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Sigma model instantons and singular tau function

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Abstract

The generating series for the instanton contribution to Green functions of the $2D$ sigma model was found in the works of Schwarz, Fateev and Frolov. We show that this series can be written as a formal tau function of the two-component KP hierarchy. The higher times of the two-component tau function allow to consider various multi-parameter insertions into the instanton partition function, therefore the tau function can be treated as the generating function for various correlators. The construction can be generalized to the multicomponent case, which gives more parameters for the generating function of the correlators. We call it formal singular tau function because this tau function is a sum where each term is the infrared and ultraviolet divergent one exactly as the series found by the mentioned authors. However, one can regularize each divergent term of this singular tau function in such a way that it is still a tau function. Thus, we enlarge the families of tau functions to work with.

In memory of Vladimir E. Zakharov

1 Introduction

The main purpose of this paper is to interpretate the contribution of instantons in the Euclidean Green function of the O(3) non-linear σ model (or the continuum classical Heisenberg ferromagnetic in two space dimensions) in terms of tau functions of integrable hierarchies. This model can be described by the action

$$S = \frac{1}{2f} \int \sum_{a=1}^{3} \left(\partial_\mu \sigma^a\left(x\right)\right)^2 \tag{1}$$

where σ^a, $a = 1, 2, 3$ are the components of the unit vector: $\sum_{a=1}^{3} \sigma^a\left(x\right)\sigma^a\left(x\right) = 1$; $\mu = 0, 1$.

The model is similar to a Yang-Mills theory and possesses exact multi-instanton solutions. The Euclidean Green functions can be represented in the form

$$\frac{\int \phi(\sigma) \exp(-S) \prod_x d\sigma(x)}{\int \exp(-S) \prod_x d\sigma(x)} \tag{2}$$

Here $\phi(\sigma)$ is an arbitrary functional of σ. If we parametrize $\sigma(x)$ with use of the complex function

$$\omega(z) = \frac{\sigma^1(z) + i\sigma^2(z)}{1 + \sigma^3(z)} \tag{3}$$

(the stereographic projection) obtained from the fields $\left(\sigma^1, \sigma^2, \sigma^3\right)$ and the complex variable $z = x_0 + ix^1$ instead of the time and space coordinates x_0, x_1, then the instanton is the solution of the equation $\delta S = 0$ with the topological charge $q > 0$ is given [1]

$$\omega_q(a, b, z) = c\frac{(z - a_1)\dots(z - a_q)}{(z - b_1)\dots(z - b_q)} \tag{4}$$

where c, a_i and b_i are arbitrary complex parameters.

Let us note that the classical σ-model in Minkowski space is the well-studied integrable model, see [8].

2 The instanton contribution and the τ function

In [2], the instanton contribution to the Euclidean Green functions of the fields σ using the steepest descent approximation was obtained. If ϕ is a functional of the instanton fields ω, then the evaluation of the functional integral around the instanton vacuums yields [2] the answers written in form of multiple integrals over instanton parameters:

$$\langle\phi\rangle_{\text{inst}} = \left[\frac{\sum_{q\geq 0} \frac{K^q}{(q!)^2} \int \phi(\omega_q) \prod_{i<j\leq q} \frac{|a_i-a_j|^2|b_i-b_j|^2}{|a_i-b_j|^2|b_i-a_j|^2} \prod_{i=1}^q \frac{d^2a_i d^2b_i}{|a_i-b_i|^2}}{\sum_{q\geq 0} \frac{K^q}{(q!)^2} \int \prod_{i<j\leq q} \frac{|a_i-a_j|^2|b_i-b_j|^2}{|a_i-b_j|^2|b_i-a_j|^2} \prod_{i=1}^q \frac{d^2a_i d^2b_i}{|a_i-b_i|^2}}\right]_{\text{reg}}, \tag{5}$$

where K is a real constant obtained as the result of the regularization procedure[1], and where for each q the instanton solution ω is given by (4). The denominator in (5) coincides with the partition function Ξ of the neutral classical two-dimensional Coulomb system (CCS) in the grand canonical ensemble with the definite temperature T (T=1 see [2]) (such a system was called the system of instanton quarks in [2]). The point T=1 is above the critical temperature (which is about T=1/2); this means that the Coulomb gas is in the plasma phase. (Below the critical temperature the Coulomb particles form dipoles). The symbol $[]_{\text{reg}}$ means that this expression should be regularized in the ultraviolet limit, where $a_i \to b_j$. Physical answers do not depend on the method of the regularization.

Note that, in fact, instanton-anti-instanton interaction is also significant (see Lipatov-Bukhvostov [3]) but this was not considered in the work [2], and we also will not touch on this much more involved topic.

[1] According to [2] the constant K is proportional to $k_0 f_{\text{phys}}^{-2} \exp\left(-4\pi f_{\text{phys}}^{-1}\right) \nu$ where ν is the substraction point, f_{phys} is a physical coupling constant, k_0 is a constant depending on the cutoff method.

