Open Communications in Nonlinear Mathematical Physics]ocnmp[Special Issue 1, 2024 pp 1–28

This article is part of an OCNMP Special Issue in Memory of Professor Decio Levi

High order multiscale analysis of discrete integrable equations

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Received August 2, 2023; Accepted October 6, 2023

Abstract

In this article we present the results obtained applying the multiple scale expansion up to the order ε^6 to a dispersive multilinear class of equations on a square lattice depending on 13 parameters. We show that the integrability conditions given by the multiple scale expansion give rise to 4 nonlinear equations, 3 of which seem to be new, depending at most on 2 parameters.

1 Introduction

Discrete—or difference—equations play an important role in Mathematical Physics for their double role. First, discrete space-time seems to be basic in the description of fundamental phenomena of nature, as suggested by quantum gravity. On the other hand, discrete equations are related to differential difference and differential equations through continuous limits. A well-known classification of integrable partial difference equations was given by Adler, Bobenko and Suris [2] in the particular case of equations defined on four lattice points. They used the "consistency around the cube" condition with some symmetry constrains to be able to get definite results. Due to the constraints introduced, this classification is partial and already new equations with respect to those contained in the ABS classification have been found [19, 17, 13, 10, 8, 1].

In this paper we provide necessary conditions for the integrability of a class of real, autonomous difference equations in the variable $u : \mathbb{Z}^2 \to \mathbb{R}$ defined on a $\mathbb{Z}^2$ square-lattice

$$\mathcal{Q}(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}; \beta_1, \beta_2, ...) = 0, \quad n, m \in \mathbb{Z}, \tag{1}$$

where the β_i's are real, independent parameters. Integrability conditions will be determined through a multiscale perturbative development, continuing with the theory explained in references such as [4, 6, 7, 5, 11] applicable in differential and difference equations. This approach has the distinctive advantage of providing criteria in a manner completely independent from other current approaches. Multiscale developments can be used to reinforce, enhance or augment our previous knowledge of discrete integrable systems given by other techniques.

We will assume, as in [2], that (1) is linear-affine in every variable, implying that the equation is invariant under the Möbius transformation T

$$u_{n,m} \overset{T}{\mapsto} u'_{n,m} = \frac{A u_{n,m} + B}{C u_{n,m} + D}. \tag{2}$$

In this case, (1) reduces to a polynomial equation in its variables with an at most fourth order nonlinearity

$$\begin{aligned} \mathcal{Q} \;=\; & f_0 + a_{00}\, u_{00} + a_{01}\, u_{01} + a_{10}\, u_{10} + a_{11}\, u_{11} + (\alpha_1 - \alpha_2) u_{00}\, u_{10} \\ & + (\beta_1 - \beta_2) u_{00}\, u_{01} + d_1 u_{00}\, u_{11} + d_2\, u_{01}\, u_{10} + (\beta_1 + \beta_2)\, u_{10}\, u_{11} + (\alpha_1 + \alpha_2)\, u_{01}\, u_{11} \\ & + (\tau_1 - \tau_3) u_{00}\, u_{01}\, u_{10} + (\tau_1 + \tau_3) u_{00}\, u_{10}\, u_{11} + (\tau_2 + \tau_4) u_{00}\, u_{01}\, u_{11} \\ & + (\tau_2 - \tau_4)\, u_{10}\, u_{01}\, u_{11} + f_1\, u_{00}\, u_{01}\, u_{10}\, u_{11} = 0, \end{aligned} \tag{3}$$

where all coefficients are taken to be real and independent of n and m. We consider a multiple scale expansion around the dispersive solution

$$u_{n,m} = K^n \Omega^m, \tag{4}$$

of the linearized equation of (3). Rewriting the constants K and Ω as $K = e^{\mathrm{i}k}$ and $\Omega = e^{-\mathrm{i}\omega}$, and introducing the solution (4) into the linear part of Eq. (3) we get a dispersion relation $\omega = \omega(k)$

$$\omega = \arctan\left[\frac{(a_{00}a_{11} - a_{10}a_{01})\sin(k)}{a_{00}a_{01} + a_{10}a_{11} + (a_{00}a_{11} + a_{01}a_{10})\cos(k)}\right], \tag{5}$$

if $f_0 = 0$. The solution (4) of (3) with $f_0 = 0$ is dispersive if $\omega(k)$ is a real nonlinear function of the wave number k. This leads to the constraint

$$a_{00}^2 - a_{01}^2 + a_{10}^2 - a_{11}^2 + 2(a_{00}a_{10} - a_{01}a_{11})\cos(k) = 0. \tag{6}$$

The constraint (6) implies that one of the following two conditions must be satisfied to obtain a non-trivial dispersion relation:

1. $a_{00} = a_{11} \equiv a_1$, $a_{01} = a_{10} \equiv a_2$,

2. $a_{00} = -a_{11} \equiv a_1$, $a_{01} = -a_{10} \equiv a_2$.

Then the dispersion relation (5) reduces to:

$$\omega_{\pm}(k) = \arctan\left[\pm\frac{\left(a_1^2 - a_2^2\right)\sin(k)}{2a_1a_2 \pm (a_1^2 + a_2^2)\cos(k)}\right] \tag{7}$$

We denote the family of equations (3) satisfying condition (1) with dispersion relation $\omega_+(k)$ as $\mathcal{Q}^+$ and the one with dispersion relation $\omega_-(k)$ as $\mathcal{Q}^-$. In all the cases a_1 and a_2 cannot be zero and their ratio cannot be equal to ± 1 in order to have a nontrivial dispersion relation.

We will consider integrability conditions for the class of equations $\mathcal{Q}^+$. The study of the class $\mathcal{Q}^-$ is left to a future work. The result of this work are a series of integrability theorems and a table of equations, invariant under a restricted Möbius transformations, passing the very stringent integrability conditions obtained with the multiple scale expansion up to ε^6 order.

In Section 2 we present the main result on the discrete multiscale integrability test, the conditions up to order ε^6. In Section 3 we apply it to the classification of dispersive multilinear equations defined on a square lattice $\mathcal{Q}^+$. Section 4 is devoted to some conclusive remarks.

2 The discrete multiscale integrability test

Consider a dispersive discrete equation of the form $\mathcal{Q}^+$, i.e. a completely discrete multilinear dispersive equation defined on a lattice of four points. In such a situation the discrete multiscale integrability test may be summarized as follows.

i. One considers a small amplitude solution of Eq. (3) given by $u_{n,m} = \varepsilon w_{n,m}$, $0 < |\varepsilon| \ll 1$. Then (3) splits into linear and nonlinear terms:

$$\mathcal{Q}^+ = \sum_{i=1}^{N} \varepsilon^i \mathcal{Q}_i = 0, \tag{8}$$

where $N \in \mathbb{N}$ is the nonlinearity order. A multilinear equation defined on a square can be at most quartic, i.e. $N \leq 4$. In the formal expansion (8) each term $\mathcal{Q}_i$ contains only homogeneous polynomials of degree i in $w_{n,m}$. If the discrete equation is dispersive then the linear part $\mathcal{Q}_1$ admits a solution $w_{n,m} = \exp[\mathrm{i}(\kappa n - \omega m)] = K^n \Omega^m$, where $\omega = \omega(\kappa) = \omega_+(\kappa)$, the dispersion relation, is a real function of κ given by Eq. (7).

ii. The multiscale expansion of the basic field variable $w_{n,m}$ around the harmonic $K^n\Omega^m$ reads

$$w_{n,m} = \sum_{\ell=0}^{\infty} \varepsilon^\ell \sum_{\alpha=-\ell-1}^{\ell+1} K^{\alpha n}\Omega^{\alpha m} u_{\ell+1}^{(\alpha)}, \tag{9}$$

where $u_\ell^{(\alpha)} = u_\ell^{(\alpha)}(n_1, \{m_j\})$ is a bounded slowly varying function of its arguments and $u_\ell^{(-\alpha)} = \bar{u}_\ell^{(\alpha)}$, $\bar{u}_\ell$ being the complex conjugate of u_ℓ, because we look only for real solutions. Here $n_1 = \varepsilon n$, $m_j = \varepsilon^j m$ $j = 1, 2, \ldots$ are the slow-varying lattice variables.

iii. The nearest-neighbors fields are expanded according to the following formulas:

$$w_{n+1,m} = \sum_{\ell=0}^{\infty} \varepsilon^\ell \sum_{\alpha=-\ell-1}^{\ell+1} K^{\alpha(n+1)} \Omega^{\alpha m} \sum_{j=\max(0,|\alpha|-1)}^{\ell} \mathcal{A}_{\ell-j} u_{j+1}^{(\alpha)}, \tag{10}$$

$$w_{n,m+1} = \sum_{\ell=0}^{\infty} \varepsilon^\ell \sum_{\alpha=-\ell-1}^{\ell+1} K^{\alpha n} \Omega^{\alpha(m-1)} \sum_{j=\max(0,|\alpha|-1)}^{\ell} \mathcal{B}_{\ell-j} u_{j+1}^{(\alpha)}, \tag{11}$$

$$w_{n+1,m+1} = \sum_{\ell=0}^{\infty} \varepsilon^\ell \sum_{\alpha=-\ell-1}^{\ell+1} K^{\alpha(n+1)} \Omega^{\alpha(m-1)} \sum_{j=\max(0,|\alpha|-1)}^{\ell} \mathcal{C}_{\ell-j} u_{j+1}^{(\alpha)}. \tag{12}$$

The operators $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$, are equal to 1 when $i = 0$, and for some lower values of i are:

	$i=1$	$i=2$	$i=3$	$i=4$
$\mathcal{A}_i$	δ_{n_1}	$\frac{1}{2}\delta_{n_1}^2$	$\frac{1}{6}\delta_{n_1}^3$	$\frac{1}{24}\delta_{n_1}^4$
$\mathcal{B}_i$	δ_{m_1}	$\frac{1}{2}\delta_{m_1}^2 + \delta_{m_2}$	$\frac{1}{6}\delta_{m_1}^3 + \delta_{m_1}\delta_{m_2} + \delta_{m_3}$	$\frac{1}{24}\delta_{m_1}^4 + \frac{1}{2}\delta_{m_1}^2\delta_{m_2} + \frac{1}{2}\delta_{m_2}^2 + \delta_{m_1}\delta_{m_3} + \delta_{m_4}$
$\mathcal{C}_i$	∇	$\frac{1}{2}\nabla^2 + \delta_{m_2}$	$\frac{1}{6}\nabla^3 + \nabla\delta_{m_2} + \delta_{m_3}$	$\frac{1}{24}\nabla^4 + \frac{1}{2}\nabla^2\delta_{m_2} + \frac{1}{2}\delta_{m_2}^2 + \nabla\delta_{m_3} + \delta_{m_4}$

where δ_k are the formal derivatives with respect to the index k, $\delta_k := \partial_k$ and $\nabla := \delta_{m_1} + \delta_{n_1}$. The operator δ_k can always be expressed in terms of powers of the difference operators by the well known identity

$$\delta_k = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \Delta_k^i,$$

where Δ_k is the discrete first right difference operator with respect to the variable k, i.e. $\Delta_k u_k := u_{k+1} - u_k$.

