Open Communications in Nonlinear Mathematical Physics]ocnmp[Special Issue 1, 2024 pp 1–24

This article is part of an OCNMP Special Issue in Memory of Professor Decio Levi

Algebraic Entropy for Systems of Quad Equations

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Received July 25, 2023; Accepted February 5, 2024

Abstract

In this work I discuss briefly the calculation of the algebraic entropy for systems of quad equations. In particular, I observe that since systems of multilinear equations can have algebraic solution, in some cases one might need to restrict the direction of evolution only to the pair of vertices yielding a birational evolution. Some examples from the existing literature are presented and discussed within this framework.

1 Introduction

In this paper I will address the problem of the calculation of the algebraic entropy for *systems of quad equations*. That is, I will consider systems of four-point relations for an unknown field $\mathbf{x}\colon \mathbb{Z}^2 \to \mathbb{C}^M$ of the form:

$$\mathbf{Q}(\mathbf{x}_{l,m}, \mathbf{x}_{l+1,m}, \mathbf{x}_{l,m+1}, \mathbf{x}_{l+1,m+1}) = \mathbf{0}, \tag{1}$$

and I will address their growth properties in the sense of Arnol'd [3]. Equations of this form appeared seldom in the literature, see e.g. [42, 43, 54, 55, 63, 65, 85]. More recently, the interest in these models appears to be increasing, see [9], where their generalised (continuous) symmetries are discussed. The name *quad equations* originates from the fact that the four-point relation is defined on a square-like shape, see Figure 1.

I started studying the growth properties of difference equations during my Ph.D. at Università degli Studi Roma Tre under the supervision of Prof. Decio Levi, and this paper is dedicated to his memory. Some of the computations contained in this paper date back to the time I was a Ph.D. student but were not published for several reasons, while others are new and were carried out explicitly for this occasion. For these reasons this paper is written in first person singular and it will have two introductions: one, more personal, about my relationship with Decio Levi, which lasted until the very end of his life, and a scientific one, where I will address the background of the problem and the structure of this paper.

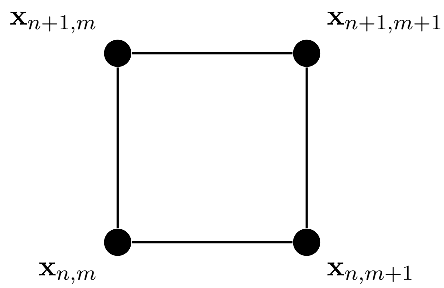

Figure 1. A quad-graph.

1.1 My relationship with Decio Levi

Before starting my Ph.D. in Mathematics at Università degli Studi Roma Tre I knew a little about Decio Levi's work after the early 90s . During the preparation of my Bachelor Thesis at Università degli Studi di Perugia [25] I studied several of the very interesting papers produced by Decio and Pavel Winternitz [13, 14, 56] regarding the optimal Lie subalgebras (see [66, §3.3]) of infinite dimensional Lie algebras. I had a strong interest in integrable systems since I attended a seminar of Mark J. Ablowitz at Università degli Studi di Perugia. Moreover, I just came from writing a Master's Thesis at Università degli Studi di Trieste under the supervision of Tamara Grava on the asymptotic properties of NLS equations with piece-wise constant initial data [16], and I was actively working with Maria Clara Nucci on Noether symmetry preserving quantisation [29]. On the other hand, I had little or no idea of discrete systems. During my very first international conference, the Twenty Years of Journal of Nonlinear Mathematical Physics conference in Nordfjordeid, I think I skipped almost all sessions about discrete systems because I understood almost nothing about that subject at that time.

So, with little or no knowledge of the work of Decio after the early 90s we had our first email exchange, where I asked for an idea of a research project to present to the Ph.D. admission test. I mentioned as a reference Maria Clara Nucci, and after asking her about me, Decio proposed me two topics:

- the problem of symmetry preserving discretisation [82];
- the study of the new quad equations classified by R. Boll in his Ph.D. thesis [6–8].

After some study of the references he provided I opted for the second one.

Our first real life encounter was after I already won a Ph.D. position at Università degli Studi Roma Tre. I went to bring the documents needed for the enrollment, and then I went to Physics Building in Via della Vasca Navale 84, got lost in the maze of the corridors and finally found his office. At that time Ravil Islamovich Yamilov was there. Our first meeting was very pleasant. To give some additional context, in those years integrable systems were a very well developed topic at Università degli Studi Roma Tre: the same year I got admitted as a Ph.D. student in Mathematics, Danilo Latini got admitted in Physics, and choose as supervisor Orlando Ragnisco, while Christian Scimiterna and Federico Zullo were postdocs, supported by the very large PRIN "Geometric and Analytic theory of Hamiltonian systems in finite and infinite dimensions". There was the local INFN unit of the project "Mathematical Methods of NonLinear Physics", which is the heir of the original Italian school of integrability led by Francesco Calogero, Giulio Soliani, and many others. In short the milieu was very fertile for developing ideas and growing as a scientist.

After almost one year of classes and study we started the real work. Since, as I told before, I was completely new to discrete integrable systems I started by reading [46]. My original task was to compute the generalised symmetries [59] of the Boll's new equations on the quad-graph. To cut a long story short, the generalised symmetry test for integrability of quad equations states that a quad-equation is integrable if its generalised symmetries are themselves integrable equations as flows, i.e. they are integrable differential-difference (or semi-discrete) equations [58]. At first I was sort of puzzled by Boll's equations, and having studied the algebraic entropy test, which looked like magic to me back then (sometimes it still does even nowadays) I proposed to Decio to check first the equations with algebraic entropy. At that time Christian Scimiterna was post-doc at Università degli Studi Roma Tre and suggested that I ask Mike Hay, who was also there with an INFN fellowship, since they recently faced a similar problem in [72]. Decio was supportive of this idea, even though his main focus was far away from the ideas of algebraic entropy.

After a few months of studying and coding, I came up with a suite of programs to compute the growth of degrees of quad equations and analyse their behaviour. The day of reckoning came and I got the unexpected result that the equations we were considering (that Christian carefully checked to be independent and "realisable" in Boll's 3D cubes configurations) possessed linear growth. Even more surprising while looking at pages and pages of output from my programs I found out that contrary to what was assumed in several papers on the topic, e.g. [75, 78], the degrees along the diagonal were different. There were several days we spent in doubt about these results, which were resolved after I sent an email to Claude Viallet, where he confirmed that the result was plausible. Thus, my first paper with Decio and Christian was born [33], even though it was not the first to be published. We further elaborated some of these ideas in [34], where upon the observation that linear(isable) equations cannot possess infinitely many conservations laws, Decio suggested to me and Christian to prove that the Lax pair of the $\mathrm{H1}_{\mathrm{t}}$ equations, the simplest equations we were considering, obtained by the Consistency Around the Cube procedure was in fact *fake* [11, 44, 45], i.e. it was just a way of rewriting the equation, and it was not giving any additional information.

