T_j s_i = T_j,\; j \neq i, i+1,\;\; \pi\, T_i = T_{i+1}\,\pi\,, \tag{11}$$

with the Bäcklund transformations s_i, $i=1,2,3,4$ and an automorphism π and the usual periodicity condition $T_{i+4} = T_i$ being imposed.

The reference [1] described construction of rational solutions of Painlevé V equation out of actions of translation operators on seed solutions that first appeared in [9]. Crucial for this construction is that rational solutions fall into two classes depending on which of the two types of seed solutions they have been derived from by actions of translation operators. These two classes of seed solutions are:

1. $q = z/2$, $p = z/2$, with the parameter $\alpha = (\mathsf{a}, 1-\mathsf{a}, \mathsf{a}, 1-\mathsf{a})$,
2. $q = z$, $p = 0$, with the parameter $\alpha_{\mathsf{a}} = (\mathsf{a}, 0, 0, 2-\mathsf{a})$ denoted here by $|q = z,\, p = 0\rangle_{\alpha_{\mathsf{a}}}$.

They both solve the Hamilton equations (5) for an arbitrary variable a. As shown in [1], the first class of seed solutions gives rise to Umemura polynomials and the second to special functions. It was also shown there that the solutions constructed with this procedure satisfy all sufficient and necessary conditions for the parameters of rational solutions of Painlevé V equation first derived in [6]. The action of the Bäcklund transformation s_i on the seed solution (1) is :

$$\big|q = z,\, p = 0\big\rangle_{\alpha_{\mathsf{a}}} \xrightarrow{s_i} \big|s_i(q = z),\, s_i(p = 0)\big\rangle_{s_i(\alpha_{\mathsf{a}})}\,,$$

and similarly for all the other Bäcklund transformations.

Acting repeatedly with the π automorphism on the seed solution (1) produces three other variants of such solution. They all serve as seed solutions in analogous way to the solution (1). Here we will limit our discussion only to the seed solution (1) and solutions generated from it as the other solutions and the corresponding structure of degeneracy follow from the same formalism under appropriate actions of π.

The Bäcklund transformations s_2, s_3 generate infinity when applied on the solution (1) and accordingly only actions by some powers of T_1, T_2, T_4 are well defined on a seed solution $|q = z,\, p = 0\rangle_{\alpha_{\mathsf{a}}}$. The allowed operations are as follows [1]:

$$T_1^{n_1} T_2^{-n_2} T_4^{n_4} |q = z,\, p = 0\rangle_{\alpha_{\mathsf{a}}},\; n_1 \in \mathbb{Z},\; n_2, n_4 \in \mathbb{Z}_+ .$$

This operation is to be understood as producing new solutions q and p of the Hamilton equations equal to $T_1^{n_1} T_2^{-n_2} T_4^{n_4}(q = z)$ and $T_1^{n_1} T_2^{-n_2} T_4^{n_4}(p = 0)$ and with a new parameter:

$$T_1^{n_1} T_2^{-n_2} T_4^{n_4}(\alpha_{\mathsf{a}}) = (\mathsf{a} + 2n_1 + 2n_2,\, -2n_2,\, -2n_4,\, 2 - \mathsf{a} + 2n_4 - 2n_1) . \tag{12}$$

Evidently, the action of $T_1^{n_1}$ only amounts to shifting a parameter a and as shown in [1] leaves the configuration $q = z, p = 0$ unchanged. Thus :

$$T_1^{n_1} |q = z,\, p = 0\rangle_{\alpha_{\mathsf{a}}} = |q = z,\, p = 0\rangle_{\alpha_{\mathsf{a}+2n_1}} . \tag{13}$$

We can therefore, largely, ignore T_1 and restrict our discussion to the solutions of Painlevé V equation of the form :

$$\begin{aligned} &\mathbb{T}(n_2, n_4; \mathsf{a}) = T_2^{-n_2} T_4^{n_4} |q = z,\, p = 0\rangle_{\alpha_{\mathsf{a}}},\; n_2, n_4 \in \mathbb{Z}_+ , \\ &\qquad \alpha_{n;\mathsf{a}} = T_2^{-n_2} T_4^{n_4}(\alpha_{\mathsf{a}}) = (\mathsf{a} + 2n_2,\, -2n_2,\, -2n_4,\, 2 - \mathsf{a} + 2n_4) , \end{aligned} \tag{14}$$

where we listed both the solution generated by translation operators and its corresponding parameter $\alpha_{n,\mathsf{a}}$. $\mathbb{Z}_+$ contains positive integers and zero.

To describe solutions $\mathbb{T}(n_2, n_4; \mathsf{a})$ we will first set $n_4 = 0$ and recall expressions for an action by T_2^{-n} [1]:

$$\begin{aligned} &T_2^{-1} : \big|q = z,\, p = 0\rangle_{\alpha_{\mathsf{a}}} \to |q = z, p = \frac{2z}{\mathsf{a} - z^2}\big\rangle_{(2+\mathsf{a},-2,0,2-\mathsf{a})} \\ &T_2^{-n} : \big|q = z,\, p = 0\rangle_{\alpha_{\mathsf{a}}} \to |q_n = z, p_n = \frac{2nzR_{n-1}(x,\mathsf{a})}{R_n(x,\mathsf{a})}\big\rangle_{(\mathsf{a}+2n,-2n,0,2-\mathsf{a})} , \end{aligned} \tag{15}$$

where $x = -z^2/2$ and $R_n(x, \mathsf{a})$ are Kummer polynomials that satisfy the recurrence relations:

$$2kR_{k-1}(x, \mathsf{a}) = R_k(x, \mathsf{a}) - R_k(x, \mathsf{a} - 2) = \frac{dR_k(x, \mathsf{a})}{dx} , \tag{16}$$

$$R_{k+1}(x, \mathsf{a}) = 2xR_k(x, \mathsf{a}) + \mathsf{a}R_k(x, \mathsf{a} + 2) , \tag{17}$$

for $k = 0, 1, 2, \ldots$ with $R_0(x, \mathsf{a}) = 1$ (see e.g. [2, 3]).

The result for $T_2^{-n_2}|q = z,\, p = 0\rangle_{\alpha_{\mathsf{a}}}$ is obtained by inserting $n = n_2$ into equation (15).