A function f_k is a *slow-varying* function of order L if $\Delta_k^{L+1} f_k = 0$. The δ_k-operators, which in principle are formal infinite series in powers of Δ_k, when acting on *slow-varying* functions of finite order L reduce to polynomials in Δ_k at most of order L. We shall assume that we are dealing with functions of an infinite slow-varying order, i.e. $L = \infty$, so the δ_k-operators may be taken as differential operators acting on the indices of the harmonics $u_j^{(\alpha)}$.

iv. Substituting the expansions (9-12) into (8), we get an equation of the following form:

$$\sum_j \varepsilon^j \sum_\alpha \mathcal{W}_j^{(\alpha)} K^{\alpha n} \Omega^{\alpha m} = 0, \tag{13}$$

i.e. we must have $\mathcal{W}_j^{(\alpha)} = 0$ for all α and j. Notice that the equations $\mathcal{W}_j^{(\alpha)} = 0$ are equations for the slowly varying functions $u_{\ell+1}^{(\alpha)}$ with $\ell \leq j$.

The multiscale expansion of the $\mathcal{Q}^+$ equation for functions of infinite order thus gives rise to a *system of continuous partial differential equations.* At the lowest order (slow-time m_2) one gets a Nonlinear Schrödinger equation (*NLS*) for the first harmonic $u_1^{(1)}$. We will use orders beyond that to define the values of the constants appearing in $\mathcal{Q}^+$ for which the equation is integrable. The first attempt to go beyond the *NLS* order in the case of partial differential equations was presented by Santini, Degasperis and Manakov in [6] and by Kodama and Mikhailov using normal forms [12]. In [6] the authors, starting from S-integrable models, through a combination of an asymptotic functional analysis and spectral methods, succeeded in removing all the secular terms from the reduced equations, order by order. Their results could be summarized in the following statements:

1. The number of slow-time variables required for the amplitudes $u_j^{(\alpha)}$ coincides with the number of non-vanishing coefficients $\omega_j(k) = \frac{1}{j!}\frac{d^j \omega(k)}{dk^j}$;

2. The amplitude $u_1^{(1)}$ evolves at the slow-times $t_\sigma := m_\sigma$, $\sigma \geq 3$ according to the $\sigma-$th equation of the *NLS* hierarchy;

3. The amplitudes of the higher perturbations of the first harmonic $u_j^{(1)}$, $j \geq 2$ evolve at the slow-times t_σ, $\sigma \geq 2$ according to certain linear, nonhomogeneous equations when taking into account some asymptotic boundary conditions.

From these statements one can conclude that the cancellation at each stage of the perturbation process of all the secular terms is a *sufficient condition* to uniquely fix the evolution equations followed by every $u_j^{(1)}$, $j \geq 1$ for each slow-time t_σ. Conversely, the results in [7] imply that this expansion is secularity-free. Thus, this procedure provides *necessary and sufficient* conditions to get secularity-free reduced equations. Following [7] we can state the following proposition:

Proposition 1. If a nonlinear dispersive partial difference equation is integrable, then under a multiscale expansion the functions $u_l^{(1)}$, $l \geq 1$ satisfy the equations

$$\partial_{t_\sigma} u_1^{(1)} = K_\sigma \left[u_1^{(1)}\right], \tag{14a}$$

$$M_\sigma u_j^{(1)} = f_\sigma(j), \quad M_\sigma := \partial_{t_\sigma} - K'_\sigma \left[u_1^{(1)}\right], \tag{14b}$$

$\forall\ j,\ \sigma \geq 2$, where $K_\sigma \left[u_1^{(1)}\right]$ is the $\sigma-$th flow in the nonlinear Schrödinger hierarchy. All the other $u_j^{(\kappa)}$, $\kappa \geq 2$ are expressed in terms of differential monomials of $u_\rho^{(1)}$, $\rho \leq j$.

In (14b) $f_\sigma(j)$ is a nonhomogeneous *nonlinear* forcing term depending on all the $u_\kappa^{(1)}$, $1 \leq \kappa \leq j-1$, their complex conjugates and their ξ-derivatives, where ξ is a variable representing the group velocity and expressed through a linear combination of the slow-space and the first slow-time t_1, while $K'_\sigma\left[u\right] v$ is the Frechet derivative of the nonlinear term

$K_\sigma[u]$ along the direction v defined by $K'_\sigma[u]v := \frac{d}{ds}K_\sigma[u+sv]\mid_{s=0}$, i.e. the linearization of the expression $K_\sigma[u]$ along the direction v near the function u.

In order to characterize the flows $K_\sigma\left[u_1^{(1)}\right]$ and the nonlinear forcing terms $f_\sigma(j)$, following [5], we introduce the finite dimensional vector spaces $\mathcal{P}_\ell$, $\ell \geq 2$, as being the set of all homogeneous, fully-nonlinear, differential polynomials in the functions $u_j^{(1)}$, $j \geq 1$, their complex conjugates and their ξ-derivatives of homogeneity order ℓ in ε and 1 in the accompanying exponential $e^{\mathrm{i}\theta} = e^{\mathrm{i}(\kappa n - \omega m)}$, where

$$\mathrm{order}_\varepsilon\left(\partial_\xi^\kappa u_j^{(1)}\right) = \mathrm{order}_\varepsilon\left(\partial_\xi^\kappa \bar{u}_j^{(1)}\right) = \kappa + j, \quad \kappa \geq 0.$$

We introduce the subspaces $\mathcal{P}_\ell(\jmath)$ of $\mathcal{P}_\ell$, $\jmath \geq 1$, $\ell \geq 2$, whose elements are homogeneous, fully-nonlinear, differential polynomials in the functions $u_k^{(1)}$, their complex conjugates and their ξ-derivatives with $1 \leq k \leq \jmath$. Firstly from these definitions it follows that $\mathcal{P}_\ell = \mathcal{P}_\ell(\ell-2)$, that is $\jmath \leq \ell - 2$. In fact the terms $u_\ell^{(1)}$ and $\bar{u}_\ell^{(1)}$, as well as $\partial_\xi u_{\ell-1}^{(1)}$ and $\partial_\xi \bar{u}_{\ell-1}^{(1)}$, are not included in $\mathcal{P}_\ell$ as any monomial should enter nonlinearly and terms like $u_{\ell-1}^{(1)}$ and $\bar{u}_{\ell-1}^{(1)}$ cannot be combined with any other of the monomials $u_1^{(1)}$ or $\bar{u}_1^{(1)}$ to give the right homogeneity degree in $e^{\mathrm{i}\theta}$. For the same reasons, terms of the types $\partial_\xi^\kappa u_{\ell-\kappa}^{(1)}$, $\partial_\xi^\kappa \bar{u}_{\ell-\kappa}^{(1)}$, $0 \leq \kappa \leq \ell - 1$ and $\partial_\xi^\kappa u_{\ell-\kappa-1}^{(1)}$, $\partial_\xi^\kappa \bar{u}_{\ell-\kappa-1}^{(1)}$, $0 \leq \kappa \leq \ell - 2$ cannot appear. So the space $\mathcal{P}_\ell(\jmath)$ is defined as that functional space generated by the base of monomials of the following types

$$\prod_{\alpha,\beta,\gamma,\delta}\left(\partial_\xi^\alpha u_\beta^{(1)}\right)^{\rho(\alpha,\beta)}\left(\partial_\xi^\gamma \bar{u}_\delta^{(1)}\right)^{\sigma(\gamma,\delta)}, \quad \rho(\alpha,\beta) \geq 0, \quad \forall\alpha,\beta, \quad \sigma(\gamma,\delta) \geq 0, \quad \forall\gamma,\delta,$$

where the product is extended for $1 \leq \beta, \delta \leq \jmath \leq \ell-2$, $0 \leq \alpha \leq \ell-\beta-2$ and $0 \leq \gamma \leq \ell-\delta-2$, so that

$$\sum_{\alpha,\beta,\gamma,\delta}(\alpha+\beta)\rho(\alpha,\beta) + (\gamma+\delta)\sigma(\gamma,\delta) = \ell, \qquad \sum_{\alpha,\beta,\gamma,\delta}\rho(\alpha,\beta) - \sigma(\gamma,\delta) = 1.$$

For $n \geq 3$ the subspaces $\mathcal{P}_\ell(\jmath)$, can be generated recursively starting from the lowest one, corresponding to $\ell = 2$ by the following relation

$$\mathcal{P}_\ell(\jmath) = \partial_\xi \mathcal{P}_{\ell-1}(\jmath) \cup \left\{\prod_{\beta,\delta}\left(u_\beta^{(1)}\right)^{\rho(\beta)}\left(\bar{u}_\delta^{(1)}\right)^{\sigma(\delta)}\right\},$$

where $\rho(\beta) \geq 0$ $\forall\beta$, $\sigma(\delta) \geq 0$ $\forall\delta$ and the product is extended for $1 \leq \beta, \delta \leq \jmath \leq \ell - 2$, so that

$$\sum_{\beta,\delta}\beta\rho(\beta) + \delta\sigma(\delta) = \ell, \quad \sum_{\beta,\delta}\rho(\beta) - \sigma(\delta) = 1.$$

It is then clear that in general $K_n\left[u_1^{(1)}\right] \in \left\{\partial_\xi^\ell u_1^{(1)}\right\} \cup \mathcal{P}_{\ell+1}(1)$ and that $f_\sigma(j) \in \mathcal{P}_{\sigma+j}(j-1)$, $\forall\sigma$, $j \geq 2$.

Eqs. (14) are a *necessary* condition for integrability and represent a hierarchy of *compatible* evolutions for the same function $u_1^{(1)}$ at different slow-times. The compatibility of Eqs. (14b) implies some commutativity conditions among their r.h.s. terms $f_\sigma(j)$. If they are satisfied the operators M_σ defined in Eq. (14b) commute with each other. Once we fix the index $j \geq 2$ in the set of Eqs. (14b), this commutativity condition implies the following *compatibility* conditions

$$M_\sigma f_{\sigma'}(j) = M_{\sigma'} f_\sigma(j), \quad \forall\, \sigma, \sigma' \geq 2, \tag{15}$$

where, as $f_\sigma(j)$ and $f_{\sigma'}(j)$ are functions of the different perturbations of the fundamental harmonic up to degree $j-1$, the time derivatives ∂_{t_σ}, $\partial_{t_{\sigma'}}$ of those harmonics appearing respectively in M_σ and $M_{\sigma'}$ have to be eliminated using the evolution equations (14) up to the index $j-1$. The commutativity conditions (15) turn out to be an **integrability test**.