During the rest of my Ph.D. I spent most of my time trying to understand more about the results we gathered in [33]. After I first presented our work together at the Physics and Mathematics of Nonlinear Phenomena 2015, we went back to the original idea and computed the generalised symmetries of Boll's equations [37]. When Decio unexpectedly remembered his first work with Ravil [57] we realised that the flows of all the non-autonomous generalised symmetries we computed were actually particular cases of an equation introduced by Decio and Ravil themselves! That observation, and a comparison with the works of Xenitidis and Papageorgiou [84] and Xenitidis [83], led us to build a two-periodic extension of the Q_{V} equation [79] in [36]. This is just one of the many cases when Decio's experience and cunning were fundamental.

In writing [37] the help of Christian in this task was again fundamental, as he noticed that the trapezoidal $\mathrm{H1}_{\varepsilon}$ equation had a very peculiar property: it admitted generalised symmetries depending on *arbitrary functions*. This led us to the concept of *Darboux integrability* for quad equations as exposed in [2]. This notion was the subject of the last year of my Ph.D., and of the last paper I wrote together with Decio and Christian [35]. Therein, we proposed a conjecture, strongly advocated by Decio, that all quad equations admitting generalised symmetries depending on arbitrary functions were in fact Darboux integrable. This conjecture was shortly afterwards proved true by Startsev [73]. Also during the last year of my Ph.D. Ravil was visiting Università degli Studi di Roma Tre. He had won an INdAM visiting position, and he was supposed to give a Ph.D. course. However, it turned out that I was the only student. So, after a few classic lectures, upon Decio's idea we started developing our ideas on the Darboux integrability of Boll's equations instead of doing lectures. The result of these anomalous lectures was two papers [38, 39], where we solved the mystery of Boll's equations. Indeed, we explained the linear growth by showing that the trapezoidal H4 and the H6 equations are Darboux integrable. Moreover, we proved that they are exactly solvable, up to some discrete Riccati equations. My collaboration with Ravil continued in the later years, see [22], and it lasted until Ravil's premature demise.

After that, I got my Ph.D. and I moved to Australia. Indeed, in the last part of 2016, I won a post-doc position at the University of Sydney in the Integrable and Nearly Integrable Systems group led by Nalini Joshi. In some sense, I also owe this possibility to Decio, because my stronger connection to Australia, started when following his idea I went to Australia in 2015, touring Melbourne, Brisbane, and Sydney. In time, our encounters become sporadic: we met once in Sydney in 2017, once in Fukuoka at the SIDE13 conference in 2018, and at the special session for his 71st birthday in Montreal. On the few occasions when I came back to Università degli Studi Roma Tre after I got my Ph.D., I went back to his study (converted to a joint study for him and Orlando), he was not there, because he preferred working from home. However, we frequently exchanged emails. A few times I tried to involve him in doing some additional work together, but (unfortunately) he was always turning my proposals down, yet pushing me to follow my ideas by myself. At that point his main interests were the conditional symmetry preserving schemes for PDEs [60], and his long time project with Pavel and Ravil: the book "*Continuous symmetries and integrability of discrete equations*" [61] which was finally published in the last quarter of 2022, unfortunately after the demise of all its authors.

I always resorted to Decio when I was in need of a suggestion about my career and my scientific results. Decio was always very calm, and he was the perfect balance to my illtemper. As a supervisor, Decio was really good. It was clear that he was doing the best for both of the parties, the students and the supervisor, helping you to grow and become independent, not just some mindless calculator. My very first sole author work, the review [26], followed his invitation to give a four hours lecture at ASIDE, the summer school right before SIDE13 that he was organising with Pavel in Sainte Adèle. During my Ph.D. I was also left free to have my own scientific collaborations, for instance my ongoing collaboration with Maria Clara Nucci, which resulted in a few more works [30–32], or my new collaboration with Davide Chiuchiù [12, 27] on a completely different topic than the rest of my Ph.D. thesis.

Losing Decio, I did not just lose my supervisor and mentor. I lost one of the nicest persons you can find on earth, and I will always miss him and his soft-spoken suggestions.

1.2 Statement of the problem and outline of the paper

Analysing algorithms through their complexity is an old topic in numerical analysis. Heuristically, the idea is that algorithms performing better are those whose computational time is sub-exponential in the iteration process. The goal is then to establish the long term behaviour of the recurrence defined by the algorithm *without having to compute the complete recurrence or knowing how to express it in closed form.* Clearly, computing infinitely many terms is impossible, and the examples that one can exactly solve are very few. Moreover, having a solution does not necessarily mean that the system is asymptotically well-behaved, see for instance the solvable case of the logistic map [62, 71].

In this spirit, in [3] Arnol'd defined the complexity of intersections for diffeomorphisms. In fact the idea is rather simple: let us assume we are given a compact smooth manifold $\mathcal{X}$, a diffeomorphism $\varphi \in \mathcal{C}^\infty(\mathcal{X})$ and two compact smooth submanifolds $\mathcal{Y}$, $\mathcal{Z}$, such that $\dim \mathcal{Y} + \dim \mathcal{Z} = \dim \mathcal{X}$. Then a natural way to estimate the complexity upon iteration of the diffeomorphism φ is to consider the cardinality of the intersection of the successive images of $\mathcal{Y}$ with $\mathcal{Z}$:

$$N_k = \left| \varphi^{(k)}(\mathcal{Y}) \cap \mathcal{Z} \right|. \tag{2}$$

The "generic" growth for such intersections grows exponentially and in some pathological cases can even be super-exponential.

What was observed is that there are some particular maps (which will be later called "integrable") that are not as complex as generic ones [17, 19, 48, 77]. These maps had to be birational maps of complex projective spaces, because the drop in complexity relies on the ability of the map to enter (and possibly exit) the singularities. Then Bellon and Viallet introduced the notion of *algebraic entropy* [5] to have an invariant measure of growth of birational maps of the complex spaces. Almost at the same time [70] introduced the analogous notion of *dynamical degree*. Since then the development of the theory of algebraic entropy has been a thriving research topic, both for the discrete integrable systems community and for the algebraic dynamics one, see for instance the reviews [24, 26].

In particular, over the years the theory of algebraic entropy has been extended beyond birational maps of complex projective spaces, to several infinite-dimensional cases. For instance in [75, 78] the method was developed in the case of scalar quad equations, while in [33] a slight generalisation was proposed. For instance, a search for integrable quad equations using factorisation was used in [49]. Later the concept was extended to semi-discrete systems: in [15] for differential-difference equations, and in [80] for differential-delay equations. Lattice equations not of quad type have been considered more recently [28, 40, 50], while quad equations themselves are still the source of new findings [47].

Here, I will discuss the concept of algebraic entropy for systems of quad equations, i.e. pure difference systems of the form (1). Then, the structure of the paper is the following: In Section 2 I will present the technique of calculation of the degree sequences for systems of quad equations, and I will recall how to estimate asymptotic growth. My treatment will follow closely the construction that is done usually in the scalar case, see [33, 75, 78], with the peculiar difference that multilinear systems might not have a well defined birational evolution in all directions. In Section 3 I will present several examples of systems that pass the algebraic entropy test. When possible, I will discuss the similarities and differences with the scalar case. Finally, in Section 4 I will discuss the results obtained.

