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Algebraic curves as a source of separable multi-Hamiltonian systems

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Abstract

In this paper we systematically consider various ways of generating integrable and separable Hamiltonian systems in canonical and in non-canonical representations from algebraic curves on the plane. In particular, we consider Stäckel transform between two pairs of Stäckel systems, generated by $2n$-parameter algebraic curves on the plane, as well as Miura maps between Stäckel systems generated by $(n+N)$-parameter algebraic curves, leading to multi-Hamiltonian representation of these systems.

1 Introduction

This paper is devoted to a systematic (in the sense explained below) construction of various types of Liouville integrable and separable Hamiltonian systems from algebraic curves.

In [16] Sklyanin noted that any Liouville integrable system (that is a set of n Hamiltonians in involution on a $2n$-dimensional manifold M) separates in a given canonical coordinate system $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \equiv (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$ if and only if there exists n separation relations of the form

$$\varphi_i(\lambda_i, \mu_i, h_1, ..., h_n) = 0, \qquad i = 1, \ldots, n \tag{1}$$

(see also [11]). Alternatively, one can treat the relations (1) as an algebraic definition of n commuting, by construction, Hamiltonians h_i on M. The canonical variables $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ are then by construction separation variables for all the Hamilton-Jacobi equations associated with the Hamiltonians h_i. This shift of view yields a powerful way of generating many (in fact, all known in literature) separable Hamiltonian systems from scratch. This approach has been initiated in [3] and then fruitfully developed in many papers, see for example [4, 7, 8] as well as the review of the subject in [9].

In this paper we restrict ourselves to the important subclass of separations relations (1) where all φ_i are the same, $\varphi_i = \varphi$ for all i. In such a case the relations (1) can be interpreted as n copies of the algebraic curve on the λ-μ plane

$$\varphi(\lambda, \mu, h_1, ..., h_n) = 0, \tag{2}$$

when (λ, μ) are consecutively substituted by the pair of variables (λ_i, μ_i). One reason for restricting our attention to separation curves (2) rather than the general separation relations (1) is that in the general setting there arise problems with finding the multi-Hamiltonian formulation of the generated separable systems. Another reason for this restriction is that it allows us to skip the assumption that the corresponding coordinates $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ on M are canonical. Using this approach, in this article we systematize and develop the idea of constructing various types of finite-dimensional integrable and separable Hamiltonian systems from *parameter-dependent* planar algebraic curves. To our knowledge this is the first systematic (albeit certainly not complete) investigation of separation curves depending on more than n parameters. Relations between integrable systems and n-parameter hyperelliptic curves were extensively investigated for example in [17] (see also references there).

Below we present the structure of the article and highlight all the new results.

In Section 2 we establish a number of facts for Poisson structures in 2 dimensions and of monomial type. We first establish Lemma 2 where we find all Darboux coordinates associated with the Poisson tensor (4) on the plane with c of the monomial form $c = \lambda^\alpha \mu^\beta$ and then we prove Lemma 3 where we establish canonical maps between arbitrary pair of Darboux coordinates for π. These results will be necessary for establishing results of Section 3.

In the first subsection of Section 3 we prove (Theorem 4) that the Hamiltonians h_i obtained by algebraically solving n copies of (2) constitute a Liouville integrable system not only if the corresponding coordinates $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ on M are canonical, but in a more general case when the Poisson operator π has in the variables $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ the form (21). This is a simple generalization of the previously known result (see for example [9]). The second subsection of Section 3 contains basic information on separable Hamiltonian systems, in particular of Stäckel type. This subsection is necessary for the self-consistency of the article.

In Section 4 we use the results of Section 2 to show that each Liouville integrable Hamiltonian system generated by an algebraic curve (2) and by the non-canonical Poisson tensor (4) can also be generated by a one-parameter family of algebraic curves and the corresponding Poisson tensors in canonical form. Each class represents thus the same dynamical system written in different Darboux coordinates, related with each other by appropriate canonical transformations. We further specify these results for the case of monomial Poisson structures (with $c(\lambda_i, \mu_i) = \lambda_i^\alpha \mu_i^\beta$), see formulas (37) yielding explicit transformation to Darboux coordinates in this case.

In Section 5 we consider separable systems generated by algebraic curves depending on a set of $n+n$ rather than n parameters. Each such curve leads then to two distinct integrable Hamiltonian systems. Using the known theory ([15, 8]) we prove that these systems are related by a Stäckel transform and we also show how solutions of these two systems are related by a reciprocal (multi-time) transformations. We also specify these results to the case of Stäckel systems. These results are new.

In the final Section 6 we investigate yet another possibility of generating integrable and separable Hamiltonian systems from algebraic curves. We consider algebraic curves (70) with the block-type structure given by (71) and (72), leading to families of integrable and separable Hamiltonian systems that can be related (due to Theorem 11) with each other by a finite-dimensional analogue of Miura maps. These finite-dimensional Miura maps yield in turn multi-Hamiltonian formulation of the obtained integrable systems, as presented in Theorem 12. These are new results that generalize the particular results obtained earlier in [6] and in [14]. The section is concluded by some examples. Theorem 11 is proven in Appendix, due to a rather technical nature of the proof.

2 Poisson tensors on 2-dimensional phase space

In this section we consider Poisson structures in 2 dimensions (on a λ-μ plane) of a monomial type and find all their Darboux coordinates that can be obtained from the coordinates (λ, μ) by monomial transformations. We also find a general map between arbitrary Darboux coordinate systems of monomial type.

Let us thus consider a (λ, μ) plane P. A Poisson tensor π on P is a bi-vector with vanishing Schouten-Nijenhuis bracket. The Poisson tensor π must be of co-rank zero since $\dim P = 2$. It defines a Poisson bracket on the plane:

$$\{f, g\}_\pi := \pi(df, dg), \qquad f, g \in C^\infty(P). \tag{3}$$

Lemma 1. *The most general Poisson tensor π on P is of the form*

$$\pi = c(\lambda,\mu)\frac{\partial}{\partial\lambda}\wedge\frac{\partial}{\partial\mu}, \text{ with matrix representation } \pi = \begin{pmatrix} 0 & c(\lambda,\mu) \\ -c(\lambda,\mu) & 0 \end{pmatrix}, \quad c \in C^2(P). \tag{4}$$

Proof. The necessary and sufficient condition for being Poisson tensor is a Jacobi identity

$$\{f, \{g, h\}_\pi\}_\pi + \{g, \{h, f\}_\pi\}_\pi + \{h, \{f, g\}_\pi\}_\pi = 0. \tag{5}$$

By a direct computation one can verify the identity (5) for tensor (4), where

$$\{f, g\}_\pi = \left(\frac{\partial f}{\partial\lambda}\frac{\partial g}{\partial\mu} - \frac{\partial f}{\partial\mu}\frac{\partial g}{\partial\lambda}\right) c. \tag{6}$$

■

Now, let us change the parametrization of the plane $(\lambda, \mu) \longrightarrow (\bar\lambda, \bar\mu)$ given by

$$\bar\lambda = a(\lambda, \mu), \qquad \bar\mu = b(\lambda, \mu) \tag{7}$$

and such that new coordinates are Darboux (canonical) coordinates for π, i.e. $c(\bar{\lambda},\bar{\mu})=1$. It means that following condition

$$\left(\frac{\partial a}{\partial\lambda}\frac{\partial b}{\partial\mu}-\frac{\partial a}{\partial\mu}\frac{\partial b}{\partial\lambda}\right)c=1 \tag{8}$$

has to be fulfilled for a pair of unknown functions a and b.

Consider a particular but relevant subclass of Poisson tensors (6) defined by functions c in the monomial form $c=\lambda^{\alpha}\mu^{\beta}$ for fixed $\alpha,\beta\in\mathbb{R}$. Let us now search for transformations to Darboux coordinates of (4) within the following ansatz

$$\bar{\lambda}=\lambda^{\bar{\alpha}_1}\mu^{\bar{\alpha}_2},\qquad \bar{\mu}=\lambda^{\bar{\beta}_1}\mu^{\bar{\beta}_2}, \tag{9}$$

where $\bar{\alpha}_1,\bar{\alpha}_2,\bar{\beta}_1,\bar{\beta}_2\in\mathbb{R}$. This ansatz has the following inverse

$$\lambda=\bar{\lambda}^{\frac{\bar{\beta}_2}{\bar{\alpha}_1\bar{\beta}_2-\bar{\alpha}_2\bar{\beta}_1}}\bar{\mu}^{-\frac{\bar{\alpha}_2}{\bar{\alpha}_1\bar{\beta}_2-\bar{\alpha}_2\bar{\beta}_1}},\qquad \mu=\bar{\lambda}^{-\frac{\bar{\beta}_1}{\bar{\alpha}_1\bar{\beta}_2-\bar{\alpha}_2\bar{\beta}_1}}\bar{\mu}^{\frac{\bar{\alpha}_1}{\bar{\alpha}_1\bar{\beta}_2-\bar{\alpha}_2\bar{\beta}_1}}. \tag{10}$$

A direct calculation using the condition (8) shows that the transformation (9) turns π into canonical form if and only if

$$\bar{\alpha}_1+\ \bar{\beta}_1+\alpha=1,\quad \bar{\alpha}_2+\ \bar{\beta}_2+\beta=1,\quad \bar{\alpha}_1\bar{\beta}_2-\bar{\alpha}_2\bar{\beta}_1=1, \tag{11}$$

which leads to the following lemma.

Lemma 2. *The coordinates* $\left(\bar{\lambda},\ \bar{\mu}\right)$ *given by (9) are Darboux (canonical) coordinates for* π *if and only if the parameters* $\bar{\alpha}_1,\bar{\alpha}_2,\bar{\beta}_1,\bar{\beta}_2$ *are given by, for* $\alpha\neq1$

$$\bar{\alpha}_1=\bar{\alpha}_1,\quad \bar{\alpha}_2=\frac{\bar{\alpha}_1(1-\beta)-1}{1-\alpha},\quad \bar{\beta}_1=1-\alpha-\bar{\alpha}_1,\quad \bar{\beta}_2=\frac{(1-\beta)(1-\alpha-\bar{\alpha}_1)+1}{1-\alpha}, \tag{12}$$

and for $\beta\neq1$

$$\bar{\alpha}_1=\frac{(1-\alpha)(1-\beta-\ \bar{\beta}_2)+1}{1-\beta},\quad \bar{\alpha}_2=1-\beta-\ \bar{\beta}_2,\quad \bar{\beta}_1=\frac{\bar{\beta}_2(1-\alpha)-1}{1-\beta},\quad \bar{\beta}_2=\bar{\beta}_2. \tag{13}$$

For $\alpha=\beta=1$ there is no transformations to Darboux coordinates of (4) within the ansatz (9). An example of transformation that leads to Darboux coordinates in this case is $\bar{\lambda}=\ln|\lambda|\,,\bar{\mu}=\ln|\mu|\,.$

Notice that the solutions (12) are one-parameter, parametrized by $\bar{\alpha}_1$; similarly, the solutions (13) are one-parameter, and parametrized by $\bar{\beta}_2$. Note also that the last equation in (11) means that not only the map (9) but also its inverse (10) are in this case polynomial maps.