Regularization. Let us notice that the answer (5) was obtained [2] as the result of the calculation of the functional integral and a certain regularization procedure and, in turn, the multiple integrals in (5) are both infrared (IR) and ultraviolet (UV) divergent, and one needs an additional regularization procedure. In short, it is discussed in [2], page 11.

As for the IR divergence (the divergence in the limit $a_i, b_i \to \infty$), it just means that one should be interested in the densities of the instanton partition function (and of the correlation function) rather than the partition and the correlation functions themselves. Then it is reasonable to restrict the domain of the integration over each a_i to the $D = L\times L$ box in the complex plane, the same for b_i [2]. To get the density we divide each integral over L^2, simultaneously we send the constant K to KL^2.

As for the ultraviolet regularization in the regions $b_i \approx a_j$ there are different ways:

(A) We can do the following: we produce the replacement $b_i \to b_i + \epsilon$, $\bar{b}_i \to \bar{b}_i - \epsilon$, where ϵ is a small real number where ϵ^{-1} may be treated as a cutoff in the momentum space.

In particular, for the one-instanton partition function, we get

$$K\int \frac{d^2a d^2b}{|b-a|^2} \to \left(KL^2\right) L^{-2}\int_{D^2} \frac{d^2a d^2b}{|a-b|^2_\epsilon} \quad \text{where} \quad |a-b|^2_\epsilon := |a-b|^2 - \epsilon^2 + i\epsilon\Im(a-b) \tag{6}$$

The contribution of the region $b \approx a$ is finite and of order ϵ^{-1}. Let us notice that, thanks to the structure of the numerators inside the integrals in (5), the order of the q-instanton integral is ϵ^{-q}. Thus, to get finite expressions, we send $K \to KL^2\epsilon$.

(B) One is to replace integrals by sums, that is, to consider the Coulomb gas on the 2D lattice as mentioned in [2] with the list of references. We can do it as follows: we take a small real number h (square grid spacing) and set

$$a(n,m) = nh + imh \quad b(n,m) = (n+\gamma)h + i(m+\gamma')h \tag{7}$$

with non-integer γ, γ'. In fact, we have two lattices: one for positive and the other for negative Coulomb particles:

$$K\int \frac{d^2a d^2b}{|b-a|^2} \quad \to \quad \left(KL^2\right) L^{-2} \sum_{0\le n,n',m,m'\le L} \frac{h^{-2}}{|n'-n+im'-im+\frac{1}{2}(\gamma+i\gamma'^2} \tag{8}$$

The summation range $0 \le n, m \le L$ will also be denoted D, as in the previous case.

Our goal is to relate (5) with the regularizations (A)-(B) to classical integrable systems.

3 Tau functions

3.1 Two-sided two-component KP and the regularization (A)

In this case, we use (6) and write the correlation function as

$$\langle\phi\rangle^A_{\text{inst}} = \frac{\sum_{q\ge 0} \frac{K^q}{(q!)^2}\int_{D^{2q}} \phi(\omega_q) \prod_{i<j\le q} \frac{|a_i-a_j|^2|b_i-b_j|^2}{|a_i-b_j|^2_\epsilon |b_i-a_j|^2_\epsilon} \prod_{i=1}^q \frac{d^2a_i d^2b_i}{|a_i-b_i|^2_\epsilon}}{\sum_{q\ge 0} \frac{K^q}{(q!)^2}\int_{D^{2q}} \prod_{i<j\le q} \frac{|a_i-a_j|^2|b_i-b_j|^2}{|a_i-b_j|^2_\epsilon |b_i-a_j|^2_\epsilon} \prod_{i=1}^q \frac{d^2a_i d^2b_i}{|a_i-b_i|^2_\epsilon}} \tag{9}$$

We will see that it is a certain τ function of the two-sided two-component KP. Multi-component KP tau functions were introduced in the works of the Kyoto School [4] in terms of free fermion formalism, see also later works [6] and [5]. The construction of tau functions implies the use of free massless fermions:

$$\psi^{(\alpha)}(z) = \sum_{i\in\mathbb{Z}} \psi_i^{(\alpha)} z^i, \quad \psi^{\dagger(\alpha)}(z) = \sum_{i\in\mathbb{Z}} \psi_i^{\dagger(\alpha)} z^{-1-i} \tag{10}$$

where α is a "color" of fermions ($\alpha = 1,2$) and anti-commutation expressions for the Fermi modes $\psi_i^{(\alpha)}, \psi_i^{\dagger(\alpha)}$ are

$$\left[\psi_i^{(\alpha)}, \psi_j^{(\beta)}\right]_+ = 0 \qquad \left[\psi_i^{\dagger(\alpha)}, \psi_j^{\dagger(\beta)}\right] = 0 \qquad \left[\psi_i^{(\alpha)}, \psi_j^{\dagger(\beta)}\right]_+ = \delta_{\alpha,\beta}\delta_{i,j} \tag{11}$$