2 Algebraic entropy for systems of quad equations with staircase initial conditions

In this paper I will consider the problem of computing the algebraic entropy of systems of quad equations using initial conditions of a specific form, i.e. the so-called *staircase initial conditions*.

In the case of scalar quad equations

$$Q(x_{l,m}, x_{l+1,m}, x_{l,m+1}, x_{l+1,m+1}) = 0, \tag{3}$$

the multilinearity requirement ensures that the evolution is well-defined as a rational map in all the directions of evolution. That is, solving a quad equation with respect to one of the four corners yields a rational map. The evolution is then birational in the opposite direction with respect to the principal diagonal, i.e. solving with respect to the pair $(x_{l+1,m}, x_{l,m+1})$ and $(x_{l,m}, x_{l+1,m+1})$.

In the case of systems, this is no longer true. The reason is twofold: first I will not assume that all equations in the system (1) depend on all four vertices. Then, a first condition to have solvability in one of the four directions is:

$$\operatorname{rank} \frac{\partial \mathbf{Q}}{\partial \mathbf{x}_{l+1,m+j}} = M, \quad i,j \in \{ 0,1 \}, \tag{4}$$

since it allows to use the inverse function theorem. However, this is not enough: if a system of multilinear equations is solvable it does not necessarily imply that the solution is *rational*. A simple example is the following:

$$Q_1 = x_1 y_1 y_2 x_3 - x_2 y_3 y_4 x_4, \tag{5a}$$

$$Q_2 = x_1 y_2 + y_3 x_4 + y_1 x_2 + x_3 y_4, \tag{5b}$$

were for simplicity I used the variables x_i, y_i with $i = 1, 2, 3, 4$. Then we have:

$$\operatorname{rank} \frac{\partial(Q_1, Q_2)}{\partial(x_1, y_1)} = \operatorname{rank} \frac{\partial(Q_1, Q_2)}{\partial(x_3, y_3)} = 2, \tag{6}$$

but in the first case the solutions are algebraic:

$$x_1 = \frac{1}{2} \frac{-x_3^2 y_4 - x_3 x_4 y_3 \mp \sqrt{-4x_2^2 x_3 x_4 y_3 y_4 + x_3^4 y_4^2 + 2x_3^3 x_4 y_3 y_4 + x_3^2 x_4^2 y_3^2}}{x_3 y_2}, \tag{7a}$$

$$y_1 = \frac{1}{2} \frac{-x_3^2 y_4 - x_3 x_4 y_3 \pm \sqrt{-4x_2^2 x_3 x_4 y_3 y_4 + x_3^4 y_4^2 + 2x_3^3 x_4 y_3 y_4 + x_3^2 x_4^2 y_3^2}}{x_2 x_3} \tag{7b}$$

while in the second are rational:

$$x_3 = -x_2 y_4 \frac{x_1 y_2 + x_2 y_1}{x_1 y_1 y_2 + x_2 y_4^2}, \quad y_3 = -\frac{x_1 y_1 y_2}{x_4} \frac{x_1 y_2 + x_2 y_1}{x_1 y_1 y_2 + x_2 y_4^2}. \tag{8}$$

When solutions are rational, I will say that the direction of evolution is *admissible*, while the solutions are not rational, I will say that the direction of evolution is *not admissible*.

Then, if the condition of admissibility is satisfied, the evolution is possible on a staircase-like arrangement of initial values is possible in the quadrilateral lattice. In such a case the treatment of the problem will follow closely the scalar case (see e.g. [24, 26, 33, 75, 78]), with some relevant differences I will point out in the discussion. The first thing to rule out is overhang in the initial data configurations, since these could lead to a contradiction, giving more than one way to calculate the same value for the dependent variable. The staircases need to go from $(n = -\infty, m = -\infty)$ to $(n = \infty, m = \infty)$, or from $(n = -\infty, m = +\infty)$ to $(n = \infty, m = -\infty)$ because the space of initial conditions is infinite. I will restrict the present discussion to regular diagonals which are staircases with steps of *constant* horizontal length, and *constant* vertical height. Non-regular staircases were considered in the scalar case in [47], raising several interesting questions, which as far as I know are not completely resolved yet. Figure 2 shows four diagonals. The ones labeled (1) and (2) are regular. The one labeled (3) would be acceptable, but non-regular, so I will not consider it. Line (4) is excluded since it may lead to incompatibilities. In the scalar case, given a line of initial conditions, it is possible to calculate the values $x_{l,m}$ on all points of the two-dimensional lattice. In the system case, this is possible if both directions, are admissible. With the examples I will consider in Section 3 this is not always the case, *yet the growth might still be sub-exponential.*

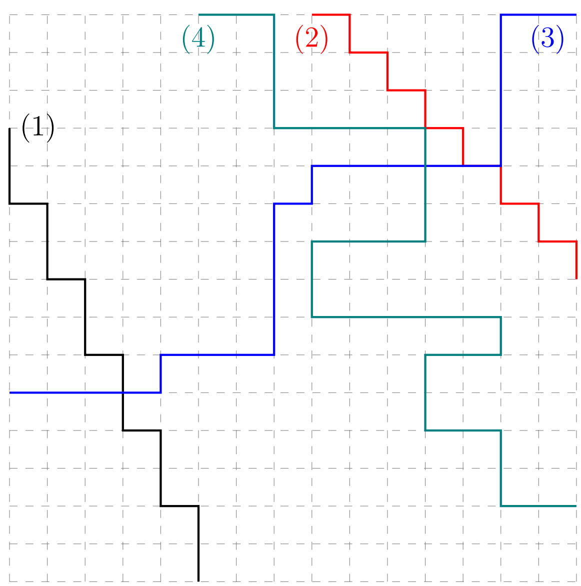

Figure 2. Regular and non-regular staircases.

The crucial point of using regular diagonals, is that *if one wants the evolution on a finite number of iterations, only a diagonal of initial conditions of finite extent is needed.* Indeed, let N be a positive integer, and each pair of relative integers $[\lambda_1, \lambda_2]$, I will denote by $\Delta^{(N)}_{[\lambda_1,\lambda_2]} \subset \mathbb{Z}^2$ a regular diagonal consisting of N steps, each having horizontal size $l_1 = |\lambda_1|$, height $l_2 = |\lambda_2|$, and going in the direction of positive (resp. negative) n_k, if $\lambda_k > 0$ (resp. $\lambda_k < 0$), for $k = 1, 2$. See Figure 3.