We finally define the **degree of integrability** of a given equation:

Definition. *If the relations (15) are satisfied up to the index j, $j \geq 2$, we say that our equation is asymptotically integrable of degree j or A_j integrable.*

Conjecturing that an A_∞ degree of asymptotic integrability actually implies integrability, we have that under this assumption the relations (14, 15) are a *sufficient* condition for the S-integrability or that integrability is a *necessary* condition to have a multiscale expansion where all the Eqs. (14) are satisfied. So the multiscale integrability test tell us that $\mathcal{Q}^+$ will be integrable if its multiscale expansion will follow all the infinite relations (14, 15). The higher the degree of asymptotic integrability, the nearer the equation will be to an integrable one. However, as we can test the conditions (14, 15) only up to a finite order (currently A_4), from them we can only derive necessary conditions for integrability, so we will not be able to state with certainty that the discrete equation is integrable. The results obtained at a finite but sufficiently high order will have a good probability to correspond to an integrable equation, but we need to use other techniques to prove it with certainty.

Let us present for completeness the lowest order conditions for asymptotic S-integrability of order k or A_k-integrability conditions. To simplify the notation, we will use for $u_j^{(1)}$ the concise form $u(j)$, $j \geq 1$. Moreover, for the convenience of the reader, we list the fluxes $K_\sigma[u]$ of the *NLS* hierarchy for u up to $\sigma = 5$:

$$K_1[u] := A u_\xi, \tag{16a}$$

$$K_2[u] := -\mathrm{i}\rho_1 \left[u_{\xi\xi} + \frac{\rho_2}{\rho_1}|u|^2 u \right], \tag{16b}$$

$$K_3[u] := B \left[u_{\xi\xi\xi} + \frac{3\rho_2}{\rho_1}|u|^2 u_\xi \right], \tag{16c}$$

$$K_4[u] := -\mathrm{i}C \left\{ u_{\xi\xi\xi\xi} + \frac{\rho_2}{\rho_1} \left[\frac{3\rho_2}{2\rho_1}|u|^4 u + 4|u|^2 u_{\xi\xi} + 3u_\xi^2 \bar{u} + 2|u_\xi|^2 u + u^2 \bar{u}_{\xi\xi} \right] \right\}, \tag{16d}$$

$$\begin{aligned} K_5[u] := D\Bigg\{ u_{\xi\xi\xi\xi\xi} + \frac{5\rho_2}{\rho_1} \Bigg[&\frac{3\rho_2}{2\rho_1}|u|^4 u_\xi + |u_\xi|^2 u_\xi + (u\bar{u}_\xi + 2\bar{u}u_\xi)\, u_{\xi\xi} \\ &+ u u_\xi \bar{u}_{\xi\xi} + |u|^2 u_{\xi\xi\xi} \Bigg] \Bigg\}, \end{aligned} \tag{16e}$$

and the corresponding $K'_\sigma[u]v$ up to $\sigma = 4$:

$$K'_1[u]v = A v_\xi, \tag{17a}$$

$$K_2'[u]v = -\mathrm{i}\rho_1 \left\{ v_{\xi\xi} + \frac{\rho_2}{\rho_1} \left[u^2 \bar{v} + 2|u|^2 v \right] \right\}, \tag{17b}$$

$$K_3'[u]v = B \left\{ v_{\xi\xi\xi} + \frac{3\rho_2}{\rho_1} \left[|u|^2 v_\xi + \bar{u} u_\xi v + u u_\xi \bar{v} \right] \right\}, \tag{17c}$$

$$\begin{aligned} K_4'\left[u\right] v = -iC \Bigg\{ v_{\xi\xi\xi\xi} + \frac{\rho_2}{\rho_1} \Big[& u^2 \bar{v}_{\xi\xi} + 4|u|^2 v_{\xi\xi} + 2u u_\xi \bar{v}_\xi + 2u \bar{u}_\xi v_\xi + 6\bar{u} u_\xi v_\xi + \\ & + 4u u_{\xi\xi} \bar{v} + 3u_\xi^2 \bar{v} + \frac{3\rho_2}{\rho_1} |u|^2 u^2 \bar{v} + 4\bar{u} u_{\xi\xi} v + 2u \bar{u}_{\xi\xi} v + \frac{9\rho_2}{2\rho_1} |u|^4 v + 2|u_\xi|^2 v \Big] \Bigg\}, \end{aligned} \tag{17d}$$

where A, ρ_1, ρ_2, B, C and D are real non null arbitrary constants.

2.1 The A_1-integrability condition.

The A_1-integrability condition is given by the reality of the coefficient ρ_2 of the nonlinear term in the *NLS*. It is obtained commuting the *NLS* flux $K_2[u]$ with the flux $B\left[u_{\xi\xi\xi} + \tau|u|^2 u_\xi + \mu u^2 \bar{u}_\xi\right]$ with τ and μ constants. This commutativity condition gives, if $\rho_2 \neq 0$,

$$\mathrm{Im}\left[\rho_2\right] = \mathrm{Im}\left[B\right] = \mathrm{Im}\left[\rho_1\right] = 0, \quad \tau = 3\rho_2/\rho_1, \quad \mu = 0. \tag{18}$$

We remark that, when $\rho_2 \neq 0$, by the same method it is possible to determine all the coefficients of all the higher *NLS*-symmetries (16) together with the reality conditions of the coefficients A, C and D.

2.2 The A_2-integrability conditions.

The A_2-integrability conditions are obtained choosing $j = 2$ in the compatibility conditions (15) with $\sigma = 2$ and $\sigma' = 3$ or alternatively $\sigma' = 4$, respectively

$$M_2 f_3\left(2\right) = M_3 f_2\left(2\right), \tag{19a}$$

$$M_2 f_4\left(2\right) = M_4 f_2\left(2\right). \tag{19b}$$

In this case $f_2(2)$, $f_3(2)$ and $f_4(2)$ will be identified by respectively two, (a, b), five, $(\alpha, \beta, \gamma, \delta, \epsilon)$, and eight, $(\theta_1, \cdots, \theta_8)$, complex constants

$$f_2(2) \coloneqq a u_\xi(1)|u(1)|^2 + b\bar{u}_\xi(1)u(1)^2, \tag{20a}$$

$$\begin{aligned} f_3(2) \coloneqq\ & \alpha|u(1)|^4 u(1) + \beta|u_\xi(1)|^2 u(1) + \gamma u_\xi(1)^2 \bar{u}(1) + \delta \bar{u}_{\xi\xi}(1) u(1)^2 \\ & + \epsilon|u(1)|^2 u_{\xi\xi}(1), \end{aligned} \tag{20b}$$

$$\begin{aligned} f_4\left(2\right) \coloneqq\ & \theta_1 |u\left(1\right)|^4 u_\xi\left(1\right) + \theta_2 |u\left(1\right)|^2 u\left(1\right)^2 \bar{u}_\xi\left(1\right) + \theta_3 |u_\xi\left(1\right)|^2 u_\xi\left(1\right) \\ & + \theta_4 u\left(1\right) \bar{u}_\xi\left(1\right) u_{\xi\xi}\left(1\right) + \theta_5 \bar{u}\left(1\right) u_\xi\left(1\right) u_{\xi\xi}\left(1\right) + \theta_6 u\left(1\right) u_\xi\left(1\right) \bar{u}_{\xi\xi}\left(1\right) \\ & + \theta_7 |u\left(1\right)|^2 u_{\xi\xi\xi}\left(1\right) + \theta_8 u\left(1\right)^2 \bar{u}_{\xi\xi\xi}\left(1\right). \end{aligned} \tag{20c}$$

As $\rho_2 \neq 0$, eliminating from Eq. (19a) the derivatives of $u(1)$ with respect to the slow-times t_2 and t_3, using the evolutions (14a) with $\sigma = 2$ and $\sigma' = 3$ and equating term by term, we obtain the following two A_2-integrability conditions

$$a = \bar{a}, \quad b = \bar{b}. \tag{21}$$

So we have two conditions obtained requiring the reality of the coefficients a and b. The expressions of α, β, γ, δ, ϵ in terms of a and b are:

$$\alpha = \frac{3\mathrm{i}Ba\rho_2}{4\rho_1^2}, \quad \beta = \frac{3\mathrm{i}Bb}{\rho_1}, \quad \gamma = \frac{3\mathrm{i}Ba}{2\rho_1}, \quad \delta = 0, \quad \epsilon = \gamma. \tag{22}$$

The same integrability conditions (21) can be derived using Eq. (19b). As in our analysis we will need them, here follows the explicit expressions of the coefficients of the forcing term $f_4\,(2)$

$$\begin{aligned}\theta_1 &= \frac{6Ca\rho_2}{\rho_1^2}, \quad \theta_2 = \frac{3Cb\rho_2}{\rho_1^2}, \quad \theta_3 = \frac{(a+3b)\,C}{\rho_1}, \quad \theta_4 = \frac{(a+4b)\,C}{\rho_1}, \\ \theta_5 &= \frac{5Ca}{\rho_1}, \quad \theta_6 = \frac{(a+2b)\,C}{\rho_1}, \quad \theta_7 = \frac{2Ca}{\rho_1}, \quad \theta_8 = \frac{Cb}{\rho_1}.\end{aligned} \tag{23}$$