The fermionic states with occupied levels up to $n^{(1)}, n^{(2)}$ satisfy the conditions

$$\langle n^{(1)}, n^{(2)} | m^{(1)}, m^{(2)} \rangle = \delta_{n^{(1)},m^{(1)}} \delta_{n^{(2)},m^{(2)}}$$

$$\psi_i^{(\alpha)} | n^{(\alpha)}, * \rangle = \langle n^{(\alpha)}, * | \psi_i^{\dagger(\alpha)} = \psi_{-1-i}^{\dagger(\alpha)} | n^{(\alpha)}, * \rangle = \langle n^{(\alpha)}, * | \psi_{-1-i}^{(\alpha)} = 0, \quad i < n^{(\alpha)} \tag{12}$$

We denote the sets $n^{(\alpha)}, t_{\pm 1}^{(\alpha)}, t_{\pm 2}^{(\alpha)}, \dots$ by $\mathbf{t}^{(\alpha)}$, $\alpha = 1,2$ where $n^{(\alpha)}$ are integers and where $t_{\pm i}^{(\alpha)}$ are complex parameters. The sets $\{t_{\pm i}^{(\alpha)}, i > 0\}$ we denote $\mathbf{t}_{\pm}^{(\alpha)}$.

The family of tau functions of the two-sided two-component KP, which is related to (9), is given by

$$\tau(n, \mathbf{t}^{(1)}, \mathbf{t}^{(2)}) =$$

$$\langle n^{(1)}, n^{(2)} | \, \Gamma_1\left(\mathbf{t}_+^{(1)}\right) \Gamma_2\left(\mathbf{t}_+^{(2)}\right) g_1 g_2 \Gamma_1^\dagger\left(\mathbf{t}_-^{(1)}\right) \Gamma_2^\dagger\left(\mathbf{t}_-^{2}\right) | n^{(2)} - n, n^{(1)} + n \rangle. \tag{13}$$

where

$$g_1 = e^{K^{\frac{1}{2}} \int_{D^2} \psi^{(1)}(a) \psi^{\dagger(2)}(\bar{a}) d^2 a}, \quad g_2 = e^{K^{\frac{1}{2}} \int_{D^2} \psi^{(2)}(\bar{b}-\epsilon) \psi^{\dagger(1)}(b+\epsilon) d^2 b} \tag{14}$$

where the "evolution operators"

$$\Gamma_\alpha\left(\mathbf{t}_\mp^{(\alpha)}\right) = e^{\sum_{i>0} t_i^{(\alpha)} J_i^{(\alpha)}}, \quad \Gamma_\alpha^\dagger\left(\mathbf{t}_\mp^{(\alpha)}\right) = e^{\sum_{i>0} t_{-i}^{(\alpha)} J_{-i}^{(\alpha)}} \tag{15}$$

are expressed in terms of the modes of the currents $J_i^{(\alpha)}$. The current is given by

$$: \psi^{(\alpha)}(z) \psi^{\dagger(\alpha)}(z) := \psi^{(\alpha)}(z) \psi^{\dagger(\alpha)}(z) - \langle 0 | \psi^{(\alpha)}(z) \psi^{\dagger(\alpha)}(z) | 0 \rangle = \sum_{i\in\mathbb{Z}} J_i^{(\alpha)} z^{i-1} \tag{16}$$

It's modes can be written as follows:

$$J_m^{(\alpha)} = \sum_{i\in\mathbb{Z}} : \psi_i^{(\alpha)} \psi_{i+m}^{(\alpha)} : \tag{17}$$

We have the Heisenberg algebra:

$$\left[J_k^{(\alpha)}, J_m^{(\beta)}\right] = k \delta_{\alpha,\beta} \delta_{k+m,0} \quad . \tag{18}$$

The discrete variables $n, n^{(\alpha)}$ and complex parameters $t^{(\alpha)}_{\pm i}$, $\alpha = 1,2,\ i = 1,2,3,\dots$ are called the higher times of the two-sided two-component KP hierarchy. In what follows, we put $n = 0$ and omit it from the notations.

Remark. We use the term "two-sided" in relation to tau functions if we are interested in the dependence of the tau function on both sets: on $\mathbf{t}_+^{(1,2)}$ and by $\mathbf{t}_-^{(1,2)}$. Otherwise, we call it just a "two-component" tau function ($\alpha = 1,2$).

Remark. The insertion of $g_1 g_2$ can be interpreted as the introduction of mass for fermions, which were initially massless. This approach is developed in the next work [9].