Suppose one fixes the initial conditions $\Delta^{(N)}_{[\lambda_1,\lambda_2]}$. Then, one can calculate the values $\mathbf{x}_{l,m}$ over a rectangle of size $(Nl_1 + 1) \times (Nl_2 + 1)$. The diagonal cuts the rectangle in two halves: one of them uses all initial values. Assuming the associated direction of evolution is admissible for the system (1) one can then use those initial values to compute the evolution

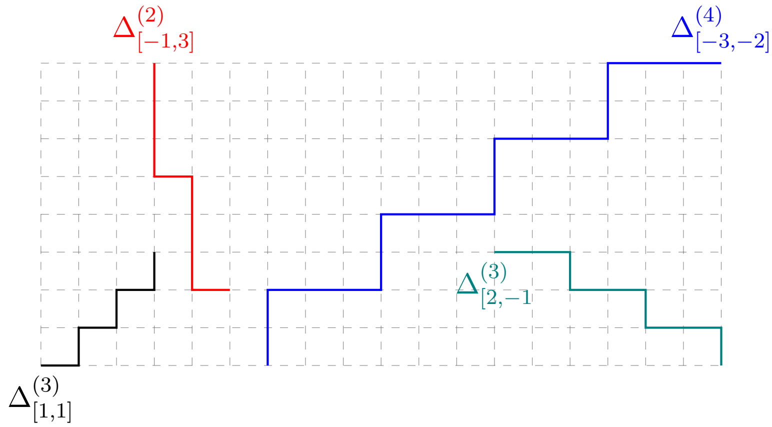

Figure 3. Varius kinds of restricted initial conditions.

on the part using all the initial values, see Figure 4. We call this set of indices $\Theta^{(N)}_{[\lambda_1,\lambda_2]}$, the *range* of $\Delta^{(N)}_{[\lambda_1,\lambda_2]}$

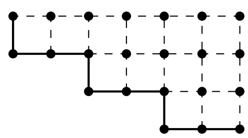

Figure 4. The range for the initial conditions $\Delta^{(3)}_{[-2,1]}$.

So, on some restricted diagonal $\Delta^{(N)}_{[\lambda_1,\lambda_2]}$, such that the direction using all the initial values is admissible, the iteration is defined. Then, by the previous discussion:

$$\left|\Delta^{(N)}_{[\lambda_1,\lambda_2]}\right| = N(l_1+l_2)+1 =: \mathcal{N}. \tag{9}$$

So, the total number of initial conditions needed is $\mathcal{N}\cdot M$, which in principle are an arbitrary vector in $\mathbb{C}^{\mathcal{N}\cdot M}$. However, as usual, it is better to consider the initial values in a *compactification* of the complex space. My choice for this paper is to consider as compactification the space:

$$\mathcal{K}_{\mathcal{N},M} = \left(\mathbb{CP}^{\mathcal{N}}\right)^{\times M}. \tag{10}$$

For instance other possibile compactifications are $\mathbb{CP}^{\mathcal{N}\cdot M}$ and $\left(\mathbb{CP}^1\right)^{\mathcal{N}\cdot M}$. However, the choice of the compactification will not change the final result. My choice of the space $\mathcal{K}_{\mathcal{N},M}$ is due to the fact that I want to keep track of every component of $\mathbf{x}_{l,m}$ as separate entities. In practice, to simplify the computations, it is usually better to consider the intial data as lying on a *line* in $\mathcal{K}_{\mathcal{N},M}$, i.e. it is possible to write them in the following form:

$$x^k_{i,j} = \frac{\alpha^k_{i,j}t_0+\beta^k_{i,j}t_1}{\alpha^k_0 t_0+\beta^k_0 t_1}, \quad (i,j)\in\Delta^{(N)}_{[\lambda_1,\lambda_2]}, [t_0:t_1]\in\mathbb{CP}^1, k=1,\dots,M, \tag{11}$$

where $\mathbf{x}_{i,j} = \left(x^1_{i,j}, \dots, x^M_{i,j}\right)$ and the coefficients $\alpha^k_{i,j}$, $\beta^k_{i,j}$, α^k_0, and β^k_0 are (fixed) integers. Then, evaluating the equation on the points reachable from $\Delta_{[\lambda_1,\lambda_2]^{(N)}}$ we obtain a two-dimensional sequence of degrees $\{\, \mathbf{d}_{l,m} \,\}_{(l,m)\in\Theta^{(N)}_{[\lambda_1,\lambda_2]}}$. Here the vector $\mathbf{d}_{l,m} = (d_{i,l,m}, \dots, d_{M,l,m})^T$, means that to any component of the field $\mathbf{x}_{l,m}$ we associate a different degree, and we whish to keep track of it through the evolution. When the number of fields is small I will use letters rather than numbering, e.g. three fields $\mathbf{x}_{l,m} = (x_{l,m}, y_{l,m}, z^T_{l,m})$. Correspondingly I will denote the degrees by letters, e.g. continuing the previous examples the vector of degrees will be $\mathbf{x}_{l,m} = (d_{x,l,m}, d_{y,l,m}, d^T_{z,l,m})$.

Among all possible restricted staircases, a special rôle is played by the simplest ones, i.e. the restricted diagonals $\Delta^{(N)}_{[\pm1,\pm1]}$. Due to their importance I denote them by the special notations $\Delta^{(N)}_{++}$, $\Delta^{(N)}_{+-}$, $\Delta^{(N)}_{-+}$ and $\Delta^{(N)}_{--}$, and name them *the fundamental diagonals*. As usual, the upper index (N) is omitted for infinite lines. The four fundamental diagonals are showed in Figure 5.

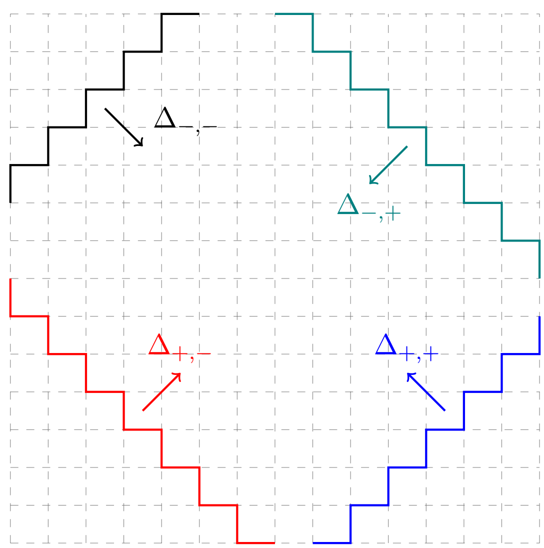

Figure 5. The four principal diagonals.

Explicitly, a degree sequence on $\Delta^{(N)}_{+-}$ has the form:

$$
\begin{array}{cccccc}
1 & \mathbf{d}_{1,N-1} & \mathbf{d}_{2,N-2} & \dots & \mathbf{d}_{N-1,2} & \mathbf{d}_{N,1} \\
1 & 1 & \mathbf{d}_{1,N-2} & \mathbf{d}^{(3)}_{2} & \dots & \mathbf{d}_{N-1,1} \\
 & 1 & 1 & \mathbf{d}_{1,3} & \mathbf{d}_{2,2} & \dots \\
 & & 1 & 1 & \mathbf{d}_{1,2} & \mathbf{d}_{2,1} \\
 & & & 1 & 1 & \mathbf{d}_{1,1} \\
 & & & & 1 & 1
\end{array}
\tag{12}
$$

In Equation (12) 1 simply means that all sequence starts with a term of homogeneous degree 1, as in Equation (11). Note that we choose the indexing in a way that the first index represents the index of iteration. Moreover, if needed, to underline the direction in the sequence of degrees we will denote by $\mathbf{d}^{\pm\pm}_{l,m}$ the sequence of degrees on corresponding to $\Delta_{\pm,\pm}$. Then, for every fixed m we define the sequence

$$1, \mathbf{d}_{1,m}^{\pm\pm}, \mathbf{d}_{2,m}^{\pm\pm}, \mathbf{d}_{3,m}^{\pm\pm}, \mathbf{d}_{4,m}^{\pm\pm}, \dots, \tag{13}$$

Usually, these sequences are just the same sequence shifted by one. In such a case we simply drop the unused index m, and write the sequence as $\left\{ \mathbf{d}_l^{\pm\pm} \right\}_{l=0}^{N}$. In the known cases where there is more than one sequence living on the lattice of Equation (12), the sequences are repeating periodically, see [33]. In the examples presented in this paper there will be only one sequence for admissible directions. To simplify our discussion of the examples in Section 3 we introduce now some notations and definitions.