The further action with $T_4^{n_4}$ utilizes expression

$$\begin{aligned} &T_4(q) = z - p - (\alpha_1 + \alpha_4)/(q + \alpha_4/(z - p)) , \\ &T_4(p) = q + \alpha_4/(z - p) - (\alpha_1 + \alpha_2 + \alpha_4)/(p + (\alpha_1 + \alpha_4)/(q + \alpha_4/(z - p))) , \end{aligned} \tag{18}$$

describing action of the translation operator T_4 on a solution q, p of the Hamilton equations (5) with $\alpha_i = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. The recurrence relations obtained from expression (18) are:

$$\begin{aligned} &q^{(k)} = T_4^k(q_0) = z - p^{(k-1)} - \frac{2(k + n_2)}{v_k} = z - u_k . \\ &p^{(k)} = T_4^k(p_0) = v_k - \frac{2k}{u_k} . \\ &\alpha^{(k)} = (\mathsf{a} + 2n_2,\, -2n_2,\, -2k,\, 2 - \mathsf{a} + 2k), \quad k = 1, 2, \ldots, n_4 , \end{aligned} \tag{19}$$

where

$$v_k = q^{(k-1)} + \frac{2k - \mathsf{a}}{z - p^{(k-1)}}, \quad u_k = p^{(k-1)} + \frac{2(k+n_2)}{v_k},$$

and $q_0 = z$ and $p_0 = \frac{2nzR_{n_2-1}(x,\mathsf{a})}{R_{n_2}(x,\mathsf{a})}$. Setting $k = n_4$ into $\alpha^{(k)}$ we recover $\alpha_{n;\mathsf{a}}$ from expression (14). The closed expressions for $q^{(k)}, p^{(k)}$ will be described in the future publication [3].

In the next section we will derive the parameters of degenerated solutions (see e.g. (22)) and compare with the above value of the parameter $\alpha_{n;\mathsf{a}}$ on the orbit of $T_2^{-n_2}T_4^{n_4}$. In section 4 we will find that for any a that is an even integer the parameter $\alpha_{n;\mathsf{a}}$ can be cast in a form of a parameter of degenerated pair of solutions.

3 Degeneracy

The above construction of solutions in section 2 did not take into account existence of any other Bäcklund transformations than translation operators. The Bäcklund transformations that are not expressible in terms of translation operators will play a role in what follows. Our construction associates the (two-fold) degeneracy to inequality (2) with two sides that are two different (finite) solutions of Painlevé V Hamilton equations that share a common Painlevé V parameter (3).

In relations (2) and (3) the symbol M denotes a Bäcklund transformation, which is not expressible in terms of translation operators only and such that $T_2^{m_2}T_4^{-m_4}M\mathbb{T}(n_2, n_4; \mathsf{a})$ is ill-defined as we will see below. For that reason the two solutions listed in (2) can not be equal. We will refer to degenerated solutions of relations (2) and (3) as M-sequence.

To determine general conditions for degeneracy let us equate for the moment expressions on the left and the right sides of the inequality (2) with each other and multiply both sides with $T_2^{m_2}T_4^{-m_4}$ to get:

$$|q = z, p = 0\rangle_{\alpha_\mathsf{b}} = T_2^{m_2}T_4^{-m_4}MT_2^{-n_2}T_4^{n_4}\,|q = z, p = 0\rangle_{\alpha_\mathsf{a}} = M\,T_3^{c_3}\,T_2^{c_2}\,T_4^{c_4}|q = z, p = 0\rangle_{\alpha_\mathsf{a}}$$

obtained after commuting $T_2^{m_2}T_4^{-m_4}$ around M and ignoring potential presence of T_1 on the right hand side since it only amounts to shifting of a. The conditions for degeneracy in this setting are

$$c_3 \neq 0, \text{ or } c_2 > 0\,, \text{ or } c_4 < 0\,, \tag{20}$$

since they correspond to presence of operators that will cause divergence when acting on $|q = z, p = 0\rangle_{\alpha_\mathsf{a}}$. We next explore several candidates for M to see if they satisfy the conditions (3) and (20).

We can easily discard $M = s_2, M = s_3$ as they do not satisfy the condition (3), as it would require $m_2 = -n_2$ for s_2 and $m_4 = -n_4$ for s_3. Further, one finds that $M = s_1, M = s_4$ do not produce infinities and accordingly fail to satisfy the conditions of relation (20).

Moving on to the quadratic expressions of the type s_is_j we find that when $j \neq i+1$ (e.g. s_1s_3 or s_2s_4) then both expressions do not satisfy the condition (3). The remaining cases are of the type s_is_{i+1} since s_is_{i-1} can be moved from the left to the right hand side of relation (2) to become s_is_{i+1}. Inspection of $s_1s_2, s_2s_3, s_3s_4, s_4s_1$ shows that only

1. $M_{12} = s_1 s_2$,
2. $M_{34} = s_3 s_4$,

satisfy the condition (3) and the condition (20) for some values of m_i, $i = 2, 4$. These conditions are also satisfied by

3. $M_1 = \pi s_1$,
4. $M_4 = \pi^{-1} s_4$,

that are effectively equivalent to the cases of $M = \pi, \pi^{-1}$ [2]. It is also easy to see that M_1 and M_4 are not invertible in the context of relation (2) since $M_i^{-1}, i = 1, 4$ acting on $\mathbb{T}(m_2, m_4; \mathsf{b})$ will cause a divergence. Thus if an equality between two solutions shown in (2) held for M_1 or M_4 then an attempt to invert M_1 or M_4 would have produced an infinity.

It suffices to consider operators M that consist of a single s_i multiplied by π or a product of two s_i's due to the following identities :

$$
\begin{aligned}
s_i s_{i+1} &= \pi s_{i+2} T_{i+2}^{-1} = \pi T_{i+3}^{-1} s_{i+2}, \quad i = 1, 2, 3, 4\,, \\
s_{i+1} s_i &= \pi^{-1} s_{i-1} T_i = \pi^{-1} T_{i-1} s_{i-1}, \quad i = 1, 2, 3, 4\,,
\end{aligned} \tag{21}
$$

for products of neighboring s_i that reduce them to one single s_i multiplied by a shift operator and an automorphism π. Accordingly, in principle, the higher products of s_i can be reduced to the lower number of s_i transformations [2].