2.3 The A_3-integrability conditions.

The A_3-integrability conditions are derived in a similar way setting $j = 3$ in the compatibility conditions (15) with $\sigma = 2$ and $\sigma' = 3$, so that $M_2 f_3\,(3) = M_3 f_2\,(3)$. In this case $f_2(3)$ and $f_3(3)$ will be respectively identified by 12 and 26 complex constants

$$\begin{aligned}f_2(3) \coloneqq{}& \tau_1|u(1)|^4u(1) + \tau_2|u_\xi(1)|^2u(1) + \tau_3|u(1)|^2u_{\xi\xi}(1) + \tau_4\bar{u}_{\xi\xi}(1)u(1)^2 \\ &+ \tau_7\bar{u}_\xi(2)u(1)^2 + \tau_8u(2)^2\bar{u}(1) + \tau_9|u(2)|^2u(1) + \tau_{10}u(2)u_\xi(1)\bar{u}(1) \\ &+ \tau_{11}u(2)\bar{u}_\xi(1)u(1) + \tau_{12}\bar{u}(2)u_\xi(1)u(1) + \tau_5u_\xi(1)^2\bar{u}(1) + \tau_6u_\xi(2)|u(1)|^2,\end{aligned} \tag{24a}$$

$$\begin{aligned}f_3(3) \coloneqq{}& \gamma_1|u(1)|^4u_\xi(1) + \gamma_2|u(1)|^2u(1)^2\bar{u}_\xi(1) + \gamma_3|u(1)|^2u_{\xi\xi\xi}(1) \\ &+ \gamma_5|u_\xi(1)|^2u_\xi(1) + \gamma_6\bar{u}_{\xi\xi}(1)u_\xi(1)u(1) + \gamma_7u_{\xi\xi}(1)\bar{u}_\xi(1)u(1) \\ &+ \gamma_9|u(1)|^4u(2) + \gamma_{10}|u(1)|^2u(1)^2\bar{u}(2) + \gamma_{11}\bar{u}_\xi(1)u(2)^2 + \gamma_{12}u_\xi(1)|u(2)|^2 \\ &+ \gamma_{13}|u_\xi(1)|^2u(2) + \gamma_{14}|u(2)|^2u(2) + \gamma_{15}u_\xi(1)^2\bar{u}(2) + \gamma_{16}|u(1)|^2u_{\xi\xi}(2) \\ &+ \gamma_{17}u(1)^2\bar{u}_{\xi\xi}(2) + \gamma_{18}u(2)\bar{u}_{\xi\xi}(1)u(1) + \gamma_{19}u(2)u_{\xi\xi}(1)\bar{u}(1) \\ &+ \gamma_{21}u(2)u_\xi(2)\bar{u}(1) + \gamma_{22}\bar{u}(2)u_\xi(2)u(1) + \gamma_{23}u_\xi(2)u_\xi(1)\bar{u}(1) \\ &+ \gamma_{25}\bar{u}_\xi(2)u_\xi(1)u(1) + \gamma_{26}\bar{u}_\xi(2)u(2)u(1) + \gamma_4u(1)^2\bar{u}_{\xi\xi\xi}(1) \\ &+ \gamma_8u_{\xi\xi}(1)u_\xi(1)\bar{u}(1) + \gamma_{20}\bar{u}(2)u_{\xi\xi}(1)u(1) + \gamma_{24}u_\xi(2)\bar{u}_\xi(1)u(1).\end{aligned} \tag{24b}$$

Eliminate from Eq. (19a) with $j = 3$ the derivatives of $u(1)$ with respect to the slow-times t_2 and t_3 using the evolutions (14a) respectively with $\sigma = 2$ and $\sigma' = 3$ and the derivatives of $u(2)$ using the evolutions (14b) with $\sigma = 2$ and $\sigma' = 3$. Equating the remaining terms term by term, with $\rho_2 \neq 0$ and, indicating with R_i and I_i the real and imaginary parts of τ_i, $i = 1, \ldots, 12$, we obtain the following 15 A_3-integrability conditions

$$
\begin{aligned}
&R_1 = -\frac{aI_6}{4\rho_1}, \quad R_3 = \frac{(b-a)I_6}{2\rho_2} - \frac{aI_{12}}{2\rho_2}, \quad R_4 = \frac{R_2}{2} + \frac{(a-b)I_6}{4\rho_2} + \frac{aI_{12}}{4\rho_2},\\
&R_5 = \frac{R_2}{2} + \frac{(a-b)I_6}{4\rho_2} + \frac{(2b-a)I_{12}}{4\rho_2}, \quad R_6 = -\frac{aI_8}{\rho_2}, \quad R_7 = R_{12} + \frac{(a-b)I_8}{\rho_2},\\
&R_8 = R_9 = 0, \quad R_{10} = R_{12}, \quad R_{11} = R_{12} + \frac{(a-2b)I_8}{\rho_2}, \\
&I_4 = \frac{(b+a)R_{12}}{4\rho_2} + \frac{\rho_1 I_1}{\rho_2} + \frac{I_2 - I_3 - 2I_5}{4} + \frac{\left[2b(a-b)+a^2\right] I_8}{4\rho_2^2}, \quad I_7 = 0,\\
&I_9 = 2I_8, \quad I_{10} = I_{12}, \quad I_{11} = I_6 + I_{12}.
\end{aligned}
\tag{25}
$$

The expressions of the γ_j, $j = 1, \ldots, 26$ as functions of the τ_i, $i = 1, \ldots, 12$ are:

$$
\begin{aligned}
&\gamma_1 = \frac{3B}{8\rho_1^2}\left[-2bR_{12} - 8\rho_1 I_1 + 2(I_2 - 2I_3 - 2I_5)\rho_2 + \mathrm{i}(b-5a)I_6 + \frac{2a^2 I_8}{\rho_2} - 3\mathrm{i}aI_{12}\right],\\
&\gamma_2 = -\frac{3Ba}{4\rho_1^2}\left[\mathrm{i}I_6 + \frac{(a-2b)I_8}{\rho_2} + \tau_{12}\right], \ \gamma_3 = \frac{3\mathrm{i}B\tau_3}{2\rho_1}, \ \gamma_4 = 0, \ \gamma_5 = \frac{3\mathrm{i}B\tau_2}{2\rho_1}, \ \gamma_6 = \frac{3\mathrm{i}B\tau_4}{\rho_1},\\
&\gamma_7 = \gamma_5, \quad \gamma_8 = \gamma_3 + \frac{3\mathrm{i}B\tau_5}{\rho_1}, \quad \gamma_9 = -\frac{3B(\rho_2 I_6 + 3a\mathrm{i}I_8)}{4\rho_1^2}, \quad \gamma_{10} = \frac{3\mathrm{i}B\rho_2 R_6}{2\rho_1^2}, \quad \gamma_{11} = 0,\\
&\gamma_{12} = \frac{3\mathrm{i}B\tau_9}{2\rho_1}, \quad \gamma_{13} = \frac{3\mathrm{i}B\tau_{11}}{2\rho_1}, \quad \gamma_{14} = 0, \quad \gamma_{15} = \frac{3\mathrm{i}B\tau_{12}}{2\rho_1}, \quad \gamma_{16} = \frac{3\mathrm{i}B\tau_6}{2\rho_1},\\
&\quad \gamma_{17} = \gamma_{18} = 0, \quad \gamma_{19} = \frac{3\mathrm{i}B\tau_{10}}{2\rho_1}, \quad \gamma_{20} = \gamma_{15}, \quad \gamma_{21} = \frac{3\mathrm{i}B\tau_8}{\rho_1}, \quad \gamma_{22} = \gamma_{12},\\
&\gamma_{23} = \gamma_{16} + \gamma_{19}, \quad \gamma_{24} = \gamma_{13}, \quad \gamma_{25} = \frac{3\mathrm{i}B\tau_7}{\rho_1}, \quad \gamma_{26} = 0.
\end{aligned}
\tag{26}
$$

2.4 The A_4-integrability conditions.

The A_4-integrability conditions are derived similarly from (15) with $j = 4$, that is $M_2 f_3\,(4) = M_3 f_2\,(4)$. Now $f_2(4)$ and $f_3(4)$ are respectively defined by 34 and 77 complex constants

$$
\begin{aligned}
f_2\,(4) \coloneqq\; &\eta_1|u(1)|^4 u_\xi(1) + \eta_2|u(1)|^2 u(1)^2 \bar{u}_\xi(1) + \eta_3|u(1)|^2 u_{\xi\xi\xi}(1)\\
&+ \eta_5|u_\xi(1)|^2 u_\xi(1) + \eta_6 \bar{u}_{\xi\xi}(1) u_\xi(1) u(1) + \eta_7 u_{\xi\xi}(1)\bar{u}_\xi(1) u(1)\\
&+ \eta_9|u(1)|^4 u(2) + \eta_{10}|u(1)|^2 u(1)^2 \bar{u}(2) + \eta_{11}\bar{u}_\xi(1) u(2)^2 + \eta_{12} u_\xi(1)|u(2)|^2\\
&+ \eta_{13}|u_\xi(1)|^2 u(2) + \eta_{14}|u(2)|^2 u(2) + \eta_{15} u_\xi(1)^2 \bar{u}(2) + \eta_{16}|u(1)|^2 u_{\xi\xi}(2)\\
&+ \eta_{17} u(1)^2 \bar{u}_{\xi\xi}(2) + \eta_{18} u(2)\bar{u}_{\xi\xi}(1) u(1) + \eta_{19} u(2) u_{\xi\xi}(1)\bar{u}(1)\\
&+ \eta_{21} u(2) u_\xi(2)\bar{u}(1) + \eta_{22}\bar{u}(2) u_\xi(2) u(1) + \eta_{23} u_\xi(2) u_\xi(1)\bar{u}(1)\\
&+ \eta_{25}\bar{u}_\xi(2) u_\xi(1) u(1) + \eta_{26}\bar{u}_\xi(2) u(2) u(1) + \eta_4 u(1)^2 \bar{u}_{\xi\xi\xi}(1)\\
&+ \eta_8 u_{\xi\xi}(1) u_\xi(1)\bar{u}(1) + \eta_{20}\bar{u}(2) u_{\xi\xi}(1) u(1) + \eta_{24} u_\xi(2)\bar{u}_\xi(1) u(1)
\end{aligned}
\tag{27a}
$$

$$
\begin{aligned}
&+ \eta_{27} u(1) \bar{u}_\xi(1) u(3) + \eta_{28} \bar{u}(1) u_\xi(1) u(3) + \eta_{29} u(1) u_\xi(1) \bar{u}(3) \\
&+ \eta_{30} u(1) \bar{u}(2) u(3) + \eta_{31} \bar{u}(1) u(2) u(3) + \eta_{32} u(1) u(2) \bar{u}(3) \\
&+ \eta_{33} |u(1)|^2 u_\xi(3) + \eta_{34} u(1)^2 \bar{u}_\xi(3),
\end{aligned}
$$