Notation 2.1. I will denote a generic element of the set $\{\pm\}^2$ by δ, and consequently the degree sequence in the direction $\delta \in \{\pm\}^2$ will be denoted by $\left\{ \mathbf{d}_{l,m}^{\delta} \right\}_{(l,m)\in\Theta_\delta^{(N)}}$.

Definition 2.2. Let us assume we are given a system of quad-equation (1). Then:

- the system is *isotropic* if for all admissible directions the sequence $\mathbf{d}_{l,m}^{\delta}$ is the same;
- the system is *strongly isotropic* if for all admissible directions the sequences $d_{i,l,m}^{\delta}$, $i = 1, \dots, M$ are the same;
- the system is *permutationally isotropic* if for all pairs of admissible directions δ_1 and δ_2 there exists a permutation $\sigma \in \mathcal{S}_M$ such that:

$$\sigma(\mathbf{d}_{l,m}^{\delta_2}) := (d_{\sigma(1),l,m}^{\delta_2}, \dots, d_{\sigma(M),l,m}^{\delta_2}) \equiv \mathbf{d}_{l,m}^{\delta_1}. \tag{14}$$

Remark 2.3. Note that the system being isotropic means that *the degree growth does not depend by the (admissible) direction*, being strongly isotropic means that the *the degree growth does not depend by the (admissible) direction* and *the field*, and finally that being permutationally isotropic means *the degree growth does is only permuted by the direction.* In all cases, the system of quad equations must have *more than one* admissible direction.

Then define the *fundamental algebraic entropies* of the lattice equation by:

$$S_{i,m}^{\delta} = \lim_{l\to\infty} \frac{1}{l} \log d_{i,l,m}^{\delta}, \quad i = 1, \dots, M. \tag{15}$$

Note that $S_{i,m}^{\delta} \geq 0$. The existence of this limit can be proved in an analogous way as for finite dimensional maps, i.e. using the sub-additivity of the logarithm and the so-called Fekete's lemma [20]. When only a degree sequence is present, we will drop the index m, and then write only S_i^{δ}.

Then we give the following definition of integrability based on the concept of algebraic entropy:

Definition 2.4. A system of quad equations (1) is said to be *directionally integrable* in the direction δ if $S_{i,m}^{\delta} = 0$ for all $i = 1, \dots, M$. If $S_{i,m}^{\delta} = 0$ for all direction we say that the equation is *integrable*.

Remark 2.5. Before going on some remarks are in order:

- The rationale of defining (directonally) integrable those systems such that $S^{\delta}_{i,m} = 0$ for all $i = 1, \dots, M$ is that the system is expected to grow *at the pace of the fastest one*. If the system can be decomposed in independent sub-systems this is clearly not a good measure of the growth of the sub-systems and they should be treated separately.

- The notion of directional integrability was elaborated in [40], even though in the cases that are presented in that paper it is not strictly needed (albeit theoretically possible). Some cases where this notion is actually *needed* will be presented in Section 3.

- The condition $S_{i,m} = 0$ implies that asymptotically as $l \to \infty$ the degree $d_{i,l,m}$ is *sub-exponential*, e.g. polynomial. Sometimes in the literature the linear asymptotic behaviour $d_{i,l,m} \sim l$ is considered to be related to *linearisation*. This is usually true for quad equations, e.g. Boll's equations [6–8], see the discussion in the Introduction. Linear growth appeared also in [28] for the type-B face-centred quad equations satisfying the consistency around the face-centred cube property. However, at present, no explicit linearsation was produced for those systems. In Section 3 I will present some examples that show that for systems of quad equations the situation can be even more complicated.

It is known, see [5, 74, 81], that except for some pathological cases [41] the sequence of iterates of degree *stabilises after a finite number of steps*. So, in this paper I will aim to give a complete proof of the growth of the examples I will be considering. I will resort to use an heuristic *yet extremely fast and efficient* method, which is to *look for a generating function for the sequences of degrees*. In particular, I will make the *ansatz* that the generating function is rational. This allows me to find it algorithmically using Padé approximants [4]. The ability of fiding such a rational generating function requires to compute a sufficiently high number of iterates, as will be the case for all the example I will consider in Section 3.

To be more specific, given a sequence $\{ d_k \}_{k\in\mathbb{N}_0}$ its generating function is the function admitting the sequence as Taylor coefficients:

$$g(s) = \sum_{k=0}^{\infty} d_k s^k \tag{16}$$

In the case that such a generating function is rational one has two immediate consequences:

- The entropy is given by the inverse of the smallest modulus of the poles of $g(s)$, since the position of the smallest pole of g governs the asymptotics of its Taylor coefficients, i.e. if $g(s) = P(s)/Q(s)$ then:

$$S = \log\left(\left[\min\left\{ |s| \in \mathbb{R}^+ \quad \middle| \quad Q(s) = 0 \right\}\right]^{-1}\right). \tag{17}$$

- The sequence $\{ d_k \}_{k\in\mathbb{N}_0}$ satisfies the constant coefficient recurrence relation:

$$Q(T_k^{-1}) d_k = 0, \quad T_k d_k = d_{k+1}, \tag{18}$$

  where we have written the polynomial $Q(s)$ as $Q(s) = \sum_{j=0}^{J} a_j s^j$, $a_0 = 1$.

If all the poles of $g = P/Q$ lie on the unit circle, the denominator $Q(s)$ factorises as:

$$Q(s) = (1-s)^{\beta_0} \prod_{i=1}^{K} \left(1 - s^{\beta_i}\right), \quad \beta_i \in \mathbb{N}, \tag{19}$$

the growth of the degree sequence is polynomial, and hence *the algebraic entropy vanishes.*

We resume some properties of rational generating functions in the following results, obtained combining results from the books [18, Chap. 6], [23, Sect. 4.4], and [21]. For further discussion on the subject and more references we refer to the reviews [24, 26].