In the special case that $\alpha=\beta=0$, (that is if the original variables (λ,μ) are already canonical for π) the transformation (9) given by (12) represents a one-parameter family of canonical transformations

$$\bar{\lambda}=\lambda^{\bar{\alpha}_1}\mu^{\bar{\alpha}_1-1},\qquad \bar{\mu}=\lambda^{1-\bar{\alpha}_1}\mu^{2-\bar{\alpha}_1}, \tag{14}$$

(parametrized by $\bar{\alpha}_1$) with the inverse

$$\lambda=\bar{\lambda}^{2-\bar{\alpha}_1}\bar{\mu}^{1-\bar{\alpha}_1},\quad \mu=\bar{\lambda}^{\bar{\alpha}_1-1}\bar{\mu}^{\bar{\alpha}_1}. \tag{15}$$

Applying Lemma 2 to two different sets of Darboux coordinates: $\left(\bar{\lambda},\ \bar{\mu}\right)$ and $(\tilde{\lambda},\tilde{\mu})$, given by (12) with $\bar{\alpha}_1$and $\tilde{\alpha}_1$ respectively, we arrive at the following lemma.

Lemma 3. *Assume, that $(\bar{\lambda},\bar{\mu})$ and $(\tilde{\lambda},\tilde{\mu})$ are two different sets of Darboux coordinates for the same Poisson tensor (4) with $c=\lambda^\alpha\mu^\beta$ related to (λ,μ) by two different solutions (12) given by $\bar{\alpha}_1$ and by $\tilde{\alpha}_1$ respectively. Then the map $(\bar{\lambda},\bar{\mu})\longrightarrow(\tilde{\lambda},\tilde{\mu})$ is canonical and takes the form*

$$\bar{\lambda}=\tilde{\lambda}^{\alpha_1}\tilde{\mu}^{\alpha_1-1},\qquad \bar{\mu}=\tilde{\lambda}^{1-\alpha_1}\tilde{\mu}^{2-\alpha_1},\qquad \alpha_1=1+\frac{\bar{\alpha}_1-\tilde{\alpha}_1}{1-\alpha},\quad \alpha\neq 1, \tag{16}$$

$$\bar{\lambda}=\tilde{\lambda}^{\alpha_2}\tilde{\mu}^{\alpha_2-1},\qquad \bar{\mu}=\tilde{\lambda}^{1-\alpha_2}\tilde{\mu}^{2-\alpha_2},\qquad \alpha_2=1+\frac{\bar{\alpha}_2-\tilde{\alpha}_2}{1-\beta},\quad \beta\neq 1. \tag{17}$$

The proof is again obtained by direct calculations. The inverse of (16) is of the form

$$\tilde{\lambda}=\bar{\lambda}^{2-\alpha_1}\bar{\mu}^{1-\alpha_1},\qquad \bar{\mu}=\bar{\lambda}^{\alpha_1-1}\bar{\mu}^{\alpha_1}, \tag{18}$$

while the inverse of (17) is

$$\tilde{\lambda}=\bar{\lambda}^{2-\alpha_2}\bar{\mu}^{1-\alpha_2},\qquad \bar{\mu}=\bar{\lambda}^{\alpha_2-1}\bar{\mu}^{\alpha_2}.$$

3 From algebraic curves to Liouville integrable and separable Hamiltonian systems

3.1 Liouville integrability

Here we show how to construct n-dimensional Liouville integrable Hamiltonian systems starting from a single n-parameter algebraic curve on a (λ,μ)-plane of the form

$$\varphi(\lambda,\mu,a_1,...,a_n)=0. \tag{19}$$

Taking n copies of (19) with (λ,μ) consecutively substituted by the pair of variables (λ_i,μ_i) we obtain the system of n equations

$$\varphi(\lambda_i,\mu_i,a_1,...,a_n)=0,\qquad i=1,\dots,n \tag{20}$$

that is assumed to be solvable with respect to the parameters a_k (at least in some open domain). In result, we obtain n functions (Hamiltonians) $a_k=h_k(\boldsymbol{\lambda},\boldsymbol{\mu})$ on $2n$-dimensional manifold M, parametrized by coordinates $\boldsymbol{\lambda}=(\lambda_1,...,\lambda_n)$, $\boldsymbol{\mu}=(\mu_1,...,\mu_n)$.

In order to turn manifold M into Poisson manifold we take n copies of the Poisson operator (4) on the plane and construct the Poisson tensor π on M as follows:

$$\pi=\sum_{i=1}^n c(\lambda_i,\mu_i)\frac{\partial}{\partial\lambda_i}\wedge\frac{\partial}{\partial\mu_i}, \tag{21}$$

so that its matrix representation is

$$\pi=\begin{pmatrix}\mathbf{0}_{n\times n} & c(\boldsymbol{\lambda},\boldsymbol{\mu})\\ -c(\boldsymbol{\lambda},\boldsymbol{\mu}) & \mathbf{0}_{n\times n}\end{pmatrix},\qquad c(\boldsymbol{\lambda},\boldsymbol{\mu})=\operatorname{diag}(c(\lambda_1,\mu_1),...,\ c(\lambda_n,\mu_n)). \tag{22}$$

Below we prove h_i generated by (2.2) commute with respect to π given by (21), which is a natural generalization of the result with $c=1$ that can be found for example in [9].

Theorem 4. *Hamiltonian functions h_i are in involution with respect to Poisson tensors (21)*

$$\{h_i, h_j\}_\pi = 0, \qquad i, j = 1, ..., n. \tag{23}$$

Proof. The Hamiltonian functions $h_i(\boldsymbol{\lambda}, \boldsymbol{\mu})$ Poisson commute as a consequence of relations (20). Indeed, differentiating equations (20) with respect to $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ coordinates we get

$$\frac{\partial \varphi_k}{\partial \lambda_i} + \sum_{r=1}^{n} \frac{\partial \varphi_k}{\partial a_r}\frac{\partial h_r}{\partial \lambda_i} = 0, \qquad \frac{\partial \varphi_k}{\partial \mu_i} + \sum_{r=1}^{n} \frac{\partial \varphi_k}{\partial a_r}\frac{\partial h_r}{\partial \mu_i} = 0,$$

so

$$\frac{\partial h_r}{\partial \lambda_i} = -\sum_{s=1}^{n} A_s^r \frac{\partial \varphi_k}{\partial \lambda_i}, \quad \frac{\partial h_r}{\partial \mu_i} = -\sum_{s=1}^{n} A_s^r \frac{\partial \varphi_k}{\partial \mu_i},$$

where (A_s^r) is a matrix being the inverse of the matrix $(\partial \varphi_s / \partial a_r)$. In consequence

$$\begin{aligned}
\{h_r, h_s\} &= \sum_{k=1}^{n} \left(\frac{\partial h_r}{\partial \lambda_k}\frac{\partial h_s}{\partial \mu_k} - \frac{\partial h_r}{\partial \mu_k}\frac{\partial h_s}{\partial \lambda_k} \right) c_k \\
&= \sum_{k=1}^{n} \left(\sum_{i,j=1}^{n} A_i^r \frac{\partial \varphi_i}{\partial \lambda_k} A_j^s \frac{\partial \varphi_j}{\partial \mu_k} - \sum_{i,j=1}^{n} A_i^r \frac{\partial \varphi_i}{\partial \mu_k} A_j^s \frac{\partial \varphi_j}{\partial \lambda_k} \right) c_k \\
&= \sum_{i,j=1}^{n} A_i^r A_j^s \sum_{k=1}^{n} \left(\frac{\partial \varphi_i}{\partial \lambda_k}\frac{\partial \varphi_j}{\partial \mu_k} - \frac{\partial \varphi_i}{\partial \mu_k}\frac{\partial \varphi_j}{\partial \lambda_k} \right) c_k \\
&= \sum_{i,j=1}^{n} A_i^r A_j^s \{\varphi_i, \varphi_j\}_\pi = 0,
\end{aligned}$$

where $c_k = c(\lambda_k, \mu_k)$. ■

In result, the system of n evolution equations on M

$$\xi_{t_i} = \pi dh_i = X_i, \quad i = 1, ..., n, \tag{24}$$

where $\xi = (\boldsymbol{\lambda}, \boldsymbol{\mu})^T$, is Liouville integrable.

3.2 Separability

Liouville integrable systems generated by algebraic curves (19) and the Poisson tensor (4) with $c = 1$ are separable in the sense of Hamilton-Jacobi theory, $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ are then their separation coordinates and equations (20) are called separation relations. Indeed, the Hamiltonian system (24) is in this case linearized through a canonical transformation

$$(\boldsymbol{\lambda}, \boldsymbol{\mu}) \longrightarrow (\boldsymbol{\beta}, \boldsymbol{\alpha}), \tag{25}$$

generated by a generating function $W(\boldsymbol{\lambda}, \boldsymbol{\alpha})$, such that it satisfies all the Hamilton-Jacobi equations $h_i = \alpha_i$ of the system. Then, the transformation (25) is given implicitly by

$$\beta_i = \frac{\partial W}{\partial \alpha_i}, \quad \mu_i = \frac{\partial W}{\partial \lambda_i}, \qquad i = 1, ..., n. \tag{26}$$

In the variables $(\boldsymbol{\beta},\boldsymbol{\alpha})$ the n evolution equations (24) linearize

$$(\beta_j)_{t_i}=\frac{\partial h_i}{\partial\alpha_j}=\delta_{ij},\qquad(\alpha_j)_{t_i}=-\frac{\partial h_i}{\partial\beta_j}=0,\qquad i,j=1,...,n,\tag{27}$$

so that

$$\beta_j(\boldsymbol{\lambda},\boldsymbol{\alpha})=\frac{\partial W}{\partial\lambda_j}=t_j+c_j,\qquad c_j\in\mathbb{R}.\tag{28}$$

The existence of separation relations (20) means that there always exists an additively separable solution

$$W(\boldsymbol{\lambda},\boldsymbol{\alpha})=\sum_{i=1}^{n}W_i(\lambda_i,\boldsymbol{\alpha}),\tag{29}$$

for the generating function $W(\boldsymbol{\lambda},\boldsymbol{\alpha})$, where functions $W_i(\lambda_i,\boldsymbol{\alpha})$ are solutions of the system of n decoupled ordinary differential equations

$$\varphi_i\left(\lambda_i,\frac{dW_i(\lambda_i,\boldsymbol{\alpha})}{d\lambda_i},\boldsymbol{\alpha}\right)=0,\qquad i=1,...,n.\tag{30}$$

In literature, Liouville integrable systems, linearizable according to Hamilton-Jacobi theory by additively separable generating function (29) are known as (generalized) Stäckel systems. A particularly interesting case of such systems (a proper Stäckel system) is generated by φ being of hyperelliptic type

$$\varphi(\lambda,\mu)\equiv\sigma(\lambda)+\sum_{k=1}^{n}h_k\lambda^{\gamma_k}-\frac{1}{2}f(\lambda)\mu^2=0,\tag{31}$$

where $\sigma(\lambda)$ and $f(\lambda)$ are Laurent polynomials in λ, $\gamma_k\in\mathbb{N}$ and are such that $\gamma_1>...>\gamma_n=0$. Then,

$$h_k=\frac{1}{2}\boldsymbol{\mu}A_kG_f\boldsymbol{\mu}^T+V_k^{(\sigma)},\qquad k=1,\dots,n\tag{32}$$

and some additional geometric structure can be related with the dynamical systems (24). The Hamiltonians h_k are considered as functions on the phase space $M=T^*Q$, where $\boldsymbol{\lambda}$ are local coordinates on a n-dimensional configuration space Q and $\boldsymbol{\mu}$ are the (fibre) momentum coordinates, G_f is treated as a contravariant metric on Q, defined by the first Hamiltonian h_1, A_k ($A_1=I$) are $(1,1)$- Killing tensors for the metric G_f (for any f) and $V_k^{(\sigma)}$ are respective potentials on Q. The quadratic in momenta $\boldsymbol{\mu}$ Hamiltonians (32) are in involution with respect to the Poisson operator $\pi=\sum_{i=1}^n\frac{\partial}{\partial\lambda_i}\wedge\frac{\partial}{\partial\mu_i}$, in accordance with the general Theorem 4 in the subsection above. By construction, the variables $(\boldsymbol{\lambda},\boldsymbol{\mu})$ are separation variables for all the Stäckel Hamiltonians h_k in (32).