$$
\begin{aligned}
f_3(4) \coloneqq\ & \kappa_1 u(1)|u(1)|^6 + \kappa_2 |u(1)|^2 \bar{u}(1) u_\xi(1)^2 + \kappa_3 |u(1)|^2 u(1) |u_\xi(1)|^2 \\
&+ \kappa_4 u(1)^3 \bar{u}_\xi(1)^2 + \kappa_5 |u(1)|^4 u_{\xi\xi}(1) + \kappa_6 |u(1)|^2 u(1)^2 \bar{u}_{\xi\xi}(1) \\
&+ \kappa_7 |u_\xi(1)|^2 u_{\xi\xi}(1) + \kappa_8 u_\xi(1)^2 \bar{u}_{\xi\xi}(1) + \kappa_9 u(1) |u_{\xi\xi}(1)|^2 + \kappa_{10} \bar{u}(1) u_{\xi\xi}(1)^2 \\
&+ \kappa_{11} \bar{u}(1) u_\xi(1) u_{\xi\xi\xi}(1) + \kappa_{12} u(1) \bar{u}_\xi(1) u_{\xi\xi\xi}(1) + \kappa_{13} u(1) u_\xi(1) \bar{u}_{\xi\xi\xi}(1) \\
&+ \kappa_{14} |u(1)|^2 u_{\xi\xi\xi\xi}(1) + \kappa_{15} u(1)^2 \bar{u}_{\xi\xi\xi\xi}(1) + \kappa_{16} |u(1)|^2 \bar{u}(1) u(2)^2 \\
&+ \kappa_{17} |u(1)|^2 u(1) |u(2)|^2 + \kappa_{18} u(1)^3 \bar{u}(2)^2 + \kappa_{19} |u(1)|^2 \bar{u}(1) u_\xi(1) u(2) \\
&+ \kappa_{20} |u(1)|^2 u(1) \bar{u}_\xi(1) u(2) + \kappa_{21} |u(1)|^2 u(1) u_\xi(1) \bar{u}(2) + \kappa_{22} u(1)^3 \bar{u}_\xi(1) \bar{u}(2) \\
&+ \kappa_{23} \bar{u}_\xi(1) u_{\xi\xi}(1) u(2) + \kappa_{24} u_\xi(1) \bar{u}_{\xi\xi}(1) u(2) + \kappa_{25} u_\xi(1) u_{\xi\xi}(1) \bar{u}(2) \\
&+ \kappa_{26} u(1) \bar{u}_{\xi\xi\xi}(1) u(2) + \kappa_{27} \bar{u}(1) u_{\xi\xi\xi}(1) u(2) + \kappa_{28} u(1) u_{\xi\xi\xi}(1) \bar{u}(2) \\
&+ \kappa_{29} \bar{u}_{\xi\xi}(1) u(2)^2 + \kappa_{30} u_{\xi\xi}(1) |u(2)|^2 + \kappa_{31} |u(1)|^4 u_\xi(2) \\
&+ \kappa_{32} |u(1)|^2 u(1)^2 \bar{u}_\xi(2) + \kappa_{33} |u_\xi(1)|^2 u_\xi(2) + \kappa_{34} u_\xi(1)^2 \bar{u}_\xi(2) \\
&+ \kappa_{35} \bar{u}(1) u_{\xi\xi}(1) u_\xi(2) + \kappa_{36} u(1) \bar{u}_{\xi\xi}(1) u_\xi(2) + \kappa_{37} u(1) u_{\xi\xi}(1) \bar{u}_\xi(2) \\
&+ \kappa_{38} u(1) \bar{u}_\xi(1) u_{\xi\xi}(2) + \kappa_{39} \bar{u}(1) u_\xi(1) u_{\xi\xi}(2) + \kappa_{40} u(1) u_\xi(1) \bar{u}_{\xi\xi}(2) \\
&+ \kappa_{41} |u(1)|^2 u_{\xi\xi\xi}(2) + \kappa_{42} u(1)^2 \bar{u}_{\xi\xi\xi}(2) + \kappa_{43} \bar{u}_\xi(1) u(2) u_\xi(2) \\
&+ \kappa_{44} u_\xi(1) \bar{u}(2) u_\xi(2) + \kappa_{45} u_\xi(1) u(2) \bar{u}_\xi(2) + \kappa_{46} u(1) |u_\xi(2)|^2 + \kappa_{47} \bar{u}(1) u_\xi(2)^2 \\
&+ \kappa_{48} \bar{u}(1) u(2) u_{\xi\xi}(2) + \kappa_{49} u(1) \bar{u}(2) u_{\xi\xi}(2) + \kappa_{50} u(1) u(2) \bar{u}_{\xi\xi}(2) \\
&+ \kappa_{51} |u(2)|^2 u_\xi(2) + \kappa_{52} u(2)^2 \bar{u}_\xi(2) + \kappa_{53} |u(1)|^4 u(3) + \kappa_{54} |u(1)|^2 u(1)^2 \bar{u}(3) \\
&+ \kappa_{55} \bar{u}(1) u(3)^2 + \kappa_{56} u(1) |u(3)|^2 + \kappa_{57} |u(2)|^2 u(3) + \kappa_{58} u(2)^2 \bar{u}(3) \\
&+ \kappa_{59} |u_\xi(1)|^2 u(3) + \kappa_{60} u_\xi(1)^2 \bar{u}(3) + \kappa_{61} u(1) \bar{u}_{\xi\xi}(1) u(3) + \kappa_{62} \bar{u}(1) u_{\xi\xi}(1) u(3) \\
&+ \kappa_{63} u(1) u_{\xi\xi}(1) \bar{u}(3) + \kappa_{64} u(1) \bar{u}_\xi(1) u_\xi(3) + \kappa_{65} \bar{u}(1) u_\xi(1) u_\xi(3) \\
&+ \kappa_{66} u(1) u_\xi(1) \bar{u}_\xi(3) + \kappa_{67} |u(1)|^2 u_{\xi\xi}(3) + \kappa_{68} u(1)^2 \bar{u}_{\xi\xi}(3) + \kappa_{69} u_\xi(1) \bar{u}(2) u(3) \\
&+ \kappa_{70} \bar{u}_\xi(1) u(2) u(3) + \kappa_{71} u_\xi(1) u(2) \bar{u}(3) + \kappa_{72} \bar{u}(1) u_\xi(2) u(3) \\
&+ \kappa_{73} u(1) \bar{u}_\xi(2) u(3) + \kappa_{74} u(1) u_\xi(2) \bar{u}(3) + \kappa_{75} u(1) \bar{u}(2) u_\xi(3) \\
&+ \kappa_{76} \bar{u}(1) u(2) u_\xi(3) + \kappa_{77} u(1) u(2) \bar{u}_\xi(3).
\end{aligned} \tag{27b}
$$

If we indicate with S_j and T_j respectively the real and imaginary parts of η_j, $j = 1, \ldots,$ 34, when $\rho_2 \neq 0$, the A_4-integrability conditions are represented by 48 real relations whose expressions we leave for a specific Appendix.

Other integrability conditions corresponding to $M_4 f_2(3) = M_2 f_4(3)$ (A_3-integrability conditions) and to $M_4 f_2(5) = M_2 f_4(5)$ (A_5-integrability conditions) in the subspaces with $u(2n) = 0$, $n \geq 1$ for purely imaginary coefficients can be found in [16]. They are respectively given by 1 and 14 real relations, the first of which can be deduced from (25) and corresponds to $I_4 = \rho_1 I_1/\rho_2 + (I_2 - I_3 - 2I_5)/4$.

The results presented in this Section will be used in the following Sections to classify integrable nonlinear equation on the square lattice.

3 Dispersive affine-linear equations on the square lattice

The aim of this Section is to derive necessary conditions for the S-integrability of the simplest class of $\mathbb{Z}^2$-lattice equations, that of dispersive and multilinear equations (3) defined on the square lattice, satisfying the condition (1) with dispersion relation $\omega_+(k)$, i.e.

$$\begin{aligned}\mathcal{Q}^+ := {} & a_1(u_{n,m}+u_{n+1,m+1})+a_2(u_{n+1,m}+u_{n,m+1}) \\ & +(\alpha_1-\alpha_2)\,u_{n,m}u_{n+1,m}+(\alpha_1+\alpha_2)\,u_{n,m+1}u_{n+1,m+1} \\ & +(\beta_1-\beta_2)\,u_{n,m}u_{n,m+1}+(\beta_1+\beta_2)\,u_{n+1,m}u_{n+1,m+1} \\ & +\gamma_1 u_{n,m}u_{n+1,m+1}+\gamma_2 u_{n+1,m}u_{n,m+1} \\ & +(\xi_1-\xi_3)\,u_{n,m}u_{n+1,m}u_{n,m+1}+(\xi_1+\xi_3)\,u_{n,m}u_{n+1,m}u_{n+1,m+1} \\ & +(\xi_2-\xi_4)\,u_{n+1,m}u_{n,m+1}u_{n+1,m+1}+(\xi_2+\xi_4)\,u_{n,m}u_{n,m+1}u_{n+1,m+1} \\ & +\zeta u_{n,m}u_{n+1,m}u_{n,m+1}u_{n+1,m+1}=0,\end{aligned} \tag{28}$$

where $a_1, a_2 \in \mathbb{R}\setminus\{0\}$, $|a_1| \neq |a_2|$, are the coefficients appearing in the linear part while $\alpha_1,\alpha_2,\beta_1,\beta_2$, γ_1,γ_2, $\xi_1,\xi_2,\xi_3,\xi_4,\zeta$ are some real parameters which enter in the nonlinear part of the system. Here we will look, by using the multiscale procedure described in Section 2, into the values of these coefficients for the class $\mathcal{Q}^+$ to be A_1 integrable.