Theorem 2.6. *A sequence* $\{d_k\}_{k\in\mathbb{N}_0}$ *admits a rational generating function* $g \in \mathbb{C}(s)$ *if and only if it solves a* linear difference equation with constant coefficients. *Moreover, if* $\rho > 0$ *is the radius of convergence of* g*, writing* g *as:*

$$g = A(s) + B(s)\left(1 - \frac{s}{\rho}\right)^{-\beta}, \quad \beta \in \mathbb{N}. \tag{20}$$

where A *and* B *are analytic functions for* $|s| < r$ *such that* $B(\rho) \neq 0$ *we have:*

$$d_k \sim \frac{B(\rho)}{\Gamma(\beta)} k^{\beta-1} \rho^{-k}, \quad k \to \infty, \tag{21}$$

where $\Gamma(s)$ *is the Euler Gamma function.*

Corollary 2.7. *With the hypotheses of Theorem 2.6 and the additional assumption that* $\rho = 1$*, then*

$$d_k \sim k^{\beta-1}, \quad k \to \infty, \tag{22}$$

where $\beta = \beta_0 + K$*, with* β_0 *and* K *as in Equation* (19) *and* γ *a constant factor.*

In short, Theorem 2.6 and Corollary 2.7 tell us that given a rational generating function we can estimate the growth, which in the integrable case is polynomial. In Section 3 we will make extensive use of these two results.

3 Examples

In this section I will consider some examples of calculation of the algebraic entropy for some known systems of quad equations. The result will illustrate the various possibilities that were described in the previous section. I decided to consider only two and three dimensional systems to discuss these properties. As mentioned earlier I will denote the components of such systems by $x_{l,m}$, $y_{l,m}$, and $z_{l,m}$.

3.1 Coupled lpKdV systems

Consider the following system of two quad equations:

$$(x_{l,m} - x_{l+1,m+1})(y_{l+1,m} - y_{l,m+1}) - \alpha + \beta = 0, \tag{23a}$$

$$(y_{l,m} - y_{l+1,m+1})(x_{l+1,m} - x_{l,m+1}) - \alpha + \beta = 0. \tag{23b}$$

This system was presented in [9] where it was discussed using the generalised symmetry method. It is a generalisation of the lattice potential KdV equation, which is obtained from the reduction $y_{l,m} = x_{l,m}$. Indeed, upon substitution the two equations collapse into a single one:

$$(x_{l,m} - x_{l+1,m+1})(x_{l+1,m} - x_{l,m+1}) - \alpha + \beta = 0, \tag{24}$$

which is known in the literature as the discrete potential KdV equation[1].

All directions are admissible for the system (23), and the system is strongly isotropic showing only the following degree sequence:

$$1, 2, 4, 7, 11, 16, 22, 29, 37 \ldots \tag{25}$$

The sequence (25) has the following rational generating function:

$$g(s) = -\frac{s^2 - s + 1}{(s-1)^3}. \tag{26}$$

From Corollary 2.7 we see immediately that the growth is quadratic. In this case we can be even more precise, and find the analytic form of the sequence (25) using for instance the $\mathcal{Z}$-transform method [18, Chap. 6]:

$$d_k = \frac{k(k+1)}{2} + 1. \tag{27}$$

Note that, the scalar equation (24) is isotropic too, and in particular has the same degree growth, see [75]. For such an equation the growth was proved rigorously using the gcd-factorisation method in [69], see also [53]. So, the system (23) behaves much like a scalar equation. This leaves open the possibility of computing such growth rigorously using the gcd-factorisation method as was done in [69] for its scalar version.

3.2 Lattice NLS system

As a second example we consider the following system:

$$x_{l+1,m} - x_{l,m+1} - \frac{\alpha - \beta}{1 + x_{l,m} y_{l+1,m+1}} x_{l,m} = 0, \tag{28a}$$

$$y_{l+1,m} - y_{l,m+1} - \frac{\alpha - \beta}{1 + x_{l,m} y_{l+1,m+1}} y_{l+1,m+1} = 0, \tag{28b}$$

This system was presented in [55] where it was introduced as a discretisation of the non-linear Schrödinger equation (NLS), then considered again more recently in [9].

The system (28) has $(+,+)$ and $(-,-)$ as admissible directions. In those directions it is again strongly isotropic showing only the degree sequence:

$$1, 3, 7, 13, 21, 31, 43, 57, 73 \ldots \tag{29}$$

[1] In the ABS classification [1] this is the H1 equation.

The sequence (29) has the following rational generating function:

$$g(s) = -\frac{s^2+1}{(s-1)^3}. \tag{30}$$

From Corollary 2.7 we see immediately that the growth is quadratic. In this case we can be even more precise, and find the analytic form of the sequence (29) using for instance the $\mathcal{Z}$-transform method:

$$d_k = k(k+1)+1. \tag{31}$$

3.3 The $\mathbf{lSG_2}$ and the $\mathbf{lmKdV_2}$ systems

In [42] a completeness study of discrete systems on the quad-graph arising from a 2×2 Lax pairs was performed. Excluding trivial, undetermined and overdetermined systems, the only two *bona fide* systems found in this classification were the lSG_2 system:

$$\begin{aligned}
&\frac{(\lambda_l^{(3)})^{(-1)^m}}{(\mu_m^{(3)})^{(-1)^l}}\frac{x_{l,m+1}}{x_{l,m}} + \lambda_l^{(1)}\mu_m^{(1)}x_{l+1,m+1}y_{l,m+1} = \\
&\qquad\qquad \frac{(\mu_m^{(3)})^{(-1)^l}}{(\lambda_l^{(3)})^{(-1)^m}}\frac{x_{l+1,m+1}}{x_{l+1,m}} + \frac{\lambda_l^{(2)}\mu_m^{(2)}}{x_{l,m}y_{l+1,m}},
\end{aligned} \tag{32a}$$

$$\begin{aligned}
&\frac{(\mu_m^{(3)})^{(-1)^l}}{(\lambda_l^{(3)})^{(-1)^m}}\frac{y_{l+1,m+1}}{y_{l,m+1}} + \frac{\lambda_l^{(2)}\mu_m^{(2)}}{x_{l,m+1}y_{l,m}} = \\
&\qquad\qquad \frac{(\lambda_l^{(3)})^{(-1)^m}}{(\mu_m^{(3)})^{(-1)^l}}\frac{y_{l+1,m}}{y_{l,m}} + \lambda_l^{(1)}\mu_m^{(1)}x_{l+1,m}y_{l+1,m+1}
\end{aligned} \tag{32b}$$

and the lmKdV_2 system:

$$\begin{aligned}
&\frac{\lambda^{(1)_l}}{(\mu_m^{(3)})^{(-1)^l}}\frac{x_{l+1,m+1}}{x_{l,m+1}} + \frac{\mu_m^{(2)}}{(\lambda_l^{(3)})^{(-1)^m}}\frac{y_{l,m}}{y_{l,m+1}} = \\
&\qquad\qquad \lambda_l^{(2)}(\mu_m^{(3)})^{(-1)^l}\frac{y_{l,m}}{y_{l+1,m}} + (\lambda_l^{(3)})^{(-1)^m}\mu_m^{(1)}\frac{x_{l+1,m+1}}{x_{l+1,m}},
\end{aligned} \tag{33a}$$

$$\begin{aligned}
&(\lambda_l^{(3)})^{(-1)^m}\mu_m^{(1)}x_{l,m+1}y_{l+1,m+1} + \lambda_l^{(2)}(\mu_m^{(3)})^{(-1)^l}x_{l,m}y_{l,m+1} = \\
&\qquad\qquad \frac{\mu_m^{(2)}}{(\lambda_l^{(3)})^{(-1)^m}}x_{l,m}y_{l+1,m} \\
&\qquad\qquad + \frac{\lambda_l^{(1)}}{(\mu_m^{(3)})^{(-1)^l}}x_{l+1,m}y_{l+1,m+1}.
\end{aligned} \tag{33b}$$