If we further assume that $\gamma_k=n-k$ then the Hamiltonians h_k in (32) become the so-called Stäckel Hamiltonians of Benenti type [1, 2, 5] and in this case

$$G_f=\operatorname{diag}\left(\frac{f(\lambda_1)}{\Delta_1},\dots,\frac{f(\lambda_n)}{\Delta_n}\right),\qquad\Delta_i=\prod_{j\neq i}(\lambda_i-\lambda_j).$$

Further, A_k are given by

$$A_k = (-1)^{k+1}\mathrm{diag}\left(\frac{\partial s_k}{\partial \lambda_1},\dots,\frac{\partial s_k}{\partial \lambda_n}\right) \qquad k=1,\dots,n.$$

They all are $(1,1)$-Killing tensors for all the metrics G_f. The function s_k is the elementary symmetric polynomial in λ_i of degree k. The potential functions $V_k^{(\sigma)}$ are given by

$$V_k = (-1)^{k+1}\sum_{i=1}^{n}\frac{\partial s_k}{\partial \lambda_i}\frac{\sigma(\lambda_i)}{\Delta_i} \qquad k=1,2,\dots \tag{33}$$

and for $\sigma(\lambda) = \sum_i a_i\lambda^i$ take the form $V_k = \sum_i a_i \mathcal{V}_k^{(i)}$, where the so-called elementary separable potentials $\mathcal{V}_k^{(i)}$ can be explicitly constructed from the recursion formula [7]

$$\mathcal{V}^{(i)} = R^i\mathcal{V}^{(0)}, \qquad \mathcal{V}^{(i)} = \left(\mathcal{V}_1^{(i)},\dots,\mathcal{V}_n^{(i)}\right)^T, \qquad \mathcal{V}^{(0)} = (0,\dots,0,-1)^T, \tag{34}$$

where

$$R = \begin{pmatrix} -q_1 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \vdots & 0 & 0 & 1 \\ -q_n & 0 & 0 & 0 \end{pmatrix}, \qquad R^{-1} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{q_n} \\ 1 & 0 & 0 & \vdots \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & -\frac{q_{n-1}}{q_n} \end{pmatrix}, \qquad q_i \equiv (-1)^i s_i. \tag{35}$$

4 Equivalence classes of algebraic curves

From results of Section 2 it follows that without loss of generality we can restrict our construction to Poisson tensors for which coordinates $(\boldsymbol{\lambda},\boldsymbol{\mu})$ are canonical coordinates, i.e. when $c(\boldsymbol{\lambda},\boldsymbol{\mu}) = I$, where I is n-dimensional identity matrix. It follows from the fact that for fixed $c(\boldsymbol{\lambda},\boldsymbol{\mu})$ we have the whole family of transformations

$$\bar{\lambda}_i = a(\lambda_i,\mu_i), \qquad \bar{\mu}_i = b(\lambda_i,\mu_i), \qquad i=1,...,n, \tag{36}$$

where $(\bar{\boldsymbol{\lambda}},\bar{\boldsymbol{\mu}})$ are canonical coordinates for Poisson tensor (21), i.e. they fulfil the condition (8) for each pair (λ_i,μ_i) of coordinates. Thus, each Liouville integrable Hamiltonian system (24), generated by an algebraic curve (19) and by the Poisson tensor (21) with a given $c(\lambda,\mu)$, can in fact be generated by a whole family (equivalence class) of algebraic curves $\varphi(\bar{\lambda},\bar{\mu},\bar{h}_1,...,\bar{h}_n) = 0$ and the corresponding Poisson tensors with $c(\bar{\lambda},\bar{\mu}) = 1$. Each class represents thus the same dynamical system written in different Darboux coordinates, related by appropriate canonical transformations.

Let us specify these considerations to the monomial case, following the case considered in Section 2. Consider thus the $2n$-dimensional Hamiltonian system (24) generated by the algebraic curve (19) and by the Poisson tensor (21) with $c(\lambda_i,\mu) = \lambda_i^{\alpha}\mu_i^{\beta}$, with fixed real α and β. Then, the transformation to its canonical (Darboux) coordinates on M and its inverse are of the form (cf. (14) and (15))

$$\bar{\lambda}_i = \lambda_i^{\bar{\alpha}_1}\mu_i^{\bar{\alpha}_2}, \quad \bar{\mu}_i = \lambda_i^{\bar{\beta}_1}\mu_i^{\bar{\beta}_2}, \quad \lambda_i = \bar{\lambda}_i^{\bar{\beta}_2}\bar{\mu}_i^{-\bar{\alpha}_2}, \quad \mu_i = \bar{\lambda}_i^{-\bar{\beta}_1}\bar{\mu}_i^{\bar{\alpha}_1} \quad i=1,...,n, \tag{37}$$

where $\bar{\alpha}_1, \bar{\alpha}_2, \bar{\beta}_1, \bar{\beta}_2$ are given either by (12) (for $\alpha \neq 1$, parameterized by $\bar{\alpha}_1$) or by (13) (for $\beta \neq 1$, parametrized by $\bar{\beta}_2$). As a result, our system can be equivalently obtained by either one of the equivalent curves

$$\varphi(\bar{\lambda}, \bar{\mu}, \bar{h}_1, ..., \bar{h}_n) = 0 \tag{38}$$

expressed in coordinates $(\bar{\lambda}, \bar{\mu})$ parametrized by $\bar{\alpha}_1$ (in case $\alpha \neq 1$) or by $\bar{\beta}_2$ (in case $\beta \neq 1$). A canonical transformation and its inverse between coordinates $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\mu}})$ and $(\tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}})$ associated with two curves from the class (38) parametrized by $\bar{\alpha}_1$ and by $\tilde{\alpha}_1$ respectively, is given by (cf. (16) and (18))

$$\bar{\lambda}_i = \tilde{\lambda}_i^{\alpha_1} \, \tilde{\mu}_i^{\alpha_1 - 1}, \quad \bar{\mu}_i = \tilde{\lambda}_i^{1-\alpha_1} \tilde{\mu}^{2-\alpha_1}, \quad \tilde{\lambda}_i = \bar{\lambda}_i^{2-\alpha_1} \bar{\mu}_i^{1-\alpha_1}, \quad \bar{\mu}_i = \bar{\lambda}_i^{\alpha_1 - 1} \bar{\mu}_i^{\alpha_1}, \quad i = 1, ..., n, \tag{39}$$

where

$$\alpha_1 = 1 + \frac{\bar{\alpha}_1 - \tilde{\alpha}_1}{1 - \alpha}.$$

Example 5. *Let us consider dynamical system (24), generated by the algebraic curve (31) but in the more general case when the Poisson tensor (4) is given by the monomial* $c(\lambda, \mu) = \lambda^m$, $m \in \mathbb{Z}$ *(so that* $\alpha = m$ *and* $\beta = 0$*) on the* (λ, μ)*-plane. Such system has one-parameter family of canonical representations in new coordinates, induced by the transformation*

$$\bar{\lambda} = \lambda^a \mu^{\frac{a-1}{1-m}}, \quad \bar{\mu} = \lambda^{1-m-a} \mu^{\frac{2-m-a}{1-m}}, \qquad a \in \mathbb{R}, \tag{40}$$

with the inverse

$$\lambda = \bar{\lambda}^{\frac{2-m-a}{1-m}} \, \bar{\mu}^{\frac{1-a}{1-m}}, \quad \mu = \bar{\lambda}^{a+m-1} \bar{\mu}^a$$

(where we now denote $\bar{\alpha}_1$*by* a*). Notice that for the distinguished choice* $a = 1$ *we have* $\bar{\lambda} = \lambda, \quad \bar{\mu} = \lambda^{-m}\mu$ *(so that* $\lambda = \bar{\lambda}, \quad \mu = \bar{\lambda}^m \bar{\mu}$*) and the algebraic curve (31) in the new variables* $(\bar{\lambda}, \bar{\mu})$ *is still of hyperelliptic type*

$$\sigma(\bar{\lambda}) + \sum_{k=1}^{n} \bar{h}_k \bar{\lambda}^{\gamma_k} = \frac{1}{2} \bar{f}(\bar{\lambda}) \bar{\mu}^2, \tag{41}$$

where $\bar{f}(\bar{\lambda}) = f(\bar{\lambda})\bar{\lambda}^{2m}$*, with the canonical Poisson tensor as* $c(\bar{\lambda}, \bar{\mu}) = 1$ *and thus generates a Stäckel system with all Hamiltonians that are again quadratic in momenta*

$$\bar{h}_k = \frac{1}{2} \bar{\boldsymbol{\mu}}^T A_k G_{\bar{f}} \bar{\boldsymbol{\mu}} + V_k^{(\sigma)}(\bar{\boldsymbol{\lambda}}), \qquad k = 1, \ldots, n. \tag{42}$$

For the choice $a = 0$

$$\bar{\lambda} = \mu^{\frac{1}{m-1}}, \quad \bar{\mu} = \lambda^{1-m} \mu^{\frac{2-m}{1-m}}, \qquad \lambda = \bar{\lambda}^{\frac{2-m}{1-m}} \bar{\mu}^{\frac{1}{1-m}}, \quad \mu = \bar{\lambda}^{m-1}, \tag{43}$$

so for the particular case $m = 2$

$$\bar{\lambda} = \mu, \quad \bar{\mu} = \lambda^{-1}, \qquad \lambda = \bar{\mu}^{-1}, \quad \mu = \bar{\lambda}, \tag{44}$$

we again obtained the hyperelliptic type curve, this time with interchanged roles of position and momenta variables:

$$\bar{\sigma}(\bar{\mu})+\sum_{k=1}^{n}\bar{h}_k\bar{\mu}^{\bar{\gamma}_k}=\frac{1}{2}\bar{f}(\bar{\mu})\bar{\lambda}^2,$$

where

$$\bar{\sigma}(\bar{\mu})=\sigma(\bar{\mu}^{-1})\ \bar{\mu}^{\gamma_1},\qquad \bar{f}(\bar{\mu})=f(\bar{\mu}^{-1})\ \bar{\mu}^{\gamma_1},\qquad \bar{\gamma}_k=\gamma_1-\gamma_k,\qquad k=1,...,n \tag{45}$$

and with the normalization $0=\bar{\gamma}_1<...<\bar{\gamma}_n$.

5 Stäckel transform and reciprocal link

In this section we will assume that the algebraic curve defining Hamiltonian system depends on a set of $n+n$, instead of just n, parameters. We show that solving this curve with respect to either the first set of n parameters or the second set of n parameters leads to two integrable systems that can be related by a Stäckel transform. We further show that solutions of these two systems are related by a reciprocal (multi-time) transformation. We further specify our results to Stäckel systems.