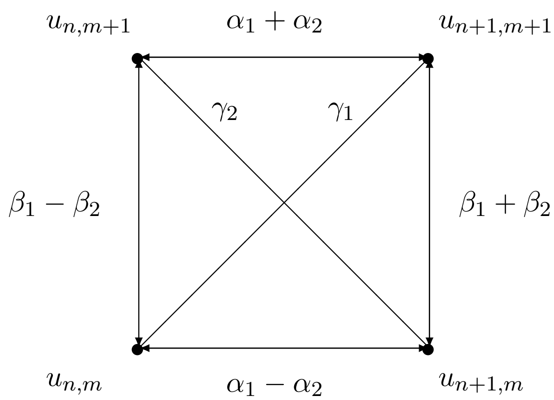

Figure 1. Representation of the quadratic nonlinearities of $\mathcal{Q}_\pm$

To perform a classification of the equations $\mathcal{Q}^+$, we need to find the set of transformations that leave it invariant, i.e. the equivalence transformation. As mentioned before, a generic multilinear equation of the form (1) is invariant under a Möbius transformation (2). The constant term f_0 and the differences $a_{00}-a_{11}$, $a_{01}-a_{10}$ transform according to

$$\begin{aligned}f_0 \overset{T}{\mapsto} f_0' &= D^4 f_0 + B^4\zeta + 2B^3D\,(\xi_1+\xi_2) + B^2D^2\,[\gamma_1+\gamma_2+2\,(\alpha_1+\beta_1)] \\ &\quad +2BD^3\,(a_{00}+a_{11}+a_{01}+a_{10})\,, \\ a_{00}-a_{11} \overset{T}{\mapsto} a_{00}'-a_{11}' &= (AD{-}BC)\left[D^2\,(a_{00}-a_{11})+B^2\,(\xi_1-\xi_2-\xi_3+\xi_4)-2BD\,(\alpha_2+\beta_2)\right] \\ a_{01}-a_{10} \overset{T}{\mapsto} a_{01}'-a_{10}' &= (AD{-}BC)\left[D^2\,(a_{01}-a_{10})-B^2\,(\xi_1-\xi_2+\xi_3-\xi_4)+2BD\,(\alpha_2-\beta_2)\right]\end{aligned} \tag{29}$$

These formulas allow to determine when a given linear-affine equation (1) can be transformed into one belonging to class $\mathcal{Q}^+$. For this to happen all three terms must be null, so setting the l.h.s. of (29) to zero we get three polynomial equations over B/D or D/B. If simultaneously solvable (over the reals), we have an equation of the class $\mathcal{Q}^+$. One could try to write the conditions over the coefficients of a general linear-affine equation (1) by using resultant calculations on the three polynomial conditions, but they turn out to be too complicated to merit further attention. Thus (29) tells that the class $\mathcal{Q}^+$ is invariant under restricted simultaneous Möbius transformations R of the form

$$u_{n,m} \mapsto u'_{n,m} = u_{n,m}/(Cu_{n,m} + D), \tag{30}$$

which will be our equivalence transformation. Under (30) the coefficients of Eq. (28) undergo the following transformations:

$$\begin{aligned}
&a_1 \overset{R}{\mapsto} a'_1 = D^3 a_1, \quad a_2 \overset{R}{\mapsto} a'_2 = D^3 a_2, \quad \alpha_1 \overset{R}{\mapsto} \alpha'_1 = D^2\left[\alpha_1 + C\left(a_1 + a_2\right)\right],\\
&\alpha_2 \overset{R}{\mapsto} \alpha'_2 = D^2\alpha_2, \quad \beta_1 \overset{R}{\mapsto} \beta'_1 = D^2\left[\beta_1 + C\left(a_1 + a_2\right)\right], \quad \beta_2 \overset{R}{\mapsto} \beta'_2 = D^2\beta_2,\\
&\gamma_1 \overset{R}{\mapsto} \gamma'_1 = D^2\left(\gamma_1 + 2Ca_1\right), \quad \gamma_2 \overset{R}{\mapsto} \gamma'_2 = D^2\left(\gamma_2 + 2Ca_2\right),\\
&\xi_1 \overset{R}{\mapsto} \xi'_1 = D\xi_1 + \tfrac{1}{2}CD\left[3C\left(a_1 + a_2\right) + \gamma_1 + \gamma_2 + 2\left(\alpha_1 - \alpha_2 + \beta_1\right)\right],\\
&\xi_2 \overset{R}{\mapsto} \xi'_2 = D\xi_2 + \tfrac{1}{2}CD\left[3C\left(a_1 + a_2\right) + \gamma_1 + \gamma_2 + 2\left(\alpha_1 + \alpha_2 + \beta_1\right)\right],\\
&\xi_3 \overset{R}{\mapsto} \xi'_3 = D\xi_3 + \tfrac{1}{2}CD\left[C(a_1 - a_2) + \gamma_1 - \gamma_2 + 2\beta_2\right],\\
&\xi_4 \overset{R}{\mapsto} \xi'_4 = D\xi_4 + \tfrac{1}{2}CD\left[C(a_1 - a_2) + \gamma_1 - \gamma_2 - 2\beta_2\right],\\
&\zeta \overset{R}{\mapsto} \zeta' = \zeta + C^2\left[2C\left(a_1 + a_2\right) + \gamma_1 + \gamma_2 + 2\left(\alpha_1 + \beta_1\right)\right] + 2C\left(\xi_1 + \xi_2\right).
\end{aligned} \tag{31}$$

We will indicate by $\mathcal{N}$ the number of free parameters (although not all of them essential under R) appearing in each subcase of (28). Its maximum number is $\mathcal{N} = 13$, the number of free coefficients in (28).

3.1 Classification at order ε^3.

By performing the multiscale expansion of Eq. (28), the following statement holds regarding A_1-asymptotic integrability

Proposition 2. The lowest order necessary conditions for the S-integrability of equations $\mathcal{Q}^+$ read:

- Case 1 ($\mathcal{N} = 9$):

$$\begin{cases} \alpha_2 = \beta_2 = 0, \\ \xi_1 = \xi_2, \quad \xi_3 = \xi_4. \end{cases} \tag{32}$$

- Case 2 ($\mathcal{N} = 7$) :

$$\begin{cases} \alpha_2 = \beta_2, \quad \alpha_1 = \beta_1, \\ a_1 = 2a_2, \\ \gamma_1 = 2\gamma_2, \\ a_1(\xi_1 - \xi_2) = -a_1(\xi_3 - \xi_4) = -2\alpha_2\gamma_2. \end{cases} \tag{33}$$

- Case 3 ($\mathcal{N}=7$):

$$\begin{cases} \alpha_2=-\beta_2, \quad \alpha_1=\beta_1, \\ a_2=2a_1, \\ \gamma_2=2\gamma_1, \\ a_1(\xi_1-\xi_2)=a_1(\xi_3-\xi_4)=-\alpha_2\gamma_1. \end{cases} \tag{34}$$

- Case 4 ($\mathcal{N}=8$):

$$\begin{cases} a_2\alpha_1=a_2\beta_1=\frac{1}{2}(a_1+a_2)\gamma_2, \\ a_2\gamma_1=a_1\gamma_2, \\ a_1(\xi_1-\xi_2)=-\alpha_2\gamma_1, \\ a_1(\xi_3-\xi_4)=\beta_2\gamma_1. \end{cases} \tag{35}$$

- Case 5 ($\mathcal{N}=8$):

$$\begin{cases} (a_2-a_1)\beta_2=(a_2+a_1)\alpha_2, \\ 2a_1a_2(a_1-a_2)\alpha_1=(a_1+a_2)(\gamma_2a_1^2-\gamma_1a_2^2), \\ 2a_1a_2\beta_1=\gamma_1a_2^2+\gamma_2a_1^2, \\ (a_2-a_1)(\xi_1-\xi_2)=(\gamma_1-\gamma_2)\alpha_2, \\ (a_2-a_1)^2(\xi_3-\xi_4)=\left[\gamma_2(a_2-3a_1)-\gamma_1(a_1-3a_2)\right]\alpha_2. \end{cases} \tag{36}$$

- Case 6 ($\mathcal{N}=8$):

$$\begin{cases} (a_2+a_1)\beta_2=(a_2-a_1)\alpha_2, \\ 2a_1a_2\alpha_1=\gamma_1a_2^2+\gamma_2a_1^2, \\ 2a_1a_2(a_1-a_2)\beta_1=(a_1+a_2)(\gamma_2a_1^2-\gamma_1a_2^2), \\ (a_2^2-a_1^2)(\xi_1-\xi_2)=\left[\gamma_1(a_1-3a_2)-\gamma_2(a_2-3a_1)\right]\alpha_2, \\ (a_1+a_2)(\xi_3-\xi_4)=(\gamma_2-\gamma_1)\alpha_2. \end{cases} \tag{37}$$

The obtained six subclasses of equation (28) are invariant under the restricted Möbius transformation (30).

Proof: Following the procedure described in Section 2 we expand the fields appearing in equation $\mathcal{Q}^+$ according to formulas (9-12). The lowest order necessary conditions for the S-integrability of $\mathcal{Q}^+$ are obtained by considering the equation $\mathcal{W}_3$ (see Eq. (13)), namely the order ε^3 of the multiscale expansion. At this order we get the m_2-evolution equation for the harmonic $u_0^{(1)}$, that is a NLS equation of the form

$$\mathrm{i}\delta_{m_2}u_1^{(1)}+\rho_1\delta_\xi^2u_1^{(1)}+\rho_2u_1^{(1)}|u_1^{(1)}|^2=0, \qquad \xi\coloneqq n_1-\frac{d\omega}{d\kappa}m_1, \tag{38}$$

where the coefficients ρ_1 and ρ_2 will depend on the parameters of the equation $\mathcal{Q}^+$ and on the wave parameters κ and $\omega=\omega_+$, with ω_+ expressed in terms of κ through the dispersion relation (7). According to our multiscale test the lowest order necessary condition for $\mathcal{Q}^+$ to be an S-integrable lattice equation is that Eq. (38) be integrable itself, namely ρ_1 and ρ_2 have to be real coefficients.

Let us outline the construction of Eq. (38). At $\mathcal{O}(\varepsilon)$ we get:

- for $\alpha = 1$ a linear equation which is identically satisfied by the dispersion relation (7).
- for $\alpha = 0$ a linear equation whose solution is $u_1^{(0)} = 0$.

At $\mathcal{O}(\varepsilon^2)$, taking into account the dispersion relation (7), we get:

- for $\alpha = 2$ an algebraic relation between $u_2^{(2)}$ and $u_1^{(1)}$.
- for $\alpha = 1$ a linear wave equation for $u_1^{(1)}$, whose solution is given by $u_1^{(1)}(n_1, m_1, m_2) = u_1^{(1)}(\xi, m_2)$, where $\xi \coloneqq n_1 - (d\omega/d\kappa)m_1$.
- for $\alpha = 0$ an algebraic relation between $u_2^{(0)}$ and $u_1^{(1)}$.

Notice that from the $\mathcal{O}(\varepsilon^2)$ we find that the dependence of all the harmonics on the slow-variables n_1 and m_1 is given by ξ.