We will now examine if there exists equivalence between the four cases with degeneracy represented by M_1, M_4, M_{12}, M_{34}. We choose as a starting point the relation (3) with $M = M_1 = \pi s_1$ and accordingly with the parameter :

$$
\pi s_1 \left(\alpha_{n;\mathsf{a}}\right) = \alpha_{m;\mathsf{b}} = 2(1 + n_2 + n_4, -m_2, m_2 - n_2, -n_4)\,, \tag{22}
$$

shared between the two solutions appearing in the inequality:

$$
\pi s_1 \mathbb{T}\left(n_2, n_4; \mathsf{a}\right) \neq \mathbb{T}\left(m_2, m_4; \mathsf{b}\right)\,. \tag{23}
$$

Expression (22) holds when the following consistency conditions are satisfied :

$$
m_4 = n_2 - m_2 \geq 0,\; n_2 \geq m_2 \geq 0,\; n_2, m_2, n_4 \in \mathbb{Z}_+\,, \tag{24}
$$

$$
\mathsf{a} = 2(m_2 - n_2) = -2m_4,\; \mathsf{b} = 2 + 2n_4 + 2m_4 = 2 + 2n_4 - \mathsf{a}\,. \tag{25}
$$

Example 3.1. We consider the case of

$$
n_2 = n_4 = 2,\; m_2 = 1 \;\to m_4 = n_2 - m_2 = 1,\; \alpha_i = 2(5, -1, -1, -2)\,, \tag{26}
$$

where we used relation (22) to calculate α_i and the consistency condition (24). For the corresponding coefficients of the Painlevé V equation we find from relation (7) for $\epsilon = 1$:

$$
\alpha = \frac{1}{2},\;\; \beta = -\frac{25}{2},\;\; \gamma = 1\,. \tag{27}
$$

According to rules of the M_1-sequence we have two degenerated solutions corresponding to the parameters given in equation (26):

$$\begin{aligned} \pi s_1 \mathbb{T}(n_2 = 2, n_4 = 2; \mathsf{a} = -2) &= \pi s_1 T_2^{-2} T_4^2 \big| q = z,\, p = 0\rangle_{\alpha_{\mathsf{a}=-2}} \,, \\ \mathbb{T}(m_2 = 1, m_4 = 1; \mathsf{b} = 8) &= T_2^{-1} T_4^1 \big| q = z,\, p = 0\rangle_{\alpha_{\mathsf{b}=8}} \,. \end{aligned} \tag{28}$$

with a and b determined from relation (25).

We first calculate $\mathbb{T}(m_2 = 1, m_4 = 1; \mathsf{b} = 8)$ from expression (28) using the first of relations (15) with the parameter b followed by action with T_4 according to (18) to get

$$\begin{aligned} q &= z \frac{(-\mathsf{b} + z^2 + 2)(z^4 - 2z^2\mathsf{b} + \mathsf{b}^2 + 2\mathsf{b})}{(-\mathsf{b} + z^2)(-2z^2\mathsf{b} + z^4 + 4z^2 - 2\mathsf{b} + \mathsf{b}^2)} \\ p &= -2z \frac{(-2z^2\mathsf{b} + z^4 + 4z^2 - 2\mathsf{b} + \mathsf{b}^2)}{(-\mathsf{b} + z^2 + 2)(-2z^2\mathsf{b} + z^4 - 2\mathsf{b} + \mathsf{b}^2)} \end{aligned} \tag{29}$$

which for $\mathsf{b} = 8$ yields

$$q = \frac{z(z^2 - 6)(z^4 - 16z^2 + 80)}{(z^2 - 8)(z^4 - 12z^2 + 48)}, \;\; p = \frac{-2z(z^4 - 12z^2 + 48)}{(z^2 - 6)(z - 2)(z + 2)(z^2 - 12)}\,, \tag{30}$$

with $\alpha_i = (10, -2, -2, -4) = 2(5, -1, -1, 2)$. To obtain a solution $y(x)$ of the Painlevé V equation we transform $q \to y = (q/z)(q/z - 1)^{-1}$ and substitute z by $x = -z^2/2$ with the result:

$$y(x) = +\frac{(x + 3)(x^2 + 8x + 20)}{(x + 2)(x + 6)}\,, \tag{31}$$

which agrees with the expression of the Painlevé V solution $w_{1,1}(x; 1)$ obtained in Example 4.11 of [4].

Next we calculate $\pi s_1 \mathbb{T}(n_2 = 2, n_4 = 2; \mathsf{a} = -2)$ from relation (28) acting first with T_2^{-2} on $q = z,\ p = 0$ that according to equation (15) for $n = 2$ yields:

$$T_2^{-2} : q = z, p = 0 \to q = z, p = \frac{4z(\mathsf{a} - z^2)}{z^4 - 2\mathsf{a}z^2 + \mathsf{a}(\mathsf{a} + 2)},\; (4 + \mathsf{a}, -4, 0, 2 - \mathsf{a})\,, \tag{32}$$

Applying T_4^2, using expression (18), on the configuration in equation (32) we get a complicated solution to Painlevé equation for $\alpha_i = (4 + \mathsf{a}, -4, -4, 6 - \mathsf{a})$. Inserting $\mathsf{a} = -2$ simplifies α_i to $(2, -4, -4, 8)$ and the expressions for q, p simplify to:

$$\begin{aligned} q &= z\, \frac{(z^4 + 12\,z^2 + 48)\,(z^8 + 16\,z^6 + 96\,z^4 + 192\,z^2 + 192)}{(z^8 + 24\,z^6 + 216\,z^4 + 768\,z^2 + 1152)\,(8\,z^2 + 24 + z^4)}\,, \\ p &= -4\, \frac{(z^6 + 6\,z^4 + 24\,z^2 + 48)\,(z^8 + 24\,z^6 + 216\,z^4 + 768\,z^2 + 1152)}{z\,(z^6 + 12\,z^4 + 72\,z^2 + 192)\,(z^8 + 16\,z^6 + 96\,z^4 + 192\,z^2 + 192)}\,, \end{aligned} \tag{33}$$

Applying then πs_1 that transforms : $(2, -4, -4, 8) \to (10, -2, -2, -4)$ we are being taken from solution (33) to:

$$\begin{aligned} q &= z\, \frac{(8\,z^2 + 24 + z^4)\,(z^6 + 18\,z^4 + 144\,z^2 + 480)}{(z^4 + 12\,z^2 + 48)\,(z^6 + 12\,z^4 + 72\,z^2 + 192)}\,, \\ p &= z\, \frac{(z^4 + 12\,z^2 + 48)\,(z^8 + 16\,z^6 + 96\,z^4 + 192\,z^2 + 192)}{(z^8 + 24\,z^6 + 216\,z^4 + 768\,z^2 + 1152)\,(8\,z^2 + 24 + z^4)}\,, \end{aligned} \tag{34}$$

which, as it was the case with expressions (30), solves the Painlevé V Hamilton equation with $\alpha_i = (10, -2, -2, -4)$.