As a result of direct evaluation (using the relations in Appendix 4), we obtain the following τ function of the two-sided two-component KP:

$$\tau^A\left(\mathbf{t}^1,\mathbf{t}^2|D,\epsilon\right) = \sum_{q\geq 0}\frac{K^q}{(q!)^2}\int_{D^{2q}}\Phi_q\left(\mathbf{a},\mathbf{b},\mathbf{t}^1,\mathbf{t}^2\right)\prod_{i<j\leq q}\frac{|a_i-a_j|^2|b_i-b_j|^2}{|a_i-b_j|^2_\epsilon|b_i-a_j|^2_\epsilon}\prod_{i=1}^{q}\frac{d^2a_i d^2b_i}{|a_i-b_i|^2_\epsilon} \tag{19}$$

where the function

$$\Phi_q\left(\mathbf{a},\mathbf{b},\mathbf{t}^1,\mathbf{t}^2\right) = \prod_{i=1}^{q}\left(\frac{a_i}{b_i}\right)^{n^{(1)}}\left(\frac{\bar a_i}{\bar b_i}\right)^{-n^{(2)}} e^{\theta(a_i,\mathbf{t}^1)-\theta(\bar a_i,\mathbf{t}^2)+\theta(b_i,\mathbf{t}^2)-\theta(\bar b_i,\mathbf{t}^1)} \tag{20}$$

$$\theta(z,\mathbf{t}^1) = V(z,\mathbf{t}^1_+) + V(z^{-1},\mathbf{t}^1_-),\quad V(z,t) = \sum_{m>0} t_m z^m \quad . \tag{21}$$

We imply that the parameters $\mathbf{t}^{(1,2)}$ are chosen in such a way that the integrals in (19) are convergent.

Moreover, if we modify the dependence of θ on higher times, according to

$$\theta(z,\mathbf{t}) = V(z,\mathbf{t}_+) + V(z^{-1},\mathbf{t}_-) + \sum_{\alpha=1}^{P} V((z-s_\alpha,\mathbf{p}^\alpha)), \tag{22}$$

we obtain the tau function of the $(P+4)$-component tau function, where the additional sets of higher times are the sets $\mathbf{p}^\alpha = (p_1^{(\alpha)}, p_2^{(\alpha)}, p_3^{(\alpha)},\dots)$, $\alpha = 1,\dots,P$. In what follows, we will not use this additional freedom. In this short work, we shall use only the sets $\mathbf{t}^1_+$ and $\mathbf{t}^2_-$ which will be denoted $\mathbf{t}^1$ and $\mathbf{t}^2$, respectively.

Because $\Phi_q(\mathbf{a},\mathbf{b},0,0) = 1$, the tau function evaluated at $\mathbf{t}^1 = \mathbf{t}^2 = 0$ is equal to the instanton grand partition function $\tau^A(0,0|D,\epsilon) = Z_{\text{inst}}$ and, for

$$\phi_q(\mathbf{a},\mathbf{b}) = \Phi_q\left(\mathbf{a},\mathbf{b},\mathbf{t}^1,\mathbf{t}^2\right) \tag{23}$$

we observe

$$\langle\phi\rangle^A_{\text{inst}} = \frac{\tau^A\left(\mathbf{t}^1,\mathbf{t}^2|D,\epsilon\right)}{\tau^A(0,0|D,\epsilon)}, \tag{24}$$

There is another and shorter way to write down the tau function (13). Let us introduce

$$\psi^{(i)}(z,\mathbf{t}^i) = e^{\theta(z,\mathbf{t}^i)}\psi^{(i)}(z),\quad \psi^{\dagger(i)}(\bar z,\mathbf{t}^i) = e^{-\theta(\bar z,\mathbf{t}^i)}\psi^{\dagger(i)}(\bar z),\quad i=1,2 \tag{25}$$

Then, the multi-component tau function can be written as

$$\tau(\mathbf{t},n^{(1)},n^{(2)}) = c(\mathbf{t})\langle n^{(1)},n^{(2)}|g_1(\mathbf{t})g_2(\mathbf{t})|n^{(1)},n^{(2)}\rangle \tag{26}$$

where $c(\mathbf{t}) = \exp\sum_{i=1,2}\sum_{m>0} m t_m^{(i)} t_{-m}^{(i)}$ and where g_i, $i = 1,2$ are given by (14) where the Fermi fields

$$\psi^{(1)}(z),\ \psi^{\dagger(1)}(\bar{z}),\ \psi^{(2)}(z),\ \psi^{\dagger(2)}(\bar{z})$$

are replaced respectively by

$$\psi^{(1)}(z,\mathbf{t}^1),\ \psi^{\dagger(1)}(\bar{z},\ \mathbf{t}^1),\ \psi^{(2)}(z,\mathbf{t}^2),\ \psi^{\dagger(2)}(\bar{z},\mathbf{t}^i)$$

.

In the rest of the paper, we put $n^{(1)} = n^{(2)} = 0$.