At $\mathcal{O}(\varepsilon^3)$, for $\alpha = 1$, by using the results obtained at the previous orders, one gets the *NLS* equation (38) with

$$\rho_1 = \frac{a_1 a_2 (a_1^2 - a_2^2)\sin\kappa}{(a_1^2 + a_2^2 + 2a_1 a_2 \cos\kappa)^2}, \qquad \rho_2 = \mathcal{R}_1 + \mathrm{i}\mathcal{R}_2,$$

where

$$\mathcal{R}_1 = \frac{\sin\kappa \left[\mathcal{R}_1^{(0)} + \mathcal{R}_1^{(1)}\cos\kappa + \mathcal{R}_1^{(2)}\cos^2\kappa + \mathcal{R}_1^{(3)}\cos^3\kappa + \mathcal{R}_1^{(4)}\cos^4\kappa\right]}{(a_1 + a_2)(a_1^2 + a_2^2 + 2a_1 a_2\cos\kappa)^2\left[(a_1 - a_2)^2 + 2a_1 a_2 \cos\kappa(1 + \cos\kappa)\right]}, \tag{39}$$

$$\mathcal{R}_2 = \frac{\mathcal{R}_2^{(0)} + \mathcal{R}_2^{(1)}\cos\kappa + \mathcal{R}_2^{(2)}\cos^2\kappa + \mathcal{R}_2^{(3)}\cos^3\kappa + \mathcal{R}_2^{(4)}\cos^4\kappa + \mathcal{R}_2^{(5)}\cos^5\kappa}{(a_1 + a_2)(a_1^2 + a_2^2 + 2a_1 a_2\cos\kappa)^2\left[(a_1 - a_2)^2 + 2a_1 a_2 \cos\kappa(1 + \cos\kappa)\right]}. \tag{40}$$

Here the coefficients $\mathcal{R}_1^{(i)}$, $0 \le i \le 4$, and $\mathcal{R}_2^{(i)}$, $0 \le i \le 5$, are polynomials depending on the coefficients $a_1, a_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \xi_1, ..., \xi_4$ and their expressions are cumbersome, so that we omit them.

Note that ρ_1 is a real coefficient depending only on the parameters of the linear part of $\mathcal{Q}^+$, while ρ_2 is a complex one. Hence the integrability of the *NLS* equation (38) is equivalent to the request $\mathcal{R}_2 = 0\ \forall\,\kappa$, that is

$$\mathcal{R}_2^{(i)} = 0, \qquad 0 \le i \le 5. \tag{41}$$

Eq. (41) is a nonlinear algebraic system of six equations in twelve unknowns. By solving it one gets the six solutions contained in Proposition 1. These solutions are computed taking into account that $a_1, a_2 \in \mathbb{R} \setminus \{0\}$ with $|a_1| \neq |a_2|$. One can solve two of the six equations (41) for ξ_1 and ξ_3, thus expressing them in terms of the remaining ten coefficients. The resulting system of four equations turns out to be ξ_2 and ξ_4-independent and linear in the four variables α_1, β_1, γ_1 and γ_2. Therefore we may write the remaining four equations as a matrix equation with coefficients nonlinearly depending on α_2, β_2, a_1 and a_2. The rank of the matrix is three. The six solutions are obtained by requiring that the matrix be of rank 3, 2, 1 and 0, and correspond to six classes of equations (28) that pass integrability conditions up to order $\mathcal{O}(\varepsilon^3)$. A direct calculation proves the invariance of the six classes with respect to the restricted Möbius transformation R.

□

Corollary 1. If the coefficients $a_1, a_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \xi_1, ..., \xi_4$ of equation $\mathcal{Q}^+$ do not satisfy one of the conditions given in Eqs. (32–37) then $\mathcal{Q}^+$ is not integrable.

Quadratic difference equations are a subclass of $\mathcal{Q}^+$ which have attracted a deal of attention. These equations are not Möbius invariant, but we can spot those that belong to the class $\mathcal{Q}^+$ and pass our integrability conditions, just by inspection of (32–35).

3.2 Classification at order ε^4.

For what concerns the A_2-asymptotically integrable cases satisfying the integrability conditions (21), the following statement holds

Proposition 3. At order ε^4, the necessary conditions for the S-integrability of equations $\mathcal{Q}^+$ read:

- Case 1 ($\mathcal{N} = 9$):

$$\begin{cases} \alpha_2 = \beta_2 = 0, \\ \xi_1 = \xi_2, \quad \xi_3 = \xi_4. \end{cases} \tag{42}$$

- Case 4 ($\mathcal{N} = 8$):

$$\begin{cases} \alpha_1 = \beta_1 = \frac{(a_1+a_2)\gamma_1}{2a_1}, \\ \gamma_2 = \frac{a_2\gamma_1}{a_1}, \\ a_1(\xi_1 - \xi_2) = -\alpha_2\gamma_1, \\ a_1(\xi_3 - \xi_4) = \beta_2\gamma_1, \\ (\alpha_2, \beta_2) \neq (0,0)\,. \end{cases} \tag{43}$$

The corresponding two subclasses of equations are non overlapping and invariant under the restricted Möbius transformation (30).

Notice that of the six A_1-asymptotically integrable cases listed in Proposition 2, Case *1* and Case *4* automatically satisfy the A_2-integrability conditions (21), while the remaining four cases *2*, *3*, *5* and *6* specify to some subcases of theirs. Notice that only two out the previous four quadratic cases in **Remark 1** survive, the Cases Q1 and Q4: the first one is a subcase of Case 1, while the second is a subcase of Case 4.

3.3 Classification at order ε^5.

It is possible to find all the cases satisfying the A_3-integrability conditions (25). They are given by the following proposition

Proposition 4. The necessary and sufficient conditions for ε^5 asymptotic integrability are:

Case (a): $(\mathcal{N}=4)$

$$\begin{aligned}
&\alpha_2=\beta_2=0, \quad \gamma_2=\alpha_1+\beta_1-\gamma_1, \quad a_2=2a_1, \quad (2\alpha_1-3\gamma_1,\ 2\beta_1-3\gamma_1)\neq(0,\ 0)\,,\\
&\xi_1=\xi_2=\frac{\alpha_1\beta_1}{2a_1}, \quad \xi_3=\xi_4=-\frac{(\alpha_1-\gamma_1)(\beta_1-\gamma_1)}{2a_1},\\
&\zeta=\frac{\gamma_1\left[3\gamma_1^2-3\gamma_1(\alpha_1+\beta_1)+4\alpha_1\beta_1\right]}{4a_1^2};
\end{aligned}$$

Case (b): $(\mathcal{N}=4)$

$$\begin{aligned}
&\alpha_2=\beta_2=0, \quad \gamma_1=\alpha_1+\beta_1-\gamma_2, \quad a_1=2a_2, \quad (2\alpha_1-3\gamma_2,\ 2\beta_1-3\gamma_2)\neq(0,\ 0)\\
&\xi_1=\xi_2=\frac{\alpha_1\beta_1}{2a_2}, \quad \xi_3=\xi_4=\frac{(\alpha_1-\gamma_2)(\beta_1-\gamma_2)}{2a_2},\\
&\zeta=\frac{\gamma_2\left[3\gamma_2^2-3\gamma_2(\alpha_1+\beta_1)+4\alpha_1\beta_1\right]}{4a_2^2};
\end{aligned}$$

Case (c): $(\mathcal{N}=5)$

$$\begin{aligned}
&\alpha_1=\beta_1=\frac{(a_1+a_2)\gamma_1}{2a_1}, \quad \alpha_2=\beta_2=0, \quad \gamma_2=\frac{a_2\gamma_1}{a_1}, \quad \xi_1=\xi_2,\\
&\xi_3=\xi_4=\frac{(a_2-a_1)\gamma_1^2}{4a_1^2}-\frac{(a_2-a_1)}{(a_2+a_1)}\xi_2, \quad \rho:=\left[\frac{8a_1^2\xi_2}{(a_1+a_2)}-3\gamma_1^2\right]\frac{1}{(a_1+a_2)^2}\neq0;
\end{aligned}$$

Case (d): $(\mathcal{N}=5)$

$$\begin{aligned}
&\alpha_1=\beta_1=\frac{(a_1+a_2)\gamma_1}{2a_1}, \quad \alpha_2=\beta_2=0, \quad \gamma_2=\frac{a_2\gamma_1}{a_1}, \quad \xi_1=\xi_2,\\
&\xi_3=\xi_4=\frac{(a_1-a_2)\gamma_1^2}{2a_1^2}-\frac{(a_1-a_2)}{(a_1+a_2)}\xi_2, \quad \rho:=\left[\frac{8a_1^2\xi_2}{(a_1+a_2)}-3\gamma_1^2\right]\frac{1}{(a_1+a_2)^2}\neq0;
\end{aligned}$$

Case (e): $(\mathcal{N}=4)$

$$\begin{aligned}
&\alpha_1=\beta_1=\frac{\gamma_1+\gamma_2}{2}, \quad \alpha_2=\beta_2=0, \quad \gamma_2\neq\frac{a_2\gamma_1}{a_1}, \quad \frac{a_2}{a_1}\neq\frac{1}{2},\ 2,\\
&\xi_1=\xi_2=\frac{3(\gamma_1+\gamma_2)^2}{8(a_1+a_2)}, \quad \xi_3=\xi_4=\frac{9(a_1-a_2)(a_1\gamma_2-a_2\gamma_1)^2}{8a_1a_2(a_1+a_2)^2}-\frac{a_1\gamma_2^2-a_2\gamma_1^2}{8a_1a_2},\\
&\zeta=\frac{(\gamma_1+\gamma_2)^3}{4(a_1+a_2)^2}+\frac{(a_1-a_2)(a_1\gamma_2-a_2\gamma_1)^3}{a_1^2a_2^2(a_1+a_2)^2};
\end{aligned}$$

Notes: In all of the cases $a_2/a_1\neq(0,\pm1)$; the values $a_2/a_1=(2,\ \frac{1}{2})$ are excluded in Case (e) because we would obtain respectively a subcase of Case (a) or of Case (b). All of the Cases (a)–(e) are subcases of Case 1. So nothing survives out of Case 4 at order ε^5. Cases Q_α-Q_δ are subcases both of the Case Q1 and Case (a); the Cases Q_η-Q_λ are subcases both of the Case Q1 and Case (b).

3.3.1 Canonical forms for ε^5 S-asymptotically integrable cases. Comparison with the ABS list.

We will use now the Möbius transformation to reduce the equation to normal form, i.e. to eliminate the maximum number of free parameters appearing in the nonlinear difference equation and reduce the coefficients of the linear part in $v_{n,m}$ and $v_{n+1,m+1}$ to 1.