The corresponding solution $y(x) = (q/z)(q/z-1)^{-1}$ of the Painlevé V equation for coefficients (27) reads

$$y = \frac{(x^2-4x+6)(-x^3+9x^2-36x+60)}{x^4-12x^3+54x^2-96x+72}\,, \tag{35}$$

that agrees with expression for $\hat{w}_{1,2}(x;-1)$ of Example 4.11 of reference [4].

Example 3.2. Next we consider the case of

$$n_2 = 3,\ n_4 = 1,\ m_2 = 2\ \to m_4 = n_2 - m_2 = 1,\ \alpha_i = 2(5,-2,-1,-1)\,, \tag{36}$$

For the corresponding coefficients of the Painlevé V equation we find from relation (7) for $\epsilon = 1$:

$$\alpha = \frac{1}{2},\ \ \beta = -\frac{25}{2},\ \ \gamma = -1\,. \tag{37}$$

We notice that the above coefficients differ from the ones in equation (27) of Example 3.1 only by the sign of γ, which will be of importance below.

Again, according to rules of the M_1-sequence we have two degenerated solutions corresponding to the parameters given in equation (36):

$$\begin{aligned} \pi s_1 \mathbb{T}(n_2 = 3, n_4 = 1; \mathsf{a} = -2) &= \pi s_1 T_2^{-3} T_4^1 \big| q = z,\ p = 0\rangle_{\alpha_{\mathsf{a}=-2}}\,, \\ \mathbb{T}(m_2 = 2, m_4 = 1; \mathsf{b} = 6) &= T_2^{-2} T_4^1 \big| q = z,\ p = 0\rangle_{\alpha_{\mathsf{b}=6}}\,. \end{aligned} \tag{38}$$

with $\mathsf{a} = -2m_4 = -2, \mathsf{b} = 2 + 2n_4 + 2m_4 = 6$.

We first use expression (15) that gives for $n = 3$:

$$T_2^{-3} : q = z, p = 0 \to q = z, p = \frac{6zR_2(x,\mathsf{a})}{R_3(x,\mathsf{a})},\ \alpha_i = (6+\mathsf{a}, -6, 0, 2-\mathsf{a})\,, \tag{39}$$

where for $x = -z^2/2$:

$$R_1(x,\mathsf{a}) = \mathsf{a} + 2x,\ \ R_2(x,\mathsf{a}) = -4x + (2+\mathsf{a}+2x)(\mathsf{a}+2x) \tag{40}$$

and

$$R_3(x,\mathsf{a}) = (2x)^3 + 3(2x)^2\mathsf{a} + 3(2x)\mathsf{a}(\mathsf{a}+2) + \mathsf{a}(\mathsf{a}+2)(\mathsf{a}+4)$$

as follows from the recurrence relation (17). Using the transformation rule (18) and applying πs_1 and setting $\mathsf{a} = -2$ so that $\alpha_i = (10, -4, -2, -2)$ we obtain for the first of equations (38)

$$\begin{aligned} \pi s_1 \mathbb{T}(n_2 = 3, n_4 = 1; \mathsf{a} = -2) = (q &= \frac{z(z^6+22z^4+176z^2+480)}{(z^2+8)(z^4+48+12z^2)}, \\ p &= \frac{z(z^2+8)(z^4+12z^2+24)}{(6+z^2)(z^2+4)(z^2+12)})\,, \end{aligned}$$

which gives for $y = (q/z)/(q/z-1)$:

$$y = \frac{(-x+3)(x^2-8x+20)}{(-x+6)(-x+2)} \tag{41}$$

Note that going from Example 3.1 to Example 3.2 ($\alpha_i = 2(5,-1,-1,-2) \to \alpha_i = 2(5,-2,-1,-1)$) only amounts to flipping sign of γ: $\gamma \to -\gamma$ in the Painlevé V equation. However the transformation $\gamma \to -\gamma$ amounts to $x \to -x$. Thus we go from the solution (31) of Painlevé V equation to the solution (41) only by flipping the sign of x as it is easily verified by inspection.

Using (39) and the transformation rule (18) we get

$$\mathbb{T}(m_2 = 2, m_4 = 1; \mathsf{b} = 6) = (q = \frac{z(z^4-8z^2+24)(z^6-18z^4+144z^2-480)}{(z^4-12z^2+48)(72z^2-12z^4+z^6-192)},$$
$$p = \frac{-4z(z^4-12z^2+24)(72z^2-12z^4+z^6-192)}{(z^4-8z^2+24)(-24z^6+216z^4-768z^2+1152+z^8)}),$$

which results in $y = (q/z)/(q/z-1)$ equal to

$$y = -\frac{(x^2+4x+6)(-x^3-9x^2-36x-60)}{(12x^3+x^4+54x^2+96x+72)}, \tag{42}$$

which also follows from equation (35) by flipping the sign of x.

We will now discuss other choices for the transformation M and compare them to results obtained by acting with Bäcklund transformations π, s_3, s_4 on $\alpha_{n;\mathsf{a}}$ from equation (22). We will find for π, s_4 that the resulting parameters will agree with those obtained from relations (3) with $M_4 = \pi^{-1}s_4, M_{12} = s_1s_2$, respectively, each with two degenerated solutions entering inequality (2). The case of $M_{34} = s_3s_4$ will be shown to be equivalent to M_1 although it differs from the sequence obtained by acting with s_3.