Discrete KP equations. If we specify the parameters as follows:

$$t_k^{(1)}[\mathtt{n},z] := -\frac{1}{k}\sum_{i=1}^{N}\mathtt{n}_i z_i^{-k}, \quad t_k^{(2)}[\mathtt{m},y] = -\frac{1}{k}\sum_{i=1}^{M}\mathtt{m}_i y_i^{-k} \tag{27}$$

and denote such sets as $\mathbf{t}^1[\mathtt{n},z]$ and $\mathbf{t}^2[\mathtt{m},y]$, we obtain

$$\Phi_q\left(a,b,\mathbf{t}^1[\mathtt{n},z],\mathbf{t}^2[\mathtt{m},y]\right) = \prod_{i=1}^{N}\left(\omega_q(a,b,z_i)\right)^{\mathtt{n}_i}\prod_{i=1}^{M}\left(\omega_q(\bar{a},\bar{b},y_i)\right)^{-\mathtt{m}_i} \tag{28}$$

where ω_q was defined by (4). Let us notice that $\omega(a,b,z)\omega(\bar{a},\bar{b},\bar{z}) = |\omega(a,b,z)|^2$.

The tau function written in the variables defined by (27) solves the so-called discrete KP equation; see [4] (and [7] for the review). If $\sigma^{(i)}(x)$, $i = 1,2,3$ are instanton solutions of form (4)-(3), then, for the correlation function

$$G_{\mathtt{n}_1,\mathtt{n}_2,\mathtt{n}_3}(z_1,z_2,z_3) :=$$

$$\left\langle \left(\frac{\sigma^1(z_1)+i\sigma^2(z_1)}{1+\sigma^3(z_1)}\right)^{\mathtt{n}_1}\left(\frac{\sigma^1(z_2)+i\sigma^2(z_2)}{1+\sigma^3(z_2)}\right)^{\mathtt{n}_2}\left(\frac{\sigma^1(z_3)+i\sigma^2(z_3)}{1+\sigma^3(z_3)}\right)^{\mathtt{n}_3}\right\rangle_{\text{inst}}^{A} \tag{29}$$

one can write the discrete Hirota bilinear equation (in other words, as the discrete KP equation):

$$(z_2-z_3)G_{\mathtt{n}_1+1,\mathtt{n}_2,\mathtt{n}_3}(z_1,z_2,z_3)G_{\mathtt{n}_1,\mathtt{n}_2+1,\mathtt{n}_3+1}(z_1,z_2,z_3)$$

$$+(z_3-z_1)G_{\mathtt{n}_1,\mathtt{n}_2+1,\mathtt{n}_3}(z_1,z_2,z_3)G_{\mathtt{n}_1+1,\mathtt{n}_2,\mathtt{n}_3+1}(z_1,z_2,z_3)$$

$$+(z_1-z_2)G_{\mathtt{n}_1,\mathtt{n}_2,\mathtt{n}_3+1}(z_1,z_2,z_3)G_{\mathtt{n}_1+1,\mathtt{n}_2+1,\mathtt{n}_3}(z_1,z_2,z_3) = 0 \tag{30}$$

Other sets of equations may be written for general correlation functions involving (28) (this will be done in a more detailed text).

Densities. As we mentioned, the denominator in (5) coincides with the partition function Ξ of the neutral classical Coulomb system (CCS) in the grand canonical ensemble with the definite temperature T (T=1 see [2]).

$$\tau(0,0,0,0) = \Xi \tag{31}$$

The constant K plays the role of fugacity in the Coulomb system. The expression (24) also coincides with the correlation function of the CCS (at T=1). Let us consider the instanton contribution $G^{\mathrm{inst}}(x,y)$ in the Green function

$$G(x,y) = \langle \triangle_x \log|\omega(x)|, \triangle_y \log|\omega(y)|\rangle \tag{32}$$

corresponding to functional $\phi(\omega) = \triangle_x \log|\omega(x)| \triangle_y \log|\omega(y)|$ that is $\rho(x)\rho(y)$ with $\rho(x) = 2\pi\left(\sum_i \delta(x-a_i) - \sum_i \delta(x-b_i)\right)$. In order to obtain this result in terms of τ functions, we have to make the Miwa transformation of times

$$t_n^{(\alpha)} = -\frac{t}{nx^n} - \frac{t}{ny^n} \tag{33}$$

then we achieve

$$\triangle_x \triangle_y \frac{\partial}{\partial t}\left(\prod_i^q \Phi_{0,0}(a_i,b_i,\mathbf{t}^1,\mathbf{t}^2)|_{t_n^{(\alpha)}=-\frac{t}{nx^n}-\frac{t}{ny^n}}\right)|_{t=0} \tag{34}$$

$$= \triangle_x \log|\omega(x)| \triangle_y \log|\omega(y)| = \rho(x)\rho(y). \tag{35}$$

One can interpret $\rho(x)$ as the charge density. We see

$$G^{\mathrm{inst}}(x,y) = \langle \triangle_x \log|\omega(x)| \triangle_y \log|\omega(y)|\rangle_{inst} = \langle \rho(x)\rho(y)\rangle_{CCS} \tag{36}$$

$$= \frac{C \triangle_x \triangle_y \frac{\partial}{\partial t}\left(\tau(0,0,\mathbf{t}^1,\mathbf{t}^2)|_{t_n^{(\alpha)}=-\frac{t}{nx^n}-\frac{t}{ny^n}}\right)|_{t=0}}{\tau(0,0,0,0)} \tag{37}$$