In the Case (a) of Proposition 4, performing the Möbius transformation

$$u_{n,m} = \frac{\alpha v_{n,m} + \beta}{\gamma v_{n,m} + \delta},$$

with

$$\beta = 0, \quad \gamma = -\frac{\gamma_1 \delta}{2}, \quad \alpha = a_1 \delta, \quad \delta \neq 0,$$

we obtain the canonical form:

Case (a′): ($\mathcal{N} = 2$)

$$\begin{aligned} &v_{n,m} + v_{n+1,m+1} + 2\left(v_{n+1,m} + v_{n,m+1}\right) + v_{n+1,m} v_{n,m+1} \left(\tau_1 + \tau_2\right) \\ &\quad + \left(v_{n+1,m} v_{n+1,m+1} + v_{n,m} v_{n,m+1}\right) \tau_2 + \left(v_{n,m+1} v_{n+1,m+1} + v_{n,m} v_{n+1,m}\right) \tau_1 \\ &\quad + v_{n+1,m} v_{n,m+1} \left(v_{n,m} + v_{n+1,m+1}\right) \tau_1 \tau_2 = 0, \end{aligned} \tag{44}$$

where $(\tau_1, \tau_2) := \left(\alpha_1 - \frac{3\gamma_1}{2}, \beta_1 - \frac{3\gamma_1}{2}\right) \neq (0, 0)$. Performing a further rescaling on (44), we can fix, in all generality, the coefficients to either $\tau_1 = 0$ and $\tau_2 = 1$ or $\tau_1 = 1$ and we obtain the following two canonical forms respectively

$$\begin{aligned} &v_{n,m} + v_{n+1,m+1} + 2\left(v_{n+1,m} + v_{n,m+1}\right) + \\ &\quad + v_{n+1,m} v_{n,m+1} + v_{n+1,m} v_{n+1,m+1} + v_{n,m} v_{n,m+1} = 0, \end{aligned} \tag{45a}$$

$$\begin{aligned} &v_{n,m} + v_{n+1,m+1} + 2\left(v_{n+1,m} + v_{n,m+1}\right) + v_{n+1,m} v_{n,m+1} \left(1 + \tau_2\right) + \\ &\quad + \left(v_{n+1,m} v_{n+1,m+1} + v_{n,m} v_{n,m+1}\right) \tau_2 + v_{n,m+1} v_{n+1,m+1} + v_{n,m} v_{n+1,m} \\ &\quad + v_{n+1,m} v_{n,m+1} \left(v_{n,m} + v_{n+1,m+1}\right) \tau_2 = 0, \end{aligned} \tag{45b}$$

representing the two non overlapping subclasses of *Case (a)* defined respectively by the additional conditions $\alpha_1 = \frac{3\gamma_1}{2}$ and $\alpha_1 \neq \frac{3\gamma_1}{2}$. As under a restricted Möbious transformation τ_2 is invariant, we see that two canonical forms (45b), specified by two invariants τ_{2a} and τ_{2b}, form two disconnected components of the same conjugacy subclass unless $\tau_{2a} = \tau_{2b}$.

In the Case (b) of Proposition 4, performing the Möbius transformation

$$u_{n,m} = \frac{\alpha v_{n,m} + \beta}{\gamma v_{n,m} + \delta},$$

with

$$\beta = 0, \quad \gamma = -\frac{\gamma_2 \delta}{2}, \quad \alpha = a_2 \delta, \quad \delta \neq 0,$$

we obtain the canonical form:

Case (b′): ($\mathcal{N} = 2$)

$$
\begin{aligned}
&2\left(v_{n,m}+v_{n+1,m+1}\right)+v_{n+1,m}+v_{n,m+1}+v_{n,m}v_{n+1,m+1}\left(\tau_1+\tau_2\right)\\
&+\left(v_{n+1,m}v_{n+1,m+1}+v_{n,m}v_{n,m+1}\right)\tau_2+\left(v_{n,m+1}v_{n+1,m+1}+v_{n,m}v_{n+1,m}\right)\tau_1\\
&\qquad\qquad+v_{n,m}v_{n+1,m+1}\left(v_{n+1,m}+v_{n,m+1}\right)\tau_1\tau_2=0, \quad (46)
\end{aligned}
$$

where $(\tau_1,\tau_2) \coloneqq \left(\alpha_1-\frac{3\gamma_2}{2},\beta_1-\frac{3\gamma_2}{2}\right)\neq(0,0)$. Performing a further rescaling on (46) we can fix, in all generality, the parameters either to $\tau_1=0$ and $\tau_2=1$ or to $\tau_1=1$ and we obtain respectively the two canonical forms

$$
\begin{aligned}
&2\left(v_{n,m}+v_{n+1,m+1}\right)+v_{n+1,m}+v_{n,m+1}+ \qquad (47a)\\
&\quad+v_{n,m}v_{n+1,m+1}+v_{n+1,m}v_{n+1,m+1}+v_{n,m}v_{n,m+1}=0,\\
&2\left(v_{n,m}+v_{n+1,m+1}\right)+v_{n+1,m}+v_{n,m+1}+v_{n,m}v_{n+1,m+1}\left(1+\tau_2\right)+ \qquad (47b)\\
&\quad+\left(v_{n+1,m}v_{n+1,m+1}+v_{n,m}v_{n,m+1}\right)\tau_2+v_{n,m+1}v_{n+1,m+1}+v_{n,m}v_{n+1,m}\\
&\quad+v_{n,m}v_{n+1,m+1}\left(v_{n+1,m}+v_{n,m+1}\right)\tau_2=0,
\end{aligned}
$$

representing the two non overlapping subclasses of *Case (b)* defined respectively by the additional conditions $\alpha_1=\frac{3\gamma_2}{2}$ and $\alpha_1\neq\frac{3\gamma_2}{2}$. As τ_2 is invariant under a restricted Möbious transformation, we see that two canonical forms (47b), specified by two invariants τ_{2a} and τ_{2b}, form two disconnected components of the same conjugacy subclass unless $\tau_{2a}=\tau_{2b}$;

In the Cases (c) and (d) of Proposition 4, performing the Möbius transformation

$$
u_{n,m}=\frac{\alpha v_{n,m}+\beta}{\gamma v_{n,m}+\delta},
$$

with

$$
\alpha=\frac{2a_1\delta}{\left(a_1+a_2\right)\sqrt{|\rho|}},\quad \beta=0,\quad \gamma=-\frac{\gamma_1\delta}{\left(a_1+a_2\right)\sqrt{|\rho|}},\quad \delta\neq0,
$$

we obtain the canonical forms:

Case (c′): ($\mathcal{N}=2$)

$$
\begin{aligned}
&v_{n,m}+v_{n+1,m+1}+\epsilon\left(v_{n+1,m}+v_{n,m+1}\right)+ \qquad (48)\\
&\quad+\operatorname{sgn}(\rho)\left[\epsilon v_{n+1,m}v_{n,m+1}\left(v_{n,m}+v_{n+1,m+1}\right)+v_{n,m}v_{n+1,m+1}\left(v_{n+1,m}+v_{n,m+1}\right)\right]\\
&\quad+\zeta' v_{n,m}v_{n+1,m}v_{n,m+1}v_{n+1,m+1}=0,
\end{aligned}
$$

and

Case (d′): ($\mathcal{N}=2$)

$$
\begin{aligned}
&v_{n,m}+v_{n+1,m+1}+\epsilon\left(v_{n+1,m}+v_{n,m+1}\right)+ \qquad (49)\\
&\quad+\operatorname{sgn}(\rho)\left[v_{n+1,m}v_{n,m+1}\left(v_{n,m}+v_{n+1,m+1}\right)+\epsilon v_{n,m}v_{n+1,m+1}\left(v_{n+1,m}+v_{n,m+1}\right)\right]\\
&\quad+\zeta' v_{n,m}v_{n+1,m}v_{n,m+1}v_{n+1,m+1}=0,
\end{aligned}
$$

where $\epsilon\coloneqq a_2/a_1\neq0,\pm1$, $\zeta'\coloneqq 8s\left|\frac{\pi^2}{\rho^3}\right|^{\frac{1}{2}}/\left(1+\epsilon\right)^2$, $\pi\coloneqq\left[\zeta-2\frac{\gamma_1}{a_1}\xi_2+\frac{(a_1+a_2)\gamma_1^3}{2a_1^3}\right]/\left(a_1+a_2\right)$ and $s\coloneqq\pm1$. As under a restricted Möbius transformation $\rho\to\rho\left(\alpha/\delta\right)^2$ and $\pi\to\pi\left(\alpha/\delta\right)^3$, we see that the absolute value of ζ' and $\operatorname{sgn}(\rho)$ are invariant under such a transformation. With another rescaling we can always fix $\zeta'\geq0$ and the two canonical forms, specified by the two set of invariants $(\epsilon_a, \operatorname{sgn}(\rho_a), \zeta'_a)$ and $(\epsilon_b, \operatorname{sgn}(\rho_b), \zeta'_b)$, form two disconnected components of the conjugacy class unless the two sets are the same;

In the Case (e) of Proposition 4, performing the Möbius transformation

$$u_{n,m} = \frac{\alpha v_{n,m} + \beta}{\gamma v_{n,m} + \delta},$$

with

$$\beta = 0, \quad \gamma = -\frac{(\gamma_1 + \gamma_2)\,\alpha}{2\,(a_1 + a_2)}, \quad \delta = \frac{(a_2\gamma_1 - a_1\gamma_2)\,\alpha}{a_1\,(a_1 + a_2)}, \quad \alpha \neq 0,$$

we obtain the canonical form:

Case (e′): ($\mathcal{N} = 1$)

$$\begin{aligned} &v_{n,m} + v_{n+1,m+1} + \epsilon\,(v_{n+1,m} + v_{n,m+1}) + v_{n,m}v_{n+1,m+1} - v_{n+1,m}v_{n,m+1} + \\ &\quad + \left(1 - \frac{1}{\epsilon}\right)\left[v_{n+1,m}v_{n,m+1}\,(v_{n,m} + v_{n+1,m+1}) - v_{n,m}v_{n+1,m+1}\,(v_{n+1,m} + v_{n,m+1})\right] \\ &\quad + \left(1 - \frac{1}{\epsilon^2}\right) v_{n,m}v_{n+1,m}v_{n,m+1}v_{n+1,m+1} = 0, \end{aligned} \tag{50}$$

where $\epsilon \coloneqq a_2/a_1 \neq 0, \pm 1, 2, 1/2$. As ϵ is invariant under a restricted Möbius transformation, we see that two canonical forms, specified by the two invariants ϵ_a and ϵ_b, form two disconnected components of the conjugacy class unless $\epsilon_a = \epsilon_b$;

As our allowed transformations are subcases of the full Möbius transformations allowed in the ABS approach [2], any conjugacy class of ours is either completely contained into one of the ABS classification or is totally disjointed from them. Considering that no one out of the (left hand members of the) canonical forms (a′)-(e′) possesses the invariance (up to an overall sign) under $v_{n,m} \leftrightarrow v_{n+1,m}$, $v_{n,m+1} \leftrightarrow v_{n+1,m+1}$, we can conclude that no intersection can exist between our classes and those generated by the *ABS* list. Even more, no equation in our list is of Klein-type or, that is the same [14], a subcase