To trace more easily the effect of these transformations we rename the integers $n_i \to x_i$, $m_i \to y_i$ for $i = 1,2$ to obtain from expression (22), $2(1+n_2+n_4, -m_2, m_2-n_2, -n_4)$, an expression

$$\pi s_1(\alpha_{n;\mathsf{a}}) = \alpha_{m;\mathsf{b}} = 2(1+x_2+x_4, -y_2, y_2-x_2, -x_4), \tag{43}$$

with the consistency condition $x_2 \geq y_2$.

Applying π^{-1}, s_3, s_4 on the above relation we get the following expressions for the Bäcklund transforms α_i parameters:

$$\pi^{-1} : 2(-y_2, y_2-x_2, -x_4, 1+x_2+x_4), \tag{44}$$
$$s_3 : 2(1+x_2+x_4, -x_2, x_2-y_2, y_2-x_2-x_4), \tag{45}$$
$$s_4 : 2(1+x_2, -y_2, y_2-x_2-x_4, x_4). \tag{46}$$

Next, we review these expressions in the order they appeared above in equations (44)-(46) and associate a new Bäcklund transformations M_i to each of the three cases. We will be interested in whether the consistency conditions that will hold for each of the M_i sequences will be fully derivable from the consistency condition (24) by action of the Bäcklund transformations π^{-1}, s_3, s_4 used in the above relations. If the consistency relations are mapped into each other together with the parameters then we will conclude that the two sequences are fully equivalent and the mapping did not generate a new degeneracy.

3.1 Case of expression (44) with $M_4 = \pi^{-1} s_4$

Perform the following change of variables on variables of equation (44):

$$y_2 \to n_2,\ x_4 \to m_4,\ x_2 \to n_2 + n_4 - m_4 \tag{47}$$

with the condition $x_2 \geq y_2$ transforming into $n_2+n_4-m_4 > n_2$ or $n_4 \geq m_4$. The condition $y_4 = x_2 - y_2$ of M_1-sequence is set to consistently transform to $m_2 = n_4 - m_4$. This way we obtain :

$$\alpha = 2(-n_2, m_4 - n_4, -m_4, 1 + n_2 + n_4), \quad n_4 \geq m_4 \geq 0,\ \ n_2, m_4 \in \mathbb{Z}_+\,, \tag{48}$$

which is associated with $M_4 = \pi^{-1} s_4$ and

$$\pi^{-1} s_4 T_2^{-n_2} T_4^{n_4} |q = z, p = 0\rangle_{\alpha_\mathsf{a}} \neq T_2^{-m_2} T_4^{m_4} (q = z, p = 0)_\mathsf{b}\,, \tag{49}$$

with

$$\mathsf{a} = 2(1 + n_4 - m_4) = 2 + 2m_2,\ \mathsf{b} = 2(-m_2 - n_2) = 2(1 - n_2) - \mathsf{a},\ \ m_2 = n_4 - m_4\,.$$

We see that the model described by $M_1 = \pi s_1$ with its condition $n_2 \geq m_2$ is being mapped into a model described by $M_4 = \pi^{-1} s_4$ with $n_4 \geq m_4$ with only difference that negative a/positive b transforms into positive a/negative b. Thus with consistency conditions being mapped into each other the two sequences are fully equivalent. This will be illustrated in the following example.

Example 3.3. Let us choose

$$m_4 = 0,\ n_4 = 1,\ n_2 = 1,\ \to\ \mathsf{a} = 4,\ \mathsf{b} = -4,\ m_2 = n_4 - m_4 = 1\,.$$

The corresponding solutions are :

$$\pi^{-1} s_4 T_4^1 T_2^{-1} \big|q = z,\ p = 0\big\rangle_{\alpha_{\mathsf{a}=4}} \neq T_2^{-1} T_4^0 \big|q = z,\ p = 0\big\rangle_{\alpha_{\mathsf{b}=-4}} \tag{50}$$

with $\alpha_i = (-2, -2, 0, 6)$ holding for both sides.

We find for the left hand side of inequality (50):

$$q = -\frac{2z(-4z^2 + z^4 + 8)}{(-2 + z^2)(-8z^2 + z^4 + 8)}, \quad p = \frac{2z(-8z^2 + z^4 + 8)}{(z^2 - 4)(-4z^2 + z^4 + 8)}\,,$$

while on the right hand side of (50) we find:

$$q = z,\ p = \frac{2z}{-4 - z^2}\,,$$

and indeed both solutions satisfy the Painlevé V Hamilton equations (5) with $\alpha_i = 2(-1, -1, 0, 3)$.

Corresponding to the above parameters we find by inverting relations (47) that $x_2 = 2 > y_2 = 1$ and $x_4 = 0$. Further, since the condition $m_2 = n_4 - m_4$ transforms into $y_4 = x_2 - y_2$ we get $y_4 = 1$ for the $M_1 = \pi s_1$ sequence. It follows that the corresponding parameter found from expression (3) is $\alpha_i = 2(3,-1,-1,0)$. Next we find that the corresponding solutions of (2) for $M_1 = \pi s_1$ sequence are

$$\pi s_1 \mathbb{T}\,(x_2 = 2, x_4 = 0; \mathsf{a} = -2) = \pi s_1 T_2^{-2} |q = z, p = 0\rangle_{\alpha_{\mathsf{a}=-2}}$$
$$= |(q = \frac{z^6 + 6z^4}{z(z^4 + 4z^2)}, \quad p = z\rangle_{(6,-2,-2,0)}\,,$$

versus

$$\mathbb{T}\,(y_2 = 1, y_4 = 1; \mathsf{b} = 4) = T_2^{-1} T_4 |q = z, p = 0\rangle_{\alpha_{\mathsf{b}=4}}$$
$$= |q = \frac{z\left(z^2-2\right)\left(z^4 - 8z^2 + 24\right)}{\left(z^2-4\right)\left(z^4 - 4z^2 + 8\right)}, p = -\frac{2z\left(z^4 - 4z^2 + 8\right)}{\left(z^2-2\right)\left(z^4 - 8z^2 + 8\right)}\rangle_{(6,-2,-2,0)}\,,$$

with both solutions of the Painlevé V equations (5) sharing the same parameters

$$\alpha_i = (6,-2,-2,0)\;. \tag{51}$$

Thus, as announced, we have been able to map two solutions of M_1 and M_4 sequences into each other.