Similarly to the previous way, we can obtain the instanton contribution in the more general Green function corresponding to the functional

$$\phi(\omega) = \triangle_{x_1} \log|\omega(x_1)| \triangle_{x_2} \log|\omega(x_2)| \ldots \triangle_{x_m} \log|\omega(x_m)| \tag{38}$$

by

$$G^{\mathrm{inst}}(x_1,x_2,\ldots x_m) = \langle \rho(x_1)\rho(x_2)\ldots\rho(x_m)\rangle_{CCS} = \tag{39}$$

$$= \frac{C \triangle_{x_1} \triangle_{x_2}\ldots \triangle_{x_m} \frac{\partial}{\partial t}\left(\tau(0,0,\mathbf{t}^1,\mathbf{t}^2)|_{t_n^{(\alpha)}=-\frac{t}{nx_1^n}-\frac{t}{nx_2^n}\cdots\frac{t}{nx_m^n}}\right)|_{t=0}}{\tau(0,0,0,0)}$$

3.2 Two-component KP and the regularization (B)

In this case, we have

$$\langle\phi\rangle^B_{\rm inst} =$$

$$\frac{\sum_{q\geq 0}\frac{K^q}{(q!)^2}\sum_{D^{2q}}\phi_q\left(a,b\right)\prod_{i<j\leq q}\frac{|a_{n_im_i}-a_{n_jm_j}|^2|b_{n_im_i}-b_{n_jm_j}|^2}{|a_{n_im_i}-b_{n_jm_j}|^2|b_{n_im_i}-a_{n_jm_j}|^2}\prod_{i=1}^q\frac{1}{|a_{n_im_i}-b_{n_im_i}|^2}}{\sum_{q\geq 0}\frac{K^q}{(q!)^2}\sum_{D^{2q}}\prod_{i<j\leq q}\frac{|a_{n_im_i}-a_{n_jm_j}|^2|b_{n_im_i}-b_{n_jm_j}|^2}{|a_{n_im_i}-b_{n_jm_j}|^2|b_{n_im_i}-a_{n_jm_j}|^2}\prod_{i=1}^q\frac{1}{|a_{n_im_i}-b_{n_im_i}|^2}} \tag{40}$$

where $\sum_{D^{2q}}$ means $\sum_{n_1,\dots,n_q,m_1,\dots m_q\in D}$ and where a_{nm},b_{nm} are given by (7).

One just needs to replace integrals by sums according to (8) in the expression (13):

$$\tau_{2KP}(n^{(0)},n^{(1)},n^{(2)},\mathbf{t}^1,\mathbf{t}^2|D,h)=\langle n^{(1)},n^{(2)}|\,\Gamma\left(\mathbf{t}^1\right)\Gamma\left(\mathbf{t}^2\right)\,g\,|n^{(2)}-n^{(0)},n^{(1)}+n^{(0)}\rangle \tag{41}$$

where

$$g=e^{K^{\frac12}\sum_{(k,m)\in D^2}\psi^{(1)}(a_{km})\psi^{\dagger(2)}(\bar a_{km})}\,e^{K^{\frac12}\sum_{(k,m)\in D^2}\psi^{(2)}(\bar b_{km})\psi^{\dagger(1)}(b_{km})} \tag{42}$$

and where the summation range in the exponents is chosen as $0\leq n,m\leq L$. We obtain the enumerator in (40):

$$\tau^B\left(\mathbf{t}^1,\mathbf{t}^2|D,h\right)= \tag{43}$$

$$\sum_{q\geq 0}\frac{K^q}{(q!)^2}\sum_{D^{2q}}\Phi_q\left(a,b,\mathbf{t}^1,\mathbf{t}^2\right)\prod_{i<j\leq q}\frac{|a_{n_im_i}-a_{n_jm_j}|^2|b_{n_im_i}-b_{n_jm_j}|^2}{|a_{n_im_i}-b_{n_jm_j}|^2|b_{n_im_i}-a_{n_jm_j}|^2}\prod_{i=1}^q\frac{1}{|a_{n_im_i}-b_{n_im_i}|^2} \tag{44}$$

If we choose $\phi_q(a,b)=\Phi_q(a,b,\mathbf{t}^1,\mathbf{t}^2)$ we get the same relations as in the previous case A, we replace $\langle *\rangle^A_{\rm inst}$ by $\langle *\rangle^B_{\rm inst}$.