In Equations (32) and (33) the functions λ_l^i and $\mu_m^{(i)}$ are *arbitrary*. The name of the systems follows from the fact that putting

$$y_{l,m} = x_{l,m}, \quad \lambda_l^{(i)} = \lambda^{(i)}, \mu_m^{(i)} = \mu^{(i)}, i = 1,2, \quad \lambda_l^{(3)} = \mu_l^{(3)} = 1, \tag{34}$$

and applying a proper scaling of $x_{l,m}$, one gets the lattice sine-Gordon (lsG) equation [67, 68]

$$x_{l,m}x_{l+1,m}x_{l,m+1}x_{l+1,m+1} = p(x_{l,m}x_{l+1,m+1} - x_{l+1,m}x_{l,m+1}) + r \tag{35}$$

and the lattice modified KdV (lmKdV) equation [64]

$$x_{l+1,m+1}\left(px_{l,m+1} - rx_{l+1,m}\right) = x_{l,m}\left(px_{l+1,m} - rx_{l,m+1}\right) \tag{36}$$

respectively.

These systems are non-autonomous and contain arbitrary functions. To emulate such a behaviour in our computation of the degree sequences I associate to any arbitrary function a random sequence of integers defined on the range of the initial conditions. Running the computations more than once I check that we obtain always the same result, meaning that it is independent of the choice of the arbitrary functions.

Then, we have that the lSG_2 system has $(-,+)$ and $(+,-)$ as admissible directions. In such directions it is strongly isotropic with the following sequence of degrees:

$$1, 4, 10, 19, 31, 46, 64, 85, 109 \ldots. \tag{37}$$

This sequence of degrees has the following generating function:

$$g(s) = -\frac{s^2+s+1}{(s-1)^3}, \tag{38}$$

From Corollary 2.7 we see immediately that the growth is quadratic. In this case we can even more precise, and find the analytic form of the sequence (25) using for instance the $\mathcal{Z}$-transform method:

$$d_k = \frac{3}{2}k(k+1) + 1. \tag{39}$$

Analogously, the lmKdV_2 system has $(-,+)$ and $(+,-)$ as admissible directions, but differently from all other examples considered up to now it is permutationally isotropic. For instance the degree sequences in the $(-,+)$ direction are:

$$x_{l,m}\colon 1, 4, 8, 15, 23, 34, 46, 61, 77 \ldots, \tag{40a}$$
$$y_{l,m}\colon 1, 2, 6, 11, 19, 28, 40, 53, 69 \ldots, \tag{40b}$$

while in the $(+,-)$ direction the two sequences are swapped. We have then the following generating functions:

$$g_x(s) = -\frac{s^3+2s+1}{(s-1)^3(s+1)}, \tag{41a}$$
$$g_y(s) = -\frac{s^3+2s^2+1}{(s-1)^3(s+1)}, \tag{41b}$$

which show, by Corollary 2.7, that the asymptotic growth is quadratic. The factor $s+1$ in the denominators adds some two-periodic “noise” to quadratic growth.

Let me now draw a brief comparison between the system and the scalar cases. It is known [75, 78] that the degree sequence of the lSG equation (35) is given by (29) in all directions, hence it is quadratic governed by Equation (31). On the other hand it is easy to see that the for all finite $k > 0$ the degree of lSG equation is smaller than that of lSG_2 system:

$$d_k(\text{lSG}) < d_k(\text{lSG}_2). \tag{42}$$

In the same way it is known that the degree sequence of the lmKdV equation (36) is given by (25) in all directions, hence it is quadratic and governed by Equation (27), so again the growth rates at infinity are comparable. In this case it holds that for all finite $k > 1$:

$$d_k(\text{lmKdV}) < d_{y,k}(\text{lmKdV}_2) < d_{x,k}(\text{lmKdV}_2). \tag{43}$$

3.4 Boussinesq-type systems

In this final subsection, we consider three different kinds of Boussinesq-type systems. Two of these systems are defined as "incomplete" systems, where one equation is a bona fide quad-equation and the other two are defined only on three points. The latter one is a system of coupled quad equations. A comprehensive review of these kind of systems is given in [51]. We finally note that some results on the growth of a Boussinesq equation defined on a 3×3 stencil recently appeared in [50].

3.4.1 Boussinesq system

The Boussinesq system is [65]:

$$z_{l+1,m} - x_{l,m}x_{l+1,m} + y_{l,m} = 0, \tag{44a}$$
$$z_{l,m+1} - x_{l,m}x_{l,m+1} + y_{l,m} = 0, \tag{44b}$$
$$(x_{l,m+1} - x_{l+1,m})(z_{l,m} - x_{l,m}x_{l+1,m+1} + y_{l+1,m+1}) - p + q. \tag{44c}$$

Due to the fact that Equations (44a) and (44b) are defined only on the edges of the quad-graph Figure 1, we have that the only admissible direction is the direction $(+,-)$. In such a direction the degree sequences are the following:

$$x_{l,m}\colon 1, 2, 4, 7, 14, 21, 30, 43, 55, 70, 89, 106, 127, 152, 174, 201, 232\dots, \tag{45a}$$
$$y_{l,m}\colon 1, 2, 4, 9, 14, 21, 32, 43, 55, 72, 89, 106, 129, 152, 174, 203, 232\dots, \tag{45b}$$
$$z_{l,m}\colon 1, 3, 5, 8, 15, 22, 32, 44, 56, 73, 90, 107, 131, 153, 175, 206, 233\dots. \tag{45c}$$

Then the generating functions have the following form:

$$g_i(s) = -\frac{P_i(s)}{(s-1)^3(s^2+s+1)^2} \tag{46}$$

for $i = x, y, z$, where:

$$P_x(s) = 4s^6 + 3s^5 + 5s^4 + s^3 + 2s^2 + s + 1, \tag{47a}$$

$$P_y(s) = 2s^7 + 2s^6 + 3s^5 + 3s^4 + 3s^3 + 2s^2 + s + 1, \tag{47b}$$
$$P_z(s) = 5s^6 + 3s^5 + 3s^4 + s^3 + 2s^2 + 2s + 1. \tag{47c}$$

Again from Corollary 2.7 we have that all equations have quadratic growth. The factor $(s^2 + s + 1)^2$ in the denominator of Equation (46) introduces oscillations of order k.