Consider thus a $2n$-parameter algebraic curve

$$\varphi(\lambda,\mu,a_1,...,a_n,b_1,...,b_n)=0 \tag{46}$$

and the corresponding separation relations

$$\varphi(\lambda_i,\mu_i,a_1,...,a_n,b_1,...,b_n)=0,\qquad i=1,\dots,n. \tag{47}$$

Solving these relations with respect to a_k (we assume it is possible at least in some open domain) we obtain n functions (Hamiltonians)

$$a_k=h_k(\xi,b_1,...,b_n),\qquad k=1,...,n, \tag{48}$$

considered on a $2n$-dimensional manifold M (parametrized by coordinates $\xi=(\boldsymbol{\lambda},\boldsymbol{\mu})$) and depending on n parameters $b_1,...,b_n$. These Hamiltonians define n Hamiltonians systems on M of the form

$$\xi_{t_i}=\pi dh_i\equiv X_i,\quad i=1,...,n, \tag{49}$$

where π is the canonical Poisson tensor of co-rank zero given by

$$\pi=\sum_{i=1}^{n}\frac{\partial}{\partial\lambda_i}\wedge\frac{\partial}{\partial\mu_i}$$

and where $t_1,\dots,t_n$ are respective evolution parameters. The system (47) (or equivalently (48)) is assumed to be also solvable (at least in some open domain) with respect to the parameters b_k yielding

$$b_k=\bar{h}_k(\xi,a_1,...,a_n),\qquad k=1,...,n, \tag{50}$$

i.e. new Hamiltonian functions $\bar{h}_k$ depending on n parameters $a_1, ..., a_n$. The related Hamiltonian systems take the form

$$\xi_{\bar{t}_i} = \pi d\bar{h}_i \equiv \bar{X}_i, \quad i = 1, ..., n, \tag{51}$$

where $\bar{t}_1, ..., \bar{t}_n$ are respective evolution parameters.

Note that inserting the Hamiltonians h_k into the separation curve (46) yields the following identity with respect to all $\xi \in M$ and all b_k

$$\varphi(\lambda, \mu, h_1(\xi, b_1, ..., b_n), ..., h_n(\xi, b_1, ..., b_n), b_1, ..., b_n) \equiv 0. \tag{52}$$

Similarly, inserting the Hamiltonians $\bar{h}_k$ into the separation curve (46) yields the following identity with respect to all $\xi \in M$ and all a_k

$$\varphi(\lambda, \mu, a_1, ..., a_n, \bar{h}_1(\xi, a_1, ..., a_n), ..., \bar{h}_n(\xi, a_1, ..., a_n)) \equiv 0. \tag{53}$$

Definition 6. *The n Stäckel Hamiltonians $\bar{h}_k$ and the n Stäckel Hamiltonians h_k are called* Stäckel conjugate Hamiltonians *and the procedure of mapping n Hamiltonians h_k to n Hamiltonians $\bar{h}_k$ is called* Stäckel transform*.*

Below we remind a theorem explaining the mutual geometric relations between the Stäckel conjugate Hamiltonians.

Theorem 7. *[8] For a given $2n$-tuple $(a, b) = (a_1, ..., a_n, b_1, \ldots, b_n)$ of real constants, on the n-dimensional submanifold*

$$M_{a,b} = \{\xi \in M : \varphi(\lambda_k, \mu_k, a, b) = 0,\ k = 1, ..., n\} \tag{54}$$

the following relations hold

$$dh = Ad\bar{h}, \qquad X = A\bar{X}, \qquad A_{ij} = -\frac{\partial h_i}{\partial b_j}, \quad i, j = 1, \ldots, n, \tag{55}$$

where $dh = (dh_1, ..., dh_n)^T$, $d\bar{h} = (d\bar{h}_1, ..., d\bar{h}_n)^T$, $X = (X_1, ..., X_n)^T$, $\bar{X} = (\bar{X}_1, ..., \bar{X}_n)^T$.

Note that $M_{a,b}$ can equivalently be defined as

$$\begin{aligned} M_{a,b} &= \{\xi \in M : h_k(\xi, b_1, ..., b_n) = a_k,\ k = 1, ..., n\} \\ &= \{\xi \in M : \bar{h}_k(\xi, a_1, ..., a_n) = b_k,\ k = 1, ..., n\} \end{aligned}$$

and that through each point of M there pass infinitely many manifolds $M_{a,b}$. In fact, fixing all the parameters b_k in $M_{a,b}$ and letting a_k vary we obtain a particular foliation of M and, likewise, fixing all the parameters a_k in $M_{a,b}$ and letting b_k vary we obtain another particular foliation of M. Note also that each of the manifolds $M_{a,b}$ is invariant with respect to all n systems (49) and all n systems (51) which also means that all the vector fields X_i and all the vector fields $\bar{X}_i$ are tangent to each manifold $M_{a,b}$. Note that no relation exists between the vector fields X and $\bar{X}$ on the whole manifold M. Let us also remark that the transformations (55) on $M_{a,b}$ can be inverted, yielding

$$d\bar{h} = A^{-1}dh, \qquad \bar{X} = A^{-1}X, \ \text{ with } \left(A^{-1}\right)_{ij} = -\frac{\partial \bar{h}_i}{\partial a_j}, \quad i, j = 1, \ldots, n. \tag{56}$$

The second relation in (55) can be reformulated in the dual language, that of the reciprocal multi-time transformations. The reciprocal transformation

$$\bar{t}_i = \bar{t}_i(t_1, ..., t_n, \xi_0), \qquad i = 1, ..., n,$$

given on $M_{b,a}$ by

$$d\bar{t} = A^T dt, \tag{57}$$

where $dt = (dt_1, ..., dt_n)$ and $d\bar{t} = (d\bar{t}_1, ..., d\bar{t}_n)$, transforms the n-parameter solutions $\xi(t_1, ..., t_n, \xi_0)$ of the system (49) (where ξ_0 is the initial condition) to the n-parameter solutions $\bar{\xi}(\bar{t}_1, ..., \bar{t}_n, \xi_0)$ of the system (51). Transformation (57) is well defined as its r.h.s is an exact differential.

Definition 8. *The transformation (57) between dynamical systems (49) and (51) on $M_{a,b}$ is called a n-parameter reciprocal transform.*

In the remaining part of this section, we will restrict ourselves to the case of curves (46) that are affine in all the parameters a_i and b_i. In this case the relations (48) take the form

$$a_k = h_k(\xi, b_1, ..., b_n) \equiv H_k + \sum_{j=1}^{n} H_k^{(j)} b_j, \qquad k = 1, ..., n, \tag{58}$$

while the relations (50) attain the form

$$b_k = \bar{h}_k(\xi, a_1, ..., a_n) = \bar{H}_k + \sum_{j=1}^{n} \bar{H}_k^{(j)} a_j, \qquad k = 1, ..., n. \tag{59}$$

The Stäckel transform between the Hamiltonians h_k and $\bar{h}_k$ takes the explicit matrix form

$$h = A(\bar{H} - b) \text{ or } \bar{h} = A^{-1}(H - a), \tag{60}$$

where $h = (h_1, ..., h_n)^T$, $b = (b_1, ..., b_n)^T$, $H = (H_1, ..., H_n)^T$, $\bar{h} = (\bar{h}_1, \ldots, \bar{h}_n)^T$, $a = (a_1, ..., a_n)^T$ and $A_{ij} = -\frac{\partial h_i}{\partial b_j} = -H_i^{(j)}$. Note that after setting all the a_i and b_i equal to zero we obtain the following matrix formula relating Hamiltonians H_k and $\bar{H}_k$

$$H = A\bar{H}, \tag{61}$$

where $\bar{H} = (\bar{H}_1, ..., \bar{H}_n)^T$, valid on the whole M. Formula (61) is the parameter-independent part of the Stäckel transform between h_k and $\bar{h}_k$.

Consider now a specification of the above affine case when the separation curve (46) attains the following hyperelliptic-type form

$$\sigma(\lambda) + \sum_{j=1}^{n} b_j \lambda^{\gamma_j} + \sum_{k=1}^{n} a_k \lambda^{n-k} = \frac{1}{2} f(\lambda)\mu^2, \tag{62}$$

where $\sigma(\lambda)$ and $f(\lambda)$ are Laurent polynomials in λ, $\gamma_1 > ... > \gamma_n$ are natural numbers. Solving the corresponding separation relations with respect to a_k yields the separable systems belonging to the Benenti subclass of Stäckel systems. Explicitly, we obtain n quadratic in momenta Stäckel Hamiltonians

$$h_k = H_k + \sum_{j=1}^{n} b_j \mathcal{V}_k^{(\gamma_j)} = \frac{1}{2}\mu^T A_k G_f \mu + V_k^{(\sigma)} + \sum_{j=1}^{n} b_j \mathcal{V}_k^{(\gamma_j)}, \qquad k = 1, \ldots, n. \tag{63}$$

The structure and geometric meaning of the Hamiltonians h_k is as those described in subsection 3.2.

Performing the Stäckel transform on the set of n Hamiltonians (63) we obtain the set of n Hamiltonians $\bar{h}_k$ of the form

$$\bar{h}_k = \bar{H}_k + \sum_{j=1}^{n} a_j \bar{\mathcal{V}}_k^{(n-j)} = \frac{1}{2}\mu^T \bar{A}_k \bar{G}_f \mu + \bar{V}_k^{(\sigma)} + \sum_{j=1}^{n} a_j \bar{\mathcal{V}}_k^{(n-j)}, \qquad k = 1, \ldots, n, \tag{64}$$

(where $\bar{G}_f$ is defined by $\bar{H}_1$ with $\bar{A}_1 = I$) generated by the separation curve

$$\sigma(\lambda) + \sum_{j=1}^{n} a_j \lambda^{n-j} + \sum_{k=1}^{n} \bar{h}_k \lambda^{\gamma_k} = \frac{1}{2} f(\lambda) \mu^2. \tag{65}$$

The Hamiltonians $\bar{h}_k$ define the Hamiltonian evolution equations (51). Then, on n-dimensional submanifold (54) the relations (55) hold with

$$A_{kj} = -\frac{\partial h_k}{\partial b_j} = -\mathcal{V}_k^{(\gamma_j)} \tag{66}$$

and the relations (56) hold with

$$(A^{-1})_{kj} = -\frac{\partial \bar{h}_k}{\partial a_j} = -\bar{\mathcal{V}}_k^{(n-j)}.$$

Remark 9. *From the above considerations it follows that systems generated by algebraic curves (65) can always be transformed, by an appropriate reciprocal transformation, to systems from Benenti class, generated by algebraic curves (62).*

The Stäckel systems generated by curves of the type (65) have been thoroughly studied in [4].

Example 10. *Consider the algebraic curve (62) of the form*

$$b_1\lambda^2 + a_1\lambda + b_2 + a_2 = \frac{1}{2}\lambda\mu^2 + \lambda^4. \tag{67}$$

Solving the corresponding separation relations with respect to a_k we obtain Hamiltonians h_1 and h_2 as in (58). In Viéte coordinates (q, p), associated with separable coordinates (λ, μ) through the point transformation

$$q_1 = -\lambda_1 - \lambda_2, \qquad q_2 = \lambda_1\lambda_2,$$
$$p_1 = \frac{1}{\lambda_2 - \lambda_1}(\lambda_1\mu_1 - \lambda_2\mu_2), \quad p_2 = \frac{1}{\lambda_2 - \lambda_1}(\mu_1 - \mu_2),$$

the Hamiltonians h_k attain the form

$$h_1 = \frac{1}{2}p_1^2 - \frac{1}{2}q_2p_2^2 - q_1^3 + 2q_1q_2 + b_1q_1,$$
$$h_2 = -q_2p_1p_2 - \frac{1}{2}q_1q_2p_2^2 - q_1^2q_2 + q_2^2 + b_1q_2 - b_2.$$

Passing to flat coordinates [5] (x, y) defined through the point transformation

$$q_1 = -x_1,\; q_2 = -\frac{1}{4}x_2^2,\;\; p_1 = -y_1,\;\; p_2 = -\frac{2}{x_2}y_2, \tag{68}$$

we obtain h_k in the form

$$h_1 = \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + x_1^3 + \frac{1}{2}x_1x_2^2 - b_1x_1, \tag{69}$$
$$h_2 = \frac{1}{2}x_2y_1y_2 - \frac{1}{2}x_1y_2^2 + \frac{1}{4}x_1^2x_2^2 + \frac{1}{16}x_2^4 - \frac{1}{4}b_1x_2^2 - b_2.$$

Note that for $b_1 = b_2 = 0$ the Hamiltonians h_1 and h_2 represent one of the integrable cases of Hénon-Heiles systems [12]. The matrix A in (66) and its inverse attain in the (x, y)-variables the form

$$A = \begin{pmatrix} x_1 & 0 \\ \frac{1}{4}x_2^2 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \frac{1}{x_1} & 0 \\ -\frac{1}{4x_1}x_2^2 & 1 \end{pmatrix}.$$

Solving the separation relations corresponding to (67) with respect to b_k yields the Hamiltonians $\bar{h}_1, \bar{h}_2$ that in the flat coordinates (68) attain the form

$$\bar{h}_1 = \frac{1}{2x_1}y_1^2 + \frac{1}{2x_1}y_2^2 + x_1^2 + \frac{1}{2}x_2^2 - \frac{a_1}{x_1},$$
$$\bar{h}_2 = -\frac{x_2^2}{8x_1}y_1^2 + \left(-\frac{1}{2}x_1 - \frac{1}{8}\frac{x_2^2}{x_1}\right)y_2^2 + \frac{1}{2}x_2y_1y_2 - \frac{1}{16}x_2^4 + \frac{1}{4}a_1\frac{x_2^2}{x_1} - a_2.$$

The Hamiltonians $\bar{h}_1, \bar{h}_2$ and the Hamiltonians h_1, h_2 are Stäckel conjugate. Note that the variables (68) are only conformally flat for the Hamiltonians $\bar{h}_1, \bar{h}_2$. The manifolds $M_{a,b}$ are given by

$$M_{a,b} = \{(x, y) : h_1(x, y, b_1, b_2) = a_1, \quad h_2(x, y, b_1, b_2) = a_2\}$$

or by

$$M_{a,b} = \left\{(x, y) : \bar{h}_1(x, y, a_1, a_2) = b_1, \quad \bar{h}_2(x, y, a_1, a_2) = b_2\right\}$$

and one can verify by a direct computation that on $M_{a,b}$ we have $X = A\bar{X}$ as well as $\bar{X} = A^{-1}X$. The corresponding reciprocal transformation (57) between the evolution parameters takes the form

$$d\bar{t}_1 = x_1dt_1 + \frac{1}{4}x_2^2dt_2,\;\; d\bar{t}_2 = dt_2.$$

Stäckel transform was first described by J. Hietarinta et al in [13] (where it was called the coupling-constant metamorphosis) and in [10]. In this early approach this transform was only one-parameter. In its most general form Stäckel transform has been introduced in [15] and then intensively studied in [8, 9].

6 Miura maps

In this section we investigate yet another possibility of generating integrable and separable Hamiltonian systems from algebraic curves. We will consider algebraic curves depending on $n+N$ parameters having a certain block-type structure. These curves generate integrable and separable Hamiltonian systems that can be connected by a finite-dimensional analogue of Miura maps, known from soliton theory (Theorem 11, see also its proof in the Appendix). These finite-dimensional Miura maps yield in turn multi-Hamiltonian formulation of the obtained integrable systems (Theorem 12). Results of this section generalize the results for the one-block case, obtained earlier in [14] as well as the results obtained in [6].

Consider thus the $(n+N)$-parameter algebraic curve

$$\varphi(\lambda,\mu,a_1,...,a_n,c_1,...,c_N)=0, \tag{70}$$

with $1\leq N\leq n$, in the following form

$$\varphi_0(\lambda,\mu)+\sum_{k=1}^{m}\varphi_k(\lambda,\mu)\psi_k(\lambda,a_1^{(k)},...,a_{n_k}^{(k)},c_1^{(k)},...,c_\alpha^{(k)})=0,\quad n_1+...+n_m=n,\quad \alpha m=N, \tag{71}$$

where $1\leq\alpha\leq\min(n_k)$ and where

$$\psi_k(\lambda,a_1^{(k)},...,a_{n_k}^{(k)},c_1^{(k)},...,c_\alpha^{(k)})=c_\alpha^{(k)}\lambda^{n_k-1+\alpha}+...+c_1^{(k)}\lambda^{n_k}+\sum_{i=1}^{n_k}a_i^{(k)}\lambda^{n_k-i} \tag{72}$$

with the normalization $\varphi_m(\lambda,\mu)=1$. The curve (71) consists thus of m blocks of Benenti type. Solving the related separation relations with respect to $a_i^{(k)}$ we obtain n Hamiltonian functions

$$a_i^{(k)}=h_i^{(k)}(\xi,c_1^{(1)},...,c_\alpha^{(m)}),\qquad i=1,...,n_k,\quad k=1,...,m \tag{73}$$

on a $2n$-dimensional open submanifold $M\subset\boldsymbol{R}^{2n}$ parametrized by $\xi=(\boldsymbol{\lambda},\boldsymbol{\mu})$. Assume now that $\boldsymbol{c}=\left(c_1^{(1)},\ldots,c_\alpha^{(1)},\ldots,c_1^{(m)},\ldots,c_\alpha^{(m)}\right)$ are additional coordinates on the $(2n+N)$-dimensional open submanifold $\mathcal{M}\subset\boldsymbol{R}^{2n+\alpha m}$, parametrized by $(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{c})$. The Hamiltonians $h_i^{(k)}$ in (73) generate n dynamical Hamiltonian systems (a Stäckel system) on $\mathcal{M}$, given by

$$\xi_{t_i^{(k)}}=\pi_0 dh_i^{(k)}(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{c})\equiv X_i^{(k)},\quad i=1,...,n_k,\quad k=1,...,m, \tag{74}$$

where $\xi\in\mathcal{M}$, π_0 is the canonical Poisson tensor

$$\pi_0=\sum_{i=1}^{n}\frac{\partial}{\partial\lambda_i}\wedge\frac{\partial}{\partial\mu_i}$$

of co-rank $\alpha m=N$ on $\mathcal{M}$ and $c_1^{(1)},...,c_\alpha^{(m)}$ are its Casimir functions.

The goal of this section is to construct a Miura map between the Stäckel system (74), generated by the curve (71), (72), and the Stäckel system

$$\xi_{t_i^{(k)}} = \bar{\pi} d\bar{h}_r^{(k)}(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{c}}) \equiv \bar{X}_i^{(k)}, \quad i = 1, ..., n_k, \quad k = 1, ..., m, \tag{75}$$

where

$$\bar{\pi} = \textstyle\sum_{i=1}^n \frac{\partial}{\partial \bar{\lambda}_i} \wedge \frac{\partial}{\partial \bar{\mu}_i}, \tag{76}$$

generated by the curve

$$\varphi_0(\bar{\lambda}, \bar{\lambda}^s \bar{\mu})\bar{\lambda}^{-s} + \sum_{k=1}^m \varphi_k(\bar{\lambda}, \bar{\mu}\bar{\lambda}^s)\psi_k(\bar{\lambda}, \bar{h}_1^{(k)}, ..., \bar{h}_{n_k}^{(k)}, \bar{c}_1^{(k)}, ..., \bar{c}_\alpha^{(k)}) = 0, \tag{77}$$

with $n_1 + ... + n_m = n, \alpha \cdot m = N$, where

$$\begin{aligned} &\psi_k(\bar{\lambda}, \bar{h}_1^{(k)}, ..., \bar{h}_{n_k}^{(k)}, \bar{c}_1^{(k)}, ..., \bar{c}_\alpha^{(k)}) \\ &= \bar{c}_\alpha^{(k)} \bar{\lambda}^{n_k+\alpha-s-1} + \ldots + \bar{c}_{s+1}^{(k)} \bar{\lambda}^{n_k} + \sum_{i=1}^{n_k} \bar{h}_i^{(k)} \bar{\lambda}^{n_k-i} + \bar{c}_s^{(k)} \bar{\lambda}^{-1} + \ldots + \bar{c}_1^{(k)} \bar{\lambda}^{-s}, \end{aligned} \tag{78}$$

s is an integer such that $1 \le s \le \alpha$ and where the coordinates $(\bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{c}}) = (\bar{\lambda}_i, \bar{\mu}_i, \bar{c}_1^{(1)}, \ldots, \bar{c}_\alpha^{(1)}, \ldots, \bar{c}_1^{(m)}, \ldots, \bar{c}_\alpha^{(m)})_{i=1,\ldots,n}$ on $\mathcal{M}$ are some functions of coordinates $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{c})$.

Consider the following map in $\boldsymbol{R}^2$:

$$\bar{\lambda} = \lambda, \quad \bar{\mu} = \lambda^{-s}\mu. \tag{79}$$

This map transforms (algebraically) the curve (71) into the curve (77), provided that for all $k = 1, ..., m$

$$\begin{array}{ll} \bar{c}_i^{(k)} = h_{n_k-i+1}^{(k)}, & i = 1, \ldots, s \\ \bar{c}_i^{(k)} = c_i^{(k)}, & i = s+1, \ldots, \alpha \\ \bar{h}_i^{(k)} = c_{s-i+1}^{(k)}, & i = 1, \ldots, s \\ \bar{h}_i^{(k)} = h_{i-s}^{(k)}, & i = s+1, \ldots, n_k, \end{array} \tag{80}$$

The relations (80) can be inverted to

$$\begin{array}{ll} c_i^{(k)} = \bar{c}_i^{(k)}, & i = s+1, \ldots, \alpha \\ c_i^{(k)} = \bar{h}_{s-i+1}^{(k)}, & i = 1, \ldots, s \\ h_i^{(k)} = \bar{h}_{s+i}^{(k)}, & i = 1, \ldots, n_k - s \\ h_i^{(k)} = \bar{c}_{n-i+1}^{(k)}, & i = n_k - s + 1, \ldots, n_k. \end{array} \tag{81}$$

The maps (79) and (80) induce the following Miura maps $\mathfrak{M} : \mathcal{M} \to \mathcal{M}$

$$\begin{array}{ll} \bar{\lambda}_i = \lambda_i, & i = 1, \ldots, n \\ \bar{\mu}_i = \lambda_i^{-s}\mu_i, & i = 1, \ldots, n \\ \bar{c}_i^{(k)} = h_{n_k-i+1}^{(k)}(\lambda, \mu, c), & i = 1, \ldots, s \\ \bar{c}_i^{(k)} = c_i^{(k)}, & i = s+1, \ldots, \alpha, \end{array} \tag{82}$$

with the inverse $\mathfrak{M}^{-1}:\mathcal{M}\rightarrow\mathcal{M}$

$$\begin{array}{ll} \lambda_i=\bar{\lambda}_i, & i=1,\dots,n \\ \mu_i=\bar{\lambda}_i^s\bar{\mu}_i, & i=1,\dots,n \\ c_i^{(k)}=\bar{h}_{s-i+1}^{(k)}(\bar{\lambda},\bar{\mu},\bar{c}), & i=1,\dots,s \\ c_i^{(k)}=\bar{c}_i^{(k)}, & i=s+1,\dots,\alpha. \end{array} \tag{83}$$

Let us now present the main theorem of this section.

Theorem 11. *For any $s\in\{1,\dots,\alpha\}$ the n Hamiltonian vector fields $X_i^{(k)}$ in (74) and the n Hamiltonian vector fields $\bar{X}_i^{(k)}$ in (75) pairwise coincide, provided that the coordinates $(\bar{\boldsymbol{\lambda}},\bar{\boldsymbol{\mu}},\bar{\boldsymbol{c}})$ and $(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{c})$ are connected by the Miura map (82):*

$$X_i^{(k)}=\bar{X}_i^{(k)},\quad i=1,\dots,n_k,\quad k=1,\dots,m.$$

The proof of this theorem can be found in Appendix. This theorem means that all the Stäckel systems (75), generated by the curves (77) (one for each value of s between 1 and α), represent on the extended phase space $\mathcal{M}$ the same Stäckel system as the Stäckel system (74), (the one generated by the curve (71)), written in different coordinates, connected by the corresponding invertible Miura maps (82). We can thus call the Stäckel system (75) an s-representation of the Stäckel system (74), with $s=0,\dots,\alpha$ (where $s=0$-representation means simply the original Stäckel system (74)). Since all the Miura maps (82) are invertible it also means that there exists direct Miura maps (appropriate compositions of (82) and (83)) between different s-representations of our Stäckel system, see [14].

An important consequence of the above construction is the following theorem, that generalizes the corresponding one-block theorem from [14].

Theorem 12. *The Stäckel system (74) is $(\alpha+1)$-Hamiltonian, i.e. for all $s=0,\dots,\alpha$ (and for all $k=1,\dots,m$)*

$$\begin{aligned} X_i^{(k)}&=\pi_0 dh_i^{(k)}=\pi_s d^{(k)}c_{s-i+1},\quad i=1,\dots s, \\ X_i^{(k)}&=\pi_0 dh_i^{(k)}=\pi_s dh_{i-s}^{(k)},\quad i=s+1,\dots n_k, \end{aligned} \tag{84}$$

where

$$\pi_s=\sum_{i=1}^n\lambda_i^s\frac{\partial}{\partial\lambda_i}\wedge\frac{\partial}{\partial\mu_i}\;+\sum_{k=1}^m\sum_{j=1}^s X_j^{(k)}\wedge\frac{\partial}{\partial c_{s-j+1}^{(k)}},\qquad s=0,...,\alpha. \tag{85}$$

Thus, the matrix representation of π_s in the variables $(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{c})$ is given by the following $(n+\alpha m)\times(n+\alpha m)$ matrix:

$$\pi_s(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{c})=\left(\begin{array}{cccccccc} \begin{matrix} 0 & \Lambda^s \\ -\Lambda^s & 0 \end{matrix} & X_s^{(1)}\dots X_1^{(1)} & \overbrace{0...0}^{\alpha-s} & \cdots & X_s^{(m)}\dots X_1^{(m)} & \overbrace{0...0}^{\alpha-s} \\ * & & & & & \end{array}\right), \tag{86}$$

where $\Lambda=\mathrm{diag}(\lambda_1,\dots,\lambda_n)$, $X_i^{(k)}$ denote here the columns consisting of *components* of the vector field $X_i^{(k)}$ in the coordinates $(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{c})$ and where $*$ denotes transpositions of the corresponding $X_i^{(k)}$. The proof of this theorem is obtained by a direct computation of the Poisson operator $\bar{\pi} = \sum_{i=1}^n \partial_{\bar{\lambda}_i} \wedge \partial_{\bar{\mu}_i}$, for each and every case $s = 1, \ldots, \alpha$, in the variables $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{c})$ associated with the $s = 0$ representation (for the proof of the one-block version of this theorem, see [14]).

Using the notation

$$h_j^{(k)} \equiv c_{1-j}^{(k)} \quad \text{for } j = 0, -1, \ldots, -\alpha + 1,$$

we can write the formulas (84) in the more compact form

$$X_i^{(k)} = \pi_s dh_{i-s}^{(k)}, \quad s = 0, \ldots, \alpha. \tag{87}$$

Using this notation, we can formulate the following corollary.

Corollary 13. *The multi-Hamiltonian representations (87) generate, for each $k \in \{1, \ldots, m\}$, $\binom{\alpha+1}{2}$ bi-Hamiltonian chains*

$$\begin{array}{lll} \pi_i dh_{-i}^{(k)} & = 0 & \\ \pi_i dh_{-i+1}^{(k)} & = X_1^{(k)} & = \pi_j dh_{-j+1}^{(k)} \\ & \vdots & \\ \pi_i dh_{-i+r}^{(k)} & = X_r^{(k)} & = \pi_j dh_{-j+r}^{(k)} \\ & \vdots & \\ \pi_i dh_{-i+n_k}^{(k)} & = X_{n_k}^{(k)} & = \pi_j dh_{-j+n_k}^{(k)} \\ & \quad\; 0 & = \pi_j dh_{-j+n_k+1}^{(k)} \end{array} \tag{88}$$

for $0 \leq i < j \leq \alpha$.

There are two limit cases of the above construction. The first one is the case when $m = n$ (so that all blocks have length one: $n_k = 1$ for all $k = 1, \ldots, m$). Then, the vector fields $X_i^{(k)}$ in (74) on $\mathcal{M}$ are only bi-Hamiltonian, forming n one-field chains

$$\begin{array}{lll} \pi_0 dc_1^{(k)} & = \; 0 & \\ \pi_0 dh_1^{(k)} & = X_1^{(k)} & = \pi_1 dc_1^{(k)} \;, \qquad k = 1, \ldots, n, \\ & \quad\; 0 & = \pi_1 dh_1^{(k)} \end{array}$$

where

$$\pi_0 = \sum_{i=1}^n \frac{\partial}{\partial \lambda_i} \wedge \frac{\partial}{\partial \mu_i}, \quad \pi_1 = \sum_{i=1}^n \lambda_i \frac{\partial}{\partial \lambda_i} \wedge \frac{\partial}{\partial \mu_i} + \sum_{k=1}^n X_1^{(k)} \wedge \frac{\partial}{\partial c_1^{(k)}}.$$

This particular situation was considered in [6]. The opposite limit case takes place when $m = 1$ (i.e. when there is only one block in the curve (71)). Then, the considered Stäckel system is $(n+1)$-Hamiltonian

$$X_i = \pi_s dh_{i-s}, \quad s = 0, \ldots, n,$$

with

$$\pi_0=\sum_{i=1}^{n}\frac{\partial}{\partial\lambda_i}\wedge\frac{\partial}{\partial\mu_i},\quad \pi_s=\sum_{i=1}^{n}\lambda_i^s\frac{\partial}{\partial\lambda_i}\wedge\frac{\partial}{\partial\mu_i}\ +\sum_{j=1}^{s}X_j\wedge\frac{\partial}{\partial c_{s-j+1}},\qquad s=0,...,n$$

(and with the notation $X_i^{(1)}\equiv X_i,\ i=1,...,n$). In this case there is in total $\binom{N+1}{2}$ bi-Hamiltonian chains of the form (88), where $1\le N\le n$. This situation was considered in [14].

Example 14. *Consider the special case of curve (67) from Example 10 with $b_1=b_2=0$ but in the space extended by the coordinates c_1 and c_2:*

$$c_2\lambda^3+c_1\lambda^2+h_1\lambda_1+h_2=\frac{1}{2}\lambda\mu^2+\lambda^4. \tag{89}$$

This curve has a form of (71)-(72) with $n=2,m=1$ (so it is a one-block case), $\alpha=2$ (so that $N=2$), $\varphi_0=-\frac{1}{2}\lambda\mu^2-\lambda^4$, $\varphi_1=1$. The Miura map (82) for $s=1$ attains the form

$$\bar\lambda_1=\lambda_1,\quad \bar\lambda_2=\lambda_2,\quad \bar\mu_1=\lambda_1^{-1}\mu_1,\quad \bar\mu_2=\lambda_2^{-1}\mu_2,\ \ \bar c_1=h_2(\lambda,\mu,c),\ \ \bar c_2=c_2$$

and it transforms the Stäckel system generated by the curve (89) to the Stäckel system generated by the curve

$$\bar c_2\bar\lambda^2+\bar h_1\bar\lambda+\bar h_2+\bar c_1\bar\lambda^{-1}=\frac{1}{2}\bar\lambda^2\bar\mu^2+\bar\lambda^3. \tag{90}$$

Further, for $s=2$ the Miura map (82) attains the form

$$\widetilde\lambda_1=\lambda_1,\ \widetilde\lambda_2=\lambda_2,\quad \widetilde\mu_1=\lambda_1^{-2}\mu_1,\ \widetilde\mu_2=\lambda_2^{-2}\mu_2,\ \ \widetilde c_1=h_2(\lambda,\mu,c),\ \widetilde c_2=h_1(\lambda,\mu,c)$$

(in this example we have to distinguish between the two set of "bar" variables, one for $s=1$ and for $s=2$ so in the latter case we use the variables $\left(\widetilde\lambda,\widetilde\mu,\widetilde c\right)$) and it transforms the Stäckel system generated by the curve (89) to the Stäckel system generated by the curve

$$\widetilde h_1\widetilde\lambda+\widetilde h_2+\ \widetilde c_2\widetilde\lambda^{-1}+\widetilde c_1\widetilde\lambda^{-2}=\frac{1}{2}\widetilde\lambda^3\widetilde\mu^2+\widetilde\lambda. \tag{91}$$

All these three Stäckel systems are three different s-representations (with $s=0,1$ and 2) of the same three-Hamiltonian Stäckel system, with its three Poisson tensors (86) having in the $(\lambda,\mu,\mathbf{c})$-variables the form

$$\pi_0(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{c})=\begin{pmatrix}\begin{matrix}0 & I\\ -I & 0\end{matrix} & \begin{matrix}0 & 0\end{matrix}\\ * & \end{pmatrix},$$

$$\pi_1(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{c})=\begin{pmatrix}\begin{matrix}0 & \Lambda\\ -\Lambda & 0\end{matrix} & \begin{matrix}X_1 & 0\end{matrix}\\ * & \end{pmatrix},$$

$$\pi_2(\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{c})=\begin{pmatrix}\begin{matrix}0 & \Lambda^2\\ -\Lambda^2 & 0\end{matrix} & \begin{matrix}X_2 & X_1\end{matrix}\\ * & \end{pmatrix},$$

where $I = \operatorname{diag}(1,1)$, $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2)$. *Let us write these objects explicitly in the flat coordinates (68). The two Stäckel Hamiltonians* h_1 *and* h_2 *have in these coordinates the form*

$$\begin{aligned} h_1 &= \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 + x_1^3 + \frac{1}{2}x_1x_2^2 - c_2\left(x_1^2 + \frac{1}{4}x_2^2\right) - c_1x_1, \\ h_2 &= \frac{1}{2}x_2y_1y_2 - \frac{1}{2}x_1y_2^2 + \frac{1}{4}x_1^2x_2^2 + \frac{1}{16}x_2^4 - \frac{1}{4}c_2x_1x_2^2 - \frac{1}{4}c_1x_2^2 \end{aligned}$$

while the matrix representations of the Poisson operators π_0, π_1 *and* π_2 *attain the explicit form*

$$\pi_0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{c}) = \begin{pmatrix} \begin{matrix} 0 & I \\ -I & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} \\ * & \end{pmatrix},$$

$$\pi_1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{c}) = \begin{pmatrix} \begin{matrix} 0 & 0 & x_1 & \frac{1}{2}x_2 \\ 0 & 0 & \frac{1}{2}x_2 & 0 \\ -x_1 & -\frac{1}{2}x_2 & 0 & \frac{1}{2}y_2 \\ -\frac{1}{2}x_2 & 0 & -\frac{1}{2}y_2 & 0 \end{matrix} & \begin{matrix} X_1 & 0 \end{matrix} \\ * & \end{pmatrix},$$

$$\pi_2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{c}) = \begin{pmatrix} \begin{matrix} 0 & 0 & x_1^2 + \frac{1}{4}x_2^2 & \frac{1}{2}x_1x_2 \\ 0 & 0 & \frac{1}{2}x_1x_2 & \frac{1}{4}x_2^2 \\ -x_1^2 - \frac{1}{4}x_2^2 & -\frac{1}{2}x_1x_2 & 0 & \frac{1}{2}x_1y_2 \\ -\frac{1}{2}x_1x_2 & -\frac{1}{4}x_2^2 & -\frac{1}{2}x_1y_2 & 0 \end{matrix} & \begin{matrix} X_2 & X_1 \end{matrix} \\ * & \end{pmatrix},$$

where the components of the vector fields X_1 *and* X_2 *are given by*

$$X_1 = \left(y_1, y_2, -3x_1^2 - \frac{1}{2}x_2^2 + 2c_2x_1 + c_1, -x_1x_2 + \frac{1}{2}c_2x_2, 0, 0\right)^T,$$

$$\begin{aligned} X_2 = \Bigg(&\frac{1}{2}x_2y_2, \frac{1}{2}x_2y_1 - x_1y_2, \frac{1}{2}y_2^2 - \frac{1}{2}x_1x_2^2 + \frac{1}{4}c_2x_2^2, \\ &-\frac{1}{2}y_1y_2 - \frac{1}{2}x_1^2x_2 - \frac{1}{8}x_2^3 + \frac{1}{2}c_2x_1x_2 + \frac{1}{2}c_1x_2, 0, 0\Bigg)^T, \end{aligned}$$

while the corresponding bi-Hamiltonian chains are

$$\begin{array}{lll} \pi_0 dc_1 = 0 & \pi_0 dc_1 = 0 & \pi_1 dc_2 = 0 \\ \pi_0 dh_1 = X_1 = \pi_1 dc_1 & \pi_0 dh_1 = X_1 = \pi_2 dc_2 & \pi_1 dc_1 = X_1 = \pi_2 dc_2 \\ \pi_0 dh_2 = X_2 = \pi_1 dh_1 & \pi_0 dh_2 = X_2 = \pi_2 dc_1 & \pi_1 dh_1 = X_2 = \pi_2 dc_1 \\ \qquad 0 = \pi_1 dh_2 & \qquad 0 = \pi_2 dh_1 & \qquad 0 = \pi_2 dh_1 \end{array}.$$

Example 15. *Consider now another specification of the curve (67) from Example 10, this time with* $a_1 = a_2 = 0$ *and in the space extended by the coordinates* $c^{(1)}$ *and* $c^{(2)}$. *More specifically, we consider the curve*

$$\lambda^2\left(c_1^{(2)}\lambda + h_1^{(2)}\right) + c_1^{(1)}\lambda + h_1^{(1)} = \frac{1}{2}\lambda\mu^2 + \lambda^4.$$

This curve has the form of (71)-(72) with $n = 2, m = 2$. Thus, it is a two-block case, with $n_1 = n_2 = 1$, $\alpha = 1$ (so that $N = 2$), $\varphi_0 = -\frac{1}{2}\lambda\mu^2 - \lambda^4$, $\varphi_1 = 1$ and $\varphi_2 = \lambda^2$. To simplify the notation, let us denote $h_1 = h_1^{(2)}$,$h_2 = h_1^{(1)}$, $c_1 = c_1^{(2)}$, $c_2 = c_1^{(1)}$ so that the curve is

$$\lambda^2 (c_1\lambda + h_1) + c_2\lambda + h_2 = \frac{1}{2}\lambda\mu^2 + \lambda^4. \tag{92}$$

The Miura map (82) for $s = 1$ attains the form

$$\bar{\lambda}_1 = \lambda_1, \quad \bar{\lambda}_2 = \lambda_2, \quad \bar{\mu}_1 = \lambda_1^{-1}\mu_1, \quad \bar{\mu}_2 = \lambda_2^{-1}\mu_2, \quad \bar{c}_1 = h_1(\lambda, \mu, c), \quad \bar{c}_2 = h_2(\lambda, \mu, c)$$

and it transforms the Stäckel system generated by the curve (92) to the Stäckel system generated by the curve

$$\bar{\lambda}^2 \left(\bar{h}_1 + \bar{c}_1\bar{\lambda}^{-1}\right) + \bar{h}_2 + \bar{c}_2\bar{\lambda}^{-1} = \frac{1}{2}\bar{\lambda}^2\bar{\mu}^2 + \bar{\lambda}^3. \tag{93}$$

Both Stäckel systems are two different s-representations (with $s = 0{,}1$) of the same bi-Hamiltonian Stäckel system. Two Poisson tensors (86) have in the $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{c})$-variables the form

$$\pi_0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{c}) = \left(\begin{array}{cc} \begin{matrix} 0 & I \\ -I & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} \\ * & \end{array}\right),$$

$$\pi_1(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{c}) = \left(\begin{array}{cc} \begin{matrix} 0 & \Lambda \\ -\Lambda & 0 \end{matrix} & \begin{matrix} X_1 & X_2 \end{matrix} \\ * & \end{array}\right),$$

where X_i denote here the components of the vector fields $\pi_0 dh_i$. Let us write down these objects explicitly in the conformally flat coordinates (68). The Stäckel Hamiltonians h_1 and h_2 are

$$\begin{aligned} h_1 &= \frac{1}{2x_1}y_1^2 + \frac{1}{2x_1}y_2^2 + x_1^2 + \frac{1}{2}x_2^2 - c_1\left(x_1 + \frac{x_2^2}{4x_1}\right) - \frac{c_2}{x_1}, \\ h_2 &= -\frac{x_2^2}{8x_1}y_1^2 + \left(-\frac{1}{2}x_1 - \frac{1}{8}\frac{x_2^2}{x_1}\right)y_2^2 + \frac{1}{2}x_2y_1y_2 - \frac{1}{16}x_2^4 + \frac{1}{16}c_1\frac{x_2^4}{x_1} + \frac{1}{4}c_2\frac{x_2^2}{x_1}, \end{aligned}$$

while the matrix representations of the Poisson operators π_0 and π_1 attain the explicit form

$$\pi_0(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{c}) = \left(\begin{array}{cc} \begin{matrix} 0 & I \\ -I & 0 \end{matrix} & \begin{matrix} 0 & 0 \end{matrix} \\ * & \end{array}\right),$$

$$\pi_1(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{c}) = \left(\begin{array}{cc} \begin{matrix} 0 & 0 & x_1 & \frac{1}{2}x_2 \\ 0 & 0 & \frac{1}{2}x_2 & 0 \\ -x_1 & -\frac{1}{2}x_2 & 0 & \frac{1}{2}y_2 \\ -\frac{1}{2}x_2 & 0 & -\frac{1}{2}y_2 & 0 \end{matrix} & \begin{matrix} X_1 & X_2 \end{matrix} \\ * & \end{array}\right),$$

where

$$X_1=\left(\frac{y_1}{x_1},\frac{y_2}{x_1},\frac{1}{2x_1^2}y_1^2+\frac{1}{2x_1^2}y_2^2-2x_1+c_1\left(1-\frac{x_2^2}{4x_1^2}\right),-x_2+\frac{1}{2}c_1\frac{x_2}{x_1},0,0\right)^T,$$

$$\begin{aligned}X_2=\Bigg(&-\frac{1}{4}\frac{x_2^2}{x_1}y_1+\frac{1}{2}x_2y_2,-\left(x_1+\frac{1}{4}\frac{x_2^2}{x_1^2}\right)y_2+\frac{1}{2}x_2y_1,\\&-\frac{1}{8}\frac{x_2^2}{x_1^2}y_1^2-\left(-\frac{1}{2}+\frac{1}{8}\frac{x_2^2}{x_1^2}\right)y_2^2+\frac{1}{16}c_1\frac{x_2^4}{x_1^2}+\frac{1}{4}c_2\frac{x_2^2}{x_1^2},\\&\frac{1}{4}\frac{x_2}{x_1}(y_1^2+y_2^2)-\frac{1}{2}y_1y_2+\frac{1}{4}x_2^3-\frac{1}{4}c_1\frac{x_2^3}{x_1}-\frac{1}{2}c_2\frac{x_2}{x_1},0,0\Bigg)^T.\end{aligned}$$

The two corresponding bi-Hamiltonian chains are

$$\begin{array}{rcl}\pi_0dc_2&=&0\\\pi_0dh_2&=X_2=&\pi_1dc_2\\0&=&\pi_1dh_2\end{array},\qquad\begin{array}{rcl}\pi_0dc_1&=&0\\\pi_0dh_1&=X_1=&\pi_1dc_1\\0&=&\pi_1dh_1.\end{array}$$

Appendix

We prove here Theorem 11. Let us fix $k\in\{1,\ldots,m\}$ and then $r\in\{1,\ldots,n_k\}$. We want to show that the vector fields $X_r^{(k)}$ and $\bar{X}_r^{(k)}$ on $\mathcal{M}$ coincide. Obviously, $\bar{X}_r^{(k)}=\bar{\pi}d\bar{h}_r^{(k)}=\sum_{i=1}^{n}\left(\frac{\partial\bar{h}_r^{(k)}}{\partial\bar{\mu}_i}\frac{\partial}{\partial\bar{\lambda}_i}-\frac{\partial\bar{h}_r^{(k)}}{\partial\bar{\lambda}_i}\frac{\partial}{\partial\bar{\mu}_i}\right)$. Let us write this vector field in the coordinates $(\lambda_i,\mu_i,c_1^{(1)},\ldots,c_\alpha^{(1)},\ldots,c_1^{(m)},\ldots,c_\alpha^{(m)})_{i=1,\ldots,n}$ connected with the coordinates $(\bar{\lambda}_i\bar{\mu}_i,\bar{c}_1^{(1)}\ldots,\bar{c}_\alpha^{(1)},\ldots,\bar{c}_1^{(m)},\ldots,\bar{c}_\alpha^{(m)})_{i=1,\ldots,n}$ through the Miura map (83). Components of $\bar{X}_r^{(k)}$ in the non-bar coordinates will be given by

$$J\begin{pmatrix}\frac{\partial\bar{h}_r^{(k)}}{\partial\bar{\mu}}\\-\frac{\partial\bar{h}_r^{(k)}}{\partial\bar{\lambda}}\\0_{N\times1}\end{pmatrix},\tag{A.1}$$

where

$$\frac{\partial\bar{h}_r^{(k)}}{\partial\bar{\mu}}=\left(\frac{\partial\bar{h}_r^{(k)}}{\partial\bar{\mu}_1},\ldots,\frac{\partial\bar{h}_r^{(k)}}{\partial\bar{\mu}_n}\right)^T,\qquad\frac{\partial\bar{h}_r^{(k)}}{\partial\bar{\lambda}}=\left(\frac{\partial\bar{h}_r^{(k)}}{\partial\bar{\lambda}_1},\ldots,\frac{\partial\bar{h}_r^{(k)}}{\partial\bar{\lambda}_n}\right)^T$$

and where J is the Jacobian of the map (83), given explicitly by

$$J=\begin{pmatrix}I_n&0_{n\times n}&\\&&0_{2n\times N}\\s\Lambda^{s-1}U&\Lambda^s&\\B_1&&\\\vdots&&*\\B_m&&\end{pmatrix},$$

with $\Lambda=$diag$(\bar\lambda_1,\ldots,\bar\lambda_n)$, $U=$diag$(\bar\mu_1,\ldots,\bar\mu_n)$, and with B_l, $l=1,\ldots,m$ being the $\alpha\times 2n$ matrix given by

$$B_l=\begin{pmatrix} \left[\dfrac{\partial\bar h^{(l)}_{s-i+1}}{\partial\bar\lambda_j}\right]_{\substack{i=1\ldots s\\ j=1,\ldots n}} & \left[\dfrac{\partial\bar h^{(l)}_{s-i+1}}{\partial\bar\mu_j}\right]_{\substack{i=1\ldots s\\ j=1,\ldots n}} \\ \\ 0_{(\alpha-s)\times n} & 0_{(\alpha-s)\times n}\end{pmatrix}$$

while $*$ represent some $N\times N$ matrix. Thus, (A.1) has the form

$$J\begin{pmatrix}\frac{\partial\bar h^{(k)}_r}{\partial\bar\mu}\\ -\frac{\partial\bar h^{(k)}_r}{\partial\bar\lambda}\\ 0_{N\times 1}\end{pmatrix}=\begin{pmatrix}\frac{\partial\bar h^{(k)}_r}{\partial\bar\mu}\\ s\Lambda^{s-1}U\frac{\partial\bar h^{(k)}_r}{\partial\bar\mu}-\Lambda^s\frac{\partial\bar h^{(k)}_r}{\partial\bar\lambda}\\ Z_1\\ \vdots\\ Z_m\end{pmatrix}$$

where

$$Z_l=\begin{pmatrix}\left\{\bar h^{(l)}_s,\bar h^{(k)}_r\right\}_{\bar\pi}\\ \vdots\\ \left\{\bar h^{(l)}_1,\bar h^{(k)}_r\right\}_{\bar\pi}\\ 0_{(\alpha-s)\times 1}\end{pmatrix}=0_{\alpha\times 1},$$

since all the Hamiltonians $\bar h^{(k)}_r$ mutually Poisson commute with respect to $\bar\pi$. So, (A.1) is given by

$$J\begin{pmatrix}\frac{\partial\bar h^{(k)}_r}{\partial\bar\mu}\\ -\frac{\partial\bar h^{(k)}_r}{\partial\bar\lambda}\\ 0_{N\times 1}\end{pmatrix}=\begin{pmatrix}\frac{\partial\bar h^{(k)}_r}{\partial\bar\mu}\\ s\Lambda^{s-1}U\frac{\partial\bar h^{(k)}_r}{\partial\bar\mu}-\Lambda^s\frac{\partial\bar h^{(k)}_r}{\partial\bar\lambda}\\ 0_{N\times 1}\end{pmatrix}.$$

Therefore, $X^{(k)}_r=\bar X^{(k)}_r$ provided that

$$\frac{\partial\bar h^{(k)}_r}{\partial\bar\mu_p}=\frac{\partial h^{(k)}_r}{\partial\mu_p},\ \ p=1,\ldots,n \tag{A.2}$$

and provided that

$$s\bar\lambda_p^{s-1}\bar\mu_p\frac{\partial\bar h^{(k)}_r}{\partial\bar\mu_p}-\bar\lambda_p^s\frac{\partial\bar h^{(k)}_r}{\partial\bar\lambda_p}=-\frac{\partial h^{(k)}_r}{\partial\lambda_p},\quad p=1,\ldots,n.$$

Using (A.2) and the fact that $\bar\lambda_p=\lambda_p$, $\bar\mu_p=\lambda_i^{-p}\mu_i$,, this last condition is equivalent to

$$\frac{\partial\bar h^{(k)}_r}{\partial\bar\lambda_p}=\lambda_p^{-s}\frac{\partial h^{(k)}_r}{\partial\lambda_p}+s\lambda_p^{-s-1}\mu_p\frac{\partial h^{(k)}_r}{\partial\mu_p},\ \ p=1,\ldots,n. \tag{A.3}$$

We will now prove first (A.2) and then (A.3). The separation relations following from the curve (71) yield the following n identities on $\mathcal{M} \subset \boldsymbol{R}^{2n+\alpha m}$:

$$\varphi_0(\lambda_i,\mu_i)+\sum_{k=1}^{m}\varphi_k(\lambda_i,\mu_i)\psi_k(\lambda_i,h_1^{(k)},...,h_{n_k}^{(k)},c_1^{(k)},...,c_\alpha^{(k)})\equiv 0,\quad i=1,\ldots,n, \tag{A.4}$$

where now

$$\psi_k(\lambda_i,h_1^{(k)},...,h_{n_k}^{(k)},c_1^{(k)},...,c_\alpha^{(k)})=c_\alpha^{(k)}\lambda_i^{n_k-1+\alpha}+...+c_1^{(k)}\lambda_i^{n_k}+\sum_{j=1}^{n_k}h_j^{(k)}\lambda_i^{n_k-j}. \tag{A.5}$$

Differentiating each of these identities with respect to μ_p yields (no summation over i) we obtain

$$\varphi_{0,2}'(\lambda_i,\mu_i)\delta_{ip}+\sum_{k=1}^{m}\varphi_{k,2}'(\lambda_i,\mu_i)\psi_k(\ldots)\delta_{ip}+\sum_{k=1}^{m}\varphi_k(\lambda_i,\mu_i)\sum_{j=1}^{n_k}\frac{\partial h_j^{(k)}}{\partial\mu_p}\lambda_i^{n_k-j}\equiv 0,\quad i=1,\ldots,n. \tag{A.6}$$

Let us note, for later purposes, that for $i\neq p$ the above identities attain the form

$$\sum_{k=1}^{m}\varphi_k(\lambda_i,\mu_i)\sum_{j=1}^{n_k}\frac{\partial h_j^{(k)}}{\partial\mu_p}\lambda_i^{n_k-j}\equiv 0,\quad i\neq p. \tag{A.7}$$

Analogously, the separation relations following from the curve (77) yield the following n identities on $\mathcal{M}$:

$$\varphi_0(\bar\lambda_i,\bar\lambda_i^s\bar\mu_i)\bar\lambda_i^{-s}+\sum_{k=1}^{m}\varphi_k(\bar\lambda_i,\bar\mu_i\bar\lambda_i^s)\psi_k(\bar\lambda_i,\bar h_1^{(k)},...,\bar h_{n_k}^{(k)},\bar c_1^{(k)},...,\bar c_\alpha^{(k)})\equiv 0,\quad i=1,\ldots,n, \tag{A.8}$$

where now

$$\begin{aligned}\psi_k(\bar\lambda_i,\bar h_1^{(k)},...,\bar h_{n_k}^{(k)},\bar c_1^{(k)},...,\bar c_\alpha^{(k)})=&\bar c_\alpha^{(k)}\bar\lambda_i^{n_k+\alpha-s-1}+\ldots+\bar c_{s+1}^{(k)}\bar\lambda_i^{n_k}\\&+\sum_{j=1}^{n_k}\bar h_j^{(k)}\bar\lambda_i^{n_k-j}+\bar c_s^{(k)}\bar\lambda_i^{-1}+\ldots+\bar c_1^{(k)}\bar\lambda_i^{-s}.\end{aligned} \tag{A.9}$$

Differentiating each of these identities with respect to $\bar\mu_p$ yields (again, no summation over i)

$$\varphi_{0,2}'(\bar\lambda_i,\bar\lambda_i^s\bar\mu_i)\delta_{ip}+\sum_{k=1}^{m}\varphi_{k,2}'(\bar\lambda_i,\bar\lambda_i^s\bar\mu_i)\bar\lambda_i^s\psi_k(\bar{\ldots})\delta_{ip}+\sum_{k=1}^{m}\varphi_k(\bar\lambda_i,\bar\lambda_i^s\bar\mu)\sum_{j=1}^{n_k}\frac{\partial\bar h_j^{(k)}}{\partial\bar\mu_p}\lambda_i^{n_k-j}\equiv 0, \tag{A.10}$$

$i=1,\ldots,n$, where $\psi_k(\bar{\ldots})$ is the short-hand notation for (A.9). A simple identification through (81) shows that $\psi_k(...)$ in (A.5) and $\psi_k(\bar{\ldots})$ in (A.9) coincide as functions on $\mathcal{M}$. Thus, the Miura map (83) maps the identities (A.10) exactly onto the corresponding (i.e.

with the same i) identities (A.6). That means that $\frac{\partial h_j^{(k)}}{\partial \mu_p}$ and $\frac{\partial \bar{h}_j^{(k)}}{\partial \bar{\mu}_p}$ satisfy the same set of n linear equations, with the same non-degenerated system matrix, and thus they must pairwise coincide. Thus, (A.2) is proven.

Let us now prove (A.3). Differentiating all the identities (A.4) with respect to λ_p we obtain, for $i = p$

$$\begin{aligned}&\varphi'_{0,1}(\lambda_p,\mu_p)+\sum_{k=1}^{m}\varphi'_{k,1}(\lambda_p,\mu_p)\psi_k(\lambda_p,\ldots)\\&+\sum_{k=1}^{m}\varphi_k(\lambda_p,\mu_p)\left(\sum_{j=1}^{\alpha}(n_k+j-1)c_j^{(k)}\lambda_p^{n_k+j-2}+\sum_{j=1}^{n_k}(n_k-j)\lambda_p^{n_k-j-1}h_j^{(k)}\right.\\&\qquad\qquad\left.+\sum_{j=1}^{n_k}\lambda_p^{n_k-j}\frac{\partial h_j^{(k)}}{\partial\lambda_p}\right)=0\end{aligned}\tag{A.11}$$

and for $i \neq p$

$$\sum_{k=1}^{m}\varphi_k(\lambda_i,\mu_i)\sum_{j=1}^{n_k}\lambda_i^{n_k-j}\frac{\partial h_j^{(k)}}{\partial\lambda_p}\equiv 0.\tag{A.12}$$

Analogously, multiplying all the identities (A.8) by $\bar{\lambda}_i^s$ and differentiating with respect to $\bar{\lambda}_p$ we obtain, for $i = p$

$$\begin{aligned}&\varphi'_{0,1}(\bar{\lambda}_p,\bar{\mu}_p\bar{\lambda}_p^s)+\varphi'_{0,2}(\bar{\lambda}_p,\bar{\mu}_p\bar{\lambda}_p^s)s\bar{\mu}_p\bar{\lambda}_p^{s-1}+\sum_{k=1}^{m}\varphi'_{k,1}(\bar{\lambda}_p,\bar{\mu}_p\bar{\lambda}_p^s)\psi_k(\bar{\lambda}_p,\bar{\ldots})\\&+\sum_{k=1}^{m}\varphi'_{k,2}(\bar{\lambda}_p,\bar{\mu}_p\bar{\lambda}_p^s)s\bar{\mu}_p\bar{\lambda}_p^{s-1}\psi_k(\bar{\lambda}_p,\bar{\ldots})\\&+\sum_{k=1}^{m}\varphi_k(\bar{\lambda}_p,\bar{\mu}_p\bar{\lambda}_p^s)\left(\sum_{j=1}^{\alpha}(n_k+j-1)\bar{c}_j^{(k)}\bar{\lambda}_p^{n_k+j-2}+\sum_{j=1}^{s}(j-1)\bar{c}_j^{(k)}\bar{\lambda}_p^{j-2}\right.\\&\qquad\qquad\left.+\sum_{j=1}^{n_k}(n_k+s-j)\bar{\lambda}_p^{n_k+s-j-1}\bar{h}_j^{(k)}+\sum_{j=1}^{n_k}\bar{\lambda}_p^{n_k+s-j}\frac{\partial\bar{h}_j^{(k)}}{\partial\bar{\lambda}_p}\right)\equiv 0\end{aligned}\tag{A.13}$$

and for $i \neq p$

$$\sum_{k=1}^{m}\varphi_k(\bar{\lambda}_i,\bar{\mu}_i\bar{\lambda}_i^s)\sum_{j=1}^{n_k}\bar{\lambda}_i^{n_{k+s-j}}\frac{\partial\bar{h}_j^{(k)}}{\partial\bar{\lambda}_p}=0.\tag{A.14}$$

It is immediate to see that the Miura map (83) transforms all the relations (A.14) to the corresponding (i.e. with the same i) relations (A.12). Further, careful comparison of all terms in (A.11) and (A.13) using (A.7) shows that the Miura map (83) transforms (A.13) onto (A.11) if and only if the condition (A.3) holds. Thus, (A.3) is proved.

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