3.3 One-component KP and the regularization (B)

The regularization (B) can also be written as the following KP tau function:

$$\tau^{\rm B}_{\rm KP}(\mathbf{t}|D,h)=\langle n|\Gamma(t)e^{K\sum_{D^2}\frac{\psi(a_{nm})\psi^\dagger(b_{nm})}{\bar a_{nm}-b_{nm}}}|n\rangle \tag{45}$$

where a_{nm} and b_{nm} are given by (7), and

$$\Gamma(t)=e^{\sum_{m>0}t_mJ_m},\quad J_m=\sum_{i\in\mathbb{Z}}\psi_i\psi^\dagger_{i+m}$$

$\Gamma(t)$, the Fermi fields and Φ_q are the same as in subsection 3.1, where the second component is absent:

$$\Phi_q\left(a,b,\mathbf{t}\right)=\prod_{i=1}^q\left(\frac{a_i}{b_i}\right)^n e^{V(a_i,t)-V(b_i,t)},\quad \Phi_q\left(a,b,\mathbf{t}[\mathtt{n},z]\right)=\prod_i\left(\omega(z_i)\right)^{\mathtt{n}_i}$$

where V was defined in (21) and where $\mathbf{t}[\mathtt{n}, z]$ denotes the choice $n = 0$ and $t_m = -\sum \mathtt{n}_i z_i^m$, $m > 0$. We get the same equation (30) for

$$G_{\mathtt{n}_1,\mathtt{n}_2,\mathtt{n}_3}(z_1, z_2, z_3) := \langle (\omega(z_1))^{\mathtt{n}_1} (\omega(z_2))^{\mathtt{n}_2} (\omega(z_3))^{\mathtt{n}_3} \rangle^B_{\text{inst}}$$

Formula (45) yields the same answer as (41) if we put $\mathbf{t}^2 = 0$ and $\mathbf{t}^1 = \mathbf{t}$ (see Appendix 4). However, in the case of the one-component KP, we can not construct $|\omega(z)|$ by specializing the parameters $t_1, t_2, \dots$ in Φ.

3.4 Regularization C

Few words about different regularization (without details). This regularization is performed directly in the expression for the tau function.

(a) The ultri-violet regularization is achieved by the cutting out of higher Fermi modes:

$$\psi^{(i)}(z) \to \psi^{(i)}(z; M) = \sum_{j \le M} z^j \psi_j^{(i)}, \quad \psi^{\dagger(i)}(z) \to \psi^{\dagger(i)}(z; M) = \sum_{j \le M} z^j \psi_{-j-1}^{\dagger(i)}, \qquad (46)$$

where M is the cutting parameter.

(b) The infra-red regularization is done via the including of the decay to the measure

$$\int \psi^{(1)}(a)\psi^{(2)\dagger}(\bar{a})d^2a \to \int \psi^{(1)}(a)\psi^{(2)\dagger}(\bar{a})e^{-\epsilon|a|^2}d^2a \qquad (47)$$

4 Appendix. Useful relations

We use the following relations:

$$\Gamma(t)\psi(z) = e^{V(z,t)}\psi(z)\Gamma(t), \quad \Gamma(t)\psi^\dagger(z) = e^{-V(z,t)}\psi^\dagger(z)\Gamma(t)$$

and $\Gamma(t)|n\rangle = |n\rangle$. Then

$$\langle n|\psi(z_1)\psi^\dagger(y_1)\cdots\psi(z_q)\psi^\dagger(y_q)|n\rangle = \prod_{i<j}^q \frac{(z_i - z_j)(y_i - y_j)}{(z_i - y_j)(y_i - z_j)} \prod_{i=1}^q \frac{1}{z_i - y_i} \left(\frac{z_i}{y_i}\right)^n$$

Also

$$e^{\sum_{i,j} \xi_i \eta_j A_{i,j}} = 1 + \sum_{q>0} \sum_{\substack{\alpha_1 > \cdots \alpha_q \\ \beta_1 > \cdots > \beta_q}} \xi_{\alpha_1} \cdots \xi_{\alpha_q} \eta_{\beta_1} \cdots \eta_{\beta_q} \det\left(A_{\alpha_i,\beta_j}\right)$$

where ξ_i, η_i are odd variables (Fermi fields with the property $\xi_i\eta_j + \eta_j\xi_i = 0$ for each pair i, j), and $A_{i,j}$ is a (possibly infinite) matrix.

And at last

$$\det\left(\frac{1}{z_i - y_j}\right) = \prod_{i<j} \frac{(z_i - z_j)(y_i - y_j)}{(z_i - y_j)(y_i - z_j)} \prod_i \frac{1}{z_i - y_i}$$

5 Appendix. Bilinear identity for the two-component τ function

In this section we define more general τ functions in comparison with (13):

$$\tau(n_1,n_2,n,\mathbf{t}^1,\mathbf{t}^2)=\langle n_1,n_2|\,\Gamma\left(\mathbf{t}^1,\mathbf{t}^2\right)g\,|n_2-n,n_1+n\rangle\quad, \tag{48}$$

where

$$g=\,e^{\int\sum_{i,j=1,2}:\psi^{(i)}(a_1^{(i)})\psi^{\dagger(j)}(a_2^{(j)}):d\mu(a_1^{(i)},a_2^{(j)})} \tag{49}$$

where $:\psi^{(i)}(a_1^{(i)})\psi^{\dagger(j)}(a_2^{(j)}):=\psi^{(i)}(a_1^{(i)})\psi^{\dagger(j)}(a_2^{(j)})-\langle 0,0|\psi^{(i)}(a_1^{(i)})\psi^{\dagger(j)}(a_2^{(j)})|0,0\rangle$ while $\Gamma\left(\mathbf{t}^1,\mathbf{t}^2\right)$ is given by (15). Particularly interesting for us τ the following one:

$$\tau(n_1,n_2,\mathbf{t}^1,\mathbf{t}^2):=\tau(n_1,n_2,0,\mathbf{t}^1,\mathbf{t}^2)$$