3.4.2 Schwarzian Boussinesq system

The Schwarzian Boussinesq system is [63]:

$$x_{l+1,m}y_{l,m} = z_{l+1,m} - z_{l,m}, \tag{48a}$$
$$x_{l,m+1}y_{l,m} = z_{l,m+1} - z_{l,m}, \tag{48b}$$
$$x_{l,m}y_{l+1,m+1}(y_{l+1,m} - y_{l,m+1}) = y_{l,m}\left(px_{l+1,m}y_{l,m+1} - qx_{l,m+1}y_{l+1,m}\right). \tag{48c}$$

Due to the fact that Equations (48a) and (48b) are defined only on the edges of the quad-graph Figure 1, we have that the only admissible direction is the direction $(+,-)$. In such a direction the degree sequences are the following:

$$x_{l,m}\colon 1, 4, 8, 12, 23, 34, 43, 62, 80, 94, 121, 146, 165, 200, 232, 256\ldots, \tag{49a}$$
$$y_{l,m}, z_{l,m}\colon 1, 2, 7, 12, 19, 32, 43, 56, 77, 94, 113, 142, 165, 190, 227, 256\ldots. \tag{49b}$$

Then the generating functions have the same shape as (46), but with numerator polynomials indexed by $i = x, y|z$ given by:

$$P_x(s) = 2s^6 + 3s^5 + 5s^4 + 2s^3 + 4s^2 + 3s + 1, \tag{50a}$$
$$P_{y|z}(s) = 2s^6 + 3s^5 + 5s^4 + 3s^3 + 5s^2 + s + 1. \tag{50b}$$

Again from Corollary 2.7 we have that all equations have quadratic growth, with the same comment about the oscillations.

3.4.3 Modified Boussinesq system

The modified Boussinesq system is [85]:

$$x_{l+1,m+1}(py_{l+1,m} - qy_{l,m+1}) = y_{l,m}(px_{l,m+1} - qx_{l+1,m}), \tag{51a}$$
$$x_{l,m}y_{l+1,m+1}(py_{l+1,m} - qy_{l,m+1}) = y_{l,m}(px_{l,m+1}y_{l,m+1} - qx_{l+1,m}y_{l+1,m}). \tag{51b}$$

All four directions are admissible for the system (51). Moreover, the system is permutationally isotropic. The sequence of the degrees in the directions $(+,-)$ and $(-,+)$ the degree sequence is the same, while in the directions $(+,+)$ and $(-,-)$ it is permuted. For instance, in the $(+,-)$ direction we have the sequences:

$$x_{l,m}\colon 1, 4, 8, 12, 23, 34, 43, 62, 80, 94, 121, 146, 165, 200, 232, 256\ldots, \tag{52a}$$
$$y_{l,m}\colon 1, 2, 7, 12, 19, 32, 43, 56, 77, 94, 113, 142, 165, 190, 227, 256\ldots. \tag{52b}$$

Then the generating functions have the same shape as (46), but with numerator polynomials indexed by $i = x, y$ given by:

$$P_x(s) = (s^2 + 1)(s^4 + 2s^3 + 3s^2 + s + 1), \tag{53a}$$
$$P_y(s) = s^6 + 2s^5 + 4s^4 + 2s^3 + 4s^2 + 2s + 1. \tag{53b}$$

Again from Corollary 2.7 we have that all equations have quadratic growth, with the same comment about the oscillations.

4 Discussion and perspectives

In this short commemorative note I explained the construction I used over the years to compute the algebraic entropy for systems of quad equations. This construction follows closely the well known one for the scalar setting, see [75, 78]. My point of view is that the system case needs to be handled more carefully because several properties that are taken for granted in the scalar case no longer hold. Indeed, the main problem is that for systems, multilinear it does not imply a well-defined evolution in all directions any more.

I considered, the following examples:

- a coupled lattice potential KdV system;
- a lattice NLS system;
- the lSG_2 and the lmKdV_2 systems;
- the Boussinesq system, and its Schwarzian and modified versions.

These systems showed several possible different behaviours. Some systems behave much more like their scalar counterparts, like the coupled lattice potential KdV system, while some others like the lSG_2 and the lmKdV_2 systems do not. In particular, in the latter, differently from the scalar case, the evolution is not defined in all directions, and the lmKdV_2 is just permutationally isotropic and not strongly isotropic. We also remark that the Boussinesq and the Schwarzian Boussinesq systems have only one admissible direction. However, using the “augmentation” procedure described in [10], e.g. on the Schwarzian Boussinesq system (48) we obtain the following system:

$$x_{l+1,m+1}y_{l,m+1} = z_{l+1,m+1} - z_{l,m+1}, \tag{54a}$$
$$x_{l+1,m+1}y_{l+1,m} = z_{l+1,m+1} - z_{l+1,m}, \tag{54b}$$
$$x_{l,m}y_{l+1,m+1}(y_{l+1,m} - y_{l,m+1}) = y_{l,m}\left(px_{l+1,m}y_{l,m+1} - qx_{l,m+1}y_{l+1,m}\right). \tag{54c}$$

Without entering into the details, by a direct computation it is possible to see that the system (54) has admissible directions $(-,+)$, $(+,+)$, and $(-,-)$. In the direction $(-,+)$ the growth is still quadratic, but the system becomes trivially linear in the other two directions! So, the understanding of growth properties of systems of quad equations and the definition of integrability seem to be more complicated that in the scalar case.

Finally, we note that this method was originally conceived to check the growth properties of trapezoidal H^4 equations and H^6 equations [6–8] written as systems of four quad equations, see e.g. [33, 38]. These results are too lengthy to be presented in this short note, but they can be summarised by saying that all systems display linear growth, and the H_4 "systems" have two admissible directions, while the H_6 "systems" only have one. What are left out by this construction are systems of quad equations where the explicit solution is algebraic in *all directions*. The set of integrable equations of this kind is non-empty: in [54] using a multi-component equivalent of the Consistency Around the Cube procedure the following system, valid for all $M \in \mathbb{N}$ was introduced:

$$\sum_{i=1}^{M} \left(\frac{\alpha_1 - \beta_1}{x_{l,m}^{(i)} - w_k} + \frac{\beta_1 - \alpha_2}{x_{l+1,m}^{(i)} - w_k} + \frac{\alpha_2 - \beta_2}{x_{l+1,m+1}^{(i)} - w_k} + \frac{\beta_2 - \alpha_1}{x_{l,m+1}^{(i)} - w_k} \right) = 0, \tag{55}$$

where $k = 1, \dots, M$ and w_k, α_j, and β_j are constants. Characterising the growth properties of such "infinite dimensional correspondences" can be a challenging problem, which can generalise the ideas given, for instance, in [52, 76].

Acknowledgments

I thank (alphabetical ordering) Giuseppe Gaeta, Orlando Ragnisco, Paolo Maria Santini, and Federico Zullo for their help in handling the Special Issue of OCNMP in Memory of Decio Levi, and their comments on the draft of this manuscript.

Moreover, this work was supported by the GNFM INdAM through Progetto Giovani GNFM 2023: "Strutture variazionali e applicazioni delle equazioni alle differenze ordinarie" (CUP_E53C22001930001), INFN Gruppo 4, and INFN Special Initiative *Mathemathical Methods of Nonlinear Physics*.

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