3.2 Case of expression (45), $s_3(M_1)$ versus $M_{34} = s_3 s_4$

Here we consider $s_3(\alpha_i)$ given in the equation (45) and we will show that although it agrees with the parameters α_i given in formula (3) when derived from expression (2) with $M_{34} = s_3 s_4$ the consistency conditions will not match. To study $M_{34} = s_3 s_4$ we consider the inequality

$$s_3 s_4 T_2^{-n_2} T_4^{n_4}\,|q = z, p = 0\rangle_{\alpha_{\mathsf{a}}} \neq T_2^{-m_2} T_4^{m_4}\,|q = z, p = 0\rangle_{\alpha_{\mathsf{b}}}\,.$$

For parameters of solutions on both sides of this inequality to be equal we need to have

$$\begin{aligned} & s_3 s_4 T_2^{-n_2} T_4^{n_4}(\mathsf{a}, 0, 0, 2-\mathsf{a}) = s_3 s_4(\mathsf{a} + 2n_2, -2n_2, -2n_4, 2 - \mathsf{a} + 2n_4) \\ & = (2 + 2n_2 + 2n_4, 2 - \mathsf{a} - 2n_2; \mathsf{a} - 2, -2n_4) \\ & = T_2^{-m_2} T_4^{m_4}(b, 0, 0, 2-b) = (\mathsf{b} + 2m_2, -2m_2, -2m_4, 2 - \mathsf{b} + 2m_4)\,. \end{aligned} \tag{52}$$

Solving for a and b yields

$$\mathsf{a} = 2 - 2m_4 = 2 + 2m_2 - 2n_2, \;\; \mathsf{b} = 2 + 2m_4 + 2n_4 = 4 + 2n_2 - \mathsf{a} > 0\,, \tag{53}$$

with the consistency relation

$$m_4 = n_2 - m_2\,, \tag{54}$$

required for the above equations to hold.

We notice that this consistency relation ensures that b is always positive.

Inserting the values of a and b back into the relation (52) we obtain:

$$\alpha_i = 2(1 + n_2 + n_4, -m_2, m_2 - n_2, -n_4)\,, \tag{55}$$

in full agreement with equation (45) reproduced below:

$$s_3(\alpha_i) = 2(1 + x_2 + x_4, -x_2, x_2 - y_2, y_2 - x_2 - x_4)\,,$$

when we identify $x_2 = m_2$, $y_2 = n_2$, $x_4 = n_2 + n_4 - m_2$. Note however that since $n_2 - m_2 \geq 0$ it follows that (54) reads in terms of these variables as: $y_2 - x_2 \geq 0$, which is just opposite to the original condition $x_2 - y_2 \geq 0$ of the M_1-sequence seen below (45). Thus this time the consistency relations did not get mapped into each other.

Does this result mean that the M_{34}-sequence is independent of the M_1-sequence because s_3 failed to connect those two cases? It turns out that M_{34}-sequence is fully equivalent to M_1-sequence because of relation $s_3 s_4 = \pi s_1 T_1^{-1}$, which is a special case of relations (21). It follows from this relation that

$$\begin{aligned} s_3 s_4 T_2^{-n_2} T_4^{n_4} \left|q = z, p = 0\right\rangle_{\alpha_{\mathsf{a}=2-2m_4}} &= \pi s_1 T_1^{-1} T_2^{-n_2} T_4^{n_4} \left|q = z, p = 0\right\rangle_{\alpha_{\mathsf{a}=2-2m_4}} \\ &= \pi s_1 T_2^{-n_2} T_4^{n_4} \left|q = z, p = 0\right\rangle_{\alpha_{\mathsf{a}=-2m_4}}\,, \end{aligned} \tag{56}$$

where we inserted the value of a from relation (53) and used relation (13). The above expression is equal to the one given in equation (23) then one takes into account the value of the parameter a given in (25). Thus the M_{34}-sequence is fully equivalent to the M_1-sequence.

It is still warranted to consider the sequence generated by action of s_3 on the M_1-sequence. The following observation is crucial. Consider $\alpha_i = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ entering expressions for the parameters $\alpha = \alpha_3^2/8$, $\beta = -\alpha_1^2/8$ and $\gamma = (\alpha_2 - \alpha_4)/2$ of the Painlevé V equation (6). The Bäcklund transformation s_3 transforms α_i into $(\alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3)$ maintaining the parameters α, β, γ of the Painlevé V equation (6) clearly invariant. Note that the remaining Bäcklund transformations s_1, s_2, s_4 will all change the parameters α, β, γ. However the s_3 transforms q, p as follows

$$s_3 : q \to q,\; p \to p - \frac{\alpha_3}{z - q},$$

and accordingly will leave the solution y of the Painlevé V equation (6) invariant. To illustrate these considerations we will act with s_3 on configurations given in example 3.1.

Example 3.4. As an example we consider acting with s_3 on (34), which transforms parameters as follows: $2(5, -1, -1, -2) \to 2(5, -2, 1, -3)$ Accordingly, we deal with the case of

$$n_2 = 1,\; n_4 = 3,\; m_2 = 2 \;\to m_4 = n_2 - m_2 = -1,\; \alpha_i = (5, -2, 1, -3). \tag{57}$$

We note that now $m_4 = n_2 - m_2$ is negative, however the corresponding coefficients of the Painlevé V equation, for $\epsilon = 1$, are the ones in (27) as seen in Example 3.1. Acting with s_3 on solution (30) we get

$$q = \frac{z(z^2 - 6)(z^4 - 16z^2 + 80)}{(z^2 - 8)(z^4 - 12z^2 + 48)},\quad p = \frac{(z^4 - 12z^2 + 48)}{z(z^2 - 6)}\,, \tag{58}$$

while acting with s_3 on (34), we get

$$\begin{aligned} q &= z\,\frac{(8\,z^2 + 24 + z^4)\,(z^6 + 18\,z^4 + 144\,z^2 + 480)}{(z^4 + 12\,z^2 + 48)\,(z^6 + 12\,z^4 + 72\,z^2 + 192)}\,, \\ p &= -4\,\frac{(z^4 + 12\,z^2 + 48)}{(z\,(8\,z^2 + 24 + z^4)}\,. \end{aligned} \tag{59}$$

Solutions (58) and (59) satisfy the Painlevé V Hamilton equations (5) with $\alpha_i = (10, -4, 2, -6)$ that differ from solutions in Example 3.1, which satisfy the Painlevé V Hamilton equations with the $\alpha_i = 2(5, -1, -1, -2)$. However, since $s_3(10, -4, 2, -6) = 2(5, -1, -1, -2)$ and $s_3(y(x)) = y(x)$, they give rise to the identical solutions $y(x)$ as obtained in Example 3.1 for the Painlevé V equation (6) with the coefficients (27).

3.3 Case of expression (46) with $M_{12} = s_1 s_2$

In this case we consider $s_4(\alpha)$ from equation (46) and compare with an expression for the α that we obtain from (3) for $M = M_{12}$:

$$\begin{aligned}\alpha &= T_2^{-m_2} T_4^{m_4} (\mathsf{b}, 0, 0, 2-\mathsf{b}) = T_2^{-m_2} T_4^{m_4} \ (\mathsf{b}, 0, 0, 2-\mathsf{b}) \\ &= (\mathsf{b} + 2m_2, -2m_2, -2m_4, 2-\mathsf{b}+2m_4) \\ &= s_1 s_2 T_2^{-n_2} T_4^{n_4} (\mathsf{a}, 0, 0, 2-\mathsf{a}) = (-\mathsf{a}, \mathsf{a}+2n_2, -2n_2-2m_2, 2+2n_4)\ .\end{aligned} \tag{60}$$

The consistency requires this time that:

$$m_4 = n_2 + n_4\,, \tag{61}$$

which leads to the following expressions:

$$\mathsf{a} = -2n_2 - 2m_2,\ \mathsf{b} = 2n_2 = -2m_2 - \mathsf{a}\,.$$

Plugging these values back into equation (60) we obtain an expression for α:

$$\alpha = 2(n_2 + m_2, -m_2, -n_2 - n_4, 1 + n_4) \qquad m_2, n_4 \in \mathbb{Z}_+, \tag{62}$$

that also follows from inequality (2) with $M_{12} = s_1 s_2$:

$$s_1 s_2 T_2^{-n_2} T_4^{n_4} \Big| q = z, p = 0 \Big\rangle_{\alpha_\mathsf{a}} \neq T_2^{-m_2} T_4^{m_4} \Big| q = z, p = 0 \Big\rangle_{\alpha_\mathsf{b}},\ m_4 = n_2 + n_4\,. \tag{63}$$

Expression (60) agrees with the result of (46) for :

$$m_2 = y_2,\ n_4 = x_4 - 1,\ n_2 = x_2 - y_2 + 1$$

Thus the coefficients x_2, x_4, y_2 need to satisfy inequalities $x_4 \geq 1, x_2 \geq y_2$, which are consistent with conditions (25). Note that $x_2 + 1 > y_2$ always holds since $x_2 \geq y_2$ and accordingly $n_2 > 0$.

We see that both sequences will map into each other when x_4 variable of the M_1 sequence takes values $x_4 = 1, 2, \ldots$ and correspondingly the n_2 variable of the M_{12} sequence takes values $n_2 = 1, 2, \ldots$.

4 Discussion

We have examined the cases of two-fold degeneracy of the Painlevé V rational solutions connected with the Bäcklund transformations $M_1 = \pi s_1, M_4 = \pi^{-1} s_4, M_{34} = s_3 s_4, M_{12} = s_1 s_2$ that enter the basic inequality (2) that relates the two degenerated solutions with the equal parameter (3) and showed that all four sequences of degenerated solutions are fully equivalent by employing Bäcklund transformations π^{-1} and s_4 to show equivalence of M_1-sequence with those of $M_4 = \pi^{-1}s_4, M_{12} = s_1s_2$ and relation $s_3s_4 = \pi s_1 T_1^{-1}$ for equivalence between $M_1 = \pi s_1$ and $M_{34} = s_3s_4$.

In number of Examples 3.1, 3.2 and 3.4 we have considered solutions with the Painlevé V coefficients:

$$\alpha = \frac{1}{2}, \;\; \beta = -\frac{25}{2}, \;\; \gamma = \pm 1 \,. \tag{64}$$

Let us now summarize the results of these considerations in the setting of M_1-sequence.

Recalling the expression (7) for the Painlevé V equation coefficients with $\epsilon = 1$ and inserting the relevant components of α_i (22) into these expressions we find that in order to match them with the expression (64) we need to solve the following three equations

$$(1 + n_2 + n_4)^2 = 25, \; (m_2 - n_2)^2 = 1, \; (n_4 - m_2)^2 = 1 \,, \tag{65}$$

for the three variables n_2, n_4, m_2 that all need to be positive integers.

Equations (65) have 8 solutions in total but only half of them with positive integers $n_2, n_4, m_2 \in \mathbb{Z}_+$. We list these 4 relevant solutions below:

A) $n_2 = n_4 = 2, \quad m_2 = 1 \quad \to m_4 = n_2 - m_2 = 1, \; \gamma = 1, \; \alpha_i = 2(5, -1, -1, -2)$.

B) $n_2 = 3, n_4 = 1, \quad m_2 = 2 \quad \to m_4 = n_2 - m_2 = 1, \; \gamma = -1, \; \alpha_i = 2(5, -2, -1, -1)$.

C) $n_2 = 1, n_4 = 3, \quad m_2 = 2 \quad \to m_4 = n_2 - m_2 = -1, \; \gamma = 1, \; \alpha_i = 2(5, -2, 1, -3)$.

D) $n_2 = 2, n_4 = 2, \quad m_2 = 3 \quad \to m_4 = n_2 - m_2 = -1, \; \gamma = -1, \; \alpha_i = 2(5, -3, 1, -2)$.

Items A) and B) have been discussed in Examples 3.1 and 3.2, where we noticed that they satisfy the condition $n_2 \geq m_2$ (or $m_4 \geq 0$) and are therefore a part of the M_1-sequence.

We have seen that on the level of Painlevé V equation (6) the transformation of solutions obtained inside the M_1-sequence with the parameters listed in case A) to solutions of case B) was fully accomplished by flipping $\gamma \to -\gamma$ or equivalently flipping $x \to -x$. On the level of the Hamilton Painlevé V equations the corresponding q, p solutions solve the equations (5) with different α_i given above in A) and B). Recall that in [1] we have introduced x as $x = z^2/(2\epsilon)$ with $\epsilon^2 = 1$. Thus here we are exercising the freedom of changing a sign of ϵ that changes a sign of γ (see again [1]).

The cases C) and D) are mapped from A) and B) by action of s_3:

$$C) = s_3(A)), \quad D) = s_3(B)),$$

as can be verified by inspecting the parameters α_i. We have seen the case C) being discussed in Example 3.4. Each of these two cases exhibits therefore the two-fold degeneracy of the Hamilton Painlevé V equations with solutions that are an s_3 image of the corresponding solutions of M_1-sequence with parameters of case A) and B). Since s_3 keeps both the coefficients and the solution of the Painlevé V equation (6) invariant, we conclude that the Painlevé V solutions associated to cases C) and D) are fully equal to those already found in cases A) and B).

In all examples we have seen a and b are even integers and having (to some degree) an opposite sign. For the M_1-sequence $\mathsf{a} \leq 0$ and $\mathsf{b} \geq 2$ and such that $\mathsf{a}/2 + \mathsf{b}/2 = 1, 2, \ldots$.

For the M_{12} sequence $\mathsf{a} \le 0$ and $\mathsf{b} \ge 0$ and $\mathsf{a}/2+\mathsf{b}/2 = 0, -1, -2, \ldots$. For the M_4 sequence $\mathsf{a} \ge 0$ and $\mathsf{b} \le 0$ such that $\mathsf{a}/2 + \mathsf{b}/2 = 0, -1, -2, \ldots$. For the M_{34}-sequence it holds that $\mathsf{a} \le 2$ and $\mathsf{b} \ge 2$ and $\mathsf{a}/2 + \mathsf{b}/2 = 2, 3, \ldots$ as expected since the M_{34}-sequence is equivalent to the M_1-sequence only with a shifted by 2.

As we have noted in section 2 the value of the parameter a can be shifted by an even integer $2n$ through the action of T_1^n. For degenerated solutions one can use this freedom to set, for example, the parameter a to zero since it is an even integer. However the same operation will raise or lower the value of the connected parameter b and therefore maintain invariant the value of their sum.

Example 4.1. As an example consider a and b such that $\mathsf{a} = -2n$ and $\mathsf{b} = 2n + 2k$ for $n \in \mathbb{Z}$ and $k = 1, 2, 3, \ldots$. Comparing with the paragraph above we see that this case fits into the M_1-sequence of degenerated solutions. Comparing with the expressions (24) and (25) we find that $n_4 = k - 1$ and $m_4 = n$. We conclude that for any fixed integers $n \ge 0$ and $k > 0$ we find a pair of solutions belonging to M_1-sequence:

$$\pi s_1 \mathbb{T}(n_2, k-1; \mathsf{a} = -2n) \quad \text{and} \quad \mathbb{T}(n_2 - n, n; \mathsf{b} = 2n + 2k)\,,$$

that satisfy the Painlevé V equations with the same parameters

$$\alpha_i = 2(1 + n_2 + n_4, -m_2, m_2 - n_2, -n_4) = 2(n_2 + k, n - n_2, -n, 1 - k)\,, \tag{66}$$

valid for any integer n_2 such that $n_2 \ge n$.

Comparing $\alpha(m; \mathsf{b}) = (\mathsf{b} + 2m_2,\ -2m_2,\ -2m_4,\ 2 - \mathsf{b} + 2m_4)$ from expression (14). we recognize that it agrees with expression for the parameter (66) for $\mathsf{b} = 2(k + n)$ and $m_2 = n_2 - n \ge 0$, $n = m_4$.

In summary, we have developed an explicit construction that applies to the two-fold degeneracy of Painlevé V Hamilton equations and determines the two degenerated solutions and the parameters of Painlevé V equations that they share. We also found a condition for a solution $\mathbb{T}(m; \mathsf{b})$ on the orbit of $T_2^{-m_2} T_4^{m_4}$ to agree with one of the two degenerated solutions and the condition is that the parameter b is an even integer (a positive integer for the M_1-sequence and a negative for the M_4-sequence).

Recall that the Painlevé V Hamilton system is closely related to the dressing chain of even, $N = 4$ periodicity, see [1] and references therein. Our discussion based on translation operators indicates that degeneracy will exist for all dressing chains of even periodicity because of existence of exclusion rules for translation operators permitted to act on special types of seed solutions. Especially, it will occur for $N = 6$ periodic dressing chain discussed in [1]. A natural problem to investigate is whether a degree of degeneracy (how many solutions will share the parameter α_i) will change in case of higher dressing chains of even period $N > 4$.

Acknowledgements

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001 (G.V.L.) and by CNPq and FAPESP (J.F.G. and A.H.Z.).

References

[1] Aratyn H, Gomes J F, Lobo G V and Zimerman A H, On Rational Solutions of Dressing Chains of Even Periodicity, *Symmetry*, **15**, 249, 2023.

[2] Aratyn H, Gomes J F, Lobo G V and Zimerman A H Why is my rational Painlevé V solution not unique?, arXiv:2307.07825, 2023.

[3] Aratyn H, Gomes J F, Lobo G V and Zimerman A H, paper in preparation, 2024.

[4] Clarkson PA, Dunning C, Rational Solutions of the Fifth Painlevé Equation. Generalised Laguerre Polynomials, *Stud Appl Math.* **152**, 453–507, 2024.

[5] Gromak V I, Laine I and Shimomura S, *Painlevé Differential Equations in the Complex Plane* de Gruyter Studies in Mathematics, Volume 28, 2002.

[6] Kitaev A V, Law C K and McLeod J B, *Differential Integral Equations* **7**, 967-1000, 1994.

[7] Noumi M, Painlevé Equations through Symmetry, in: *Translations of Mathematical Monographs*, vol. 223, (American Mathematical Society Providence, RI), 2004.

[8] Noumi M and Yamada Y, *Physics Letters* **247A**, 65-69, 1998.

[9] Watanabe H, *Hokkaido Math. J.* **24** no.2, 231-267, 1995.

[10] Winternitz P and Levi D (eds), *Painlevé transcendents, their asymptotics and physical applications*, NATO ASI Series B: Physics 278, Plenum, New York, 1992.