The bilinear identity is valid in the following form (see [4]). For $n_1-n_1'\geq n'-n\geq n_2'-n_2+2$, we have

$$\sum_{\alpha=1}^{2}\oint\frac{dz}{2\pi iz}\langle n_1,n_2|\,\Gamma\left(\mathbf{t}^1,\mathbf{t}^2\right)\psi^{(\alpha)}(z)g\,|n_2-n-1,n_1+n\rangle \tag{50}$$
$$\times\langle n_1',n_2'|\,\Gamma\left({t'}^{(1)},t'^{(2)}\right)\psi^{\dagger(\alpha)}(z)g\,|n_2'-n'+1,n_1'+n'\rangle=0$$

and the integration is taken along a small contour at $z=\infty$ so that $\oint\frac{dz}{2\pi iz}=1$.

Rewriting this (33), we obtain

$$\oint\frac{dz}{2\pi iz}\,(-1)^{n_2+n_2'}\,z^{n_1-1-n_1'}e^{V(z,\mathbf{t}^1-t'^{(1)})} \tag{51}$$
$$\times\tau(n_1-1,n_2,n+1,\mathbf{t}^1-\theta\left(z^{-1}\right),\mathbf{t}^2)\tau(n_1'+1,n_2',n'-1,t'^{(1)}+\theta\left(z^{-1}\right),t'^{(2)})$$
$$+\oint\frac{dz}{2\pi iz}z^{n_2-1-n_2'}e^{V(z,\mathbf{t}^2-t'^{(2)})}$$
$$\times\tau(n_1,n_2-1,n,\mathbf{t}^1,\mathbf{t}^2-\theta\left(z^{-1}\right))\tau(n_1',n_2'+1,n',t'^{(1)},t'^{(2)}+\theta\left(z^{-1}\right))\quad,$$

where $\theta\left(z^{-1}\right)=\left(\frac{1}{z},\frac{1}{2z^2},...\frac{1}{nz^n},...\right)$.

As an example of (35), for

$$f=\tau(n_1,n_2,0,\mathbf{t}^1,\mathbf{t}^2)=\tau(n_1,n_2,\mathbf{t}^1,\mathbf{t}^2),$$
$$g=\tau(n_1-1,n_2+1,1,\mathbf{t}^1,\mathbf{t}^2)$$

and

$$g^*=\tau(n_1+1,n_2-1,-1,\mathbf{t}^1,\mathbf{t}^2)$$

we get the following bilinear equations:

$$\left(D_{t_2^{(1)}}-D^2_{t_1^{(1)}}\right)f\cdot g=0,\qquad\left(D_{t_2^{(1)}}-D^2_{t_1^{(1)}}\right)g^*\cdot f=0, \tag{52}$$
$$\left(D_{t_2^{(2)}}+D^2_{t_1^{(2)}}\right)f\cdot g=0,\qquad\left(D_{t_2^{(2)}}+D^2_{t_1^{(2)}}\right)g^*\cdot f=0,$$
$$D_{t_1^{(1)}}D_{t_1^{(2)}}f\cdot f-2g\cdot g^*=0,$$

where Hirota operator is $D_x\sigma\cdot\tau=\lim\limits_{\varepsilon\to0}\frac{\partial}{\partial\varepsilon}\sigma\left(x+\varepsilon\right)\tau\left(x-\varepsilon\right)=\sigma_x\tau-\sigma\tau_x$

6 Discussion

We have shown that the Fateev-Frolov-Schwartz instanton series coincides with the formal series for the tau function of the two-component KP hierarchy. This is a formal expression, the manipulation of which nevertheless has a physical meaning. Such a tau function can be treated in the same way as with expressions of quantum field theory - eliminated by divergences by introducing trims, by replacing the measure, by switching to a lattice theory. All this can be done using the fermion representation for the tau function. In this case, based on the formal expression, one can obtain various equations for the correlation functions. However, in order to calculate the correlation functions themselves it is necessary to use not massless, but massive fermions, which obey the massive Dirac equation. Note that this is related to works about the quantum model of sin-Grodon and the Thirring model [10].

This is what we are doing in our next work.

Note that new connections with the theory of classical integrable systems appear here. Firstly, these are classic works of the Kyoto School [11]. Then it is related to the so-called $\bar{\partial}$-problem for the KP equation [12],[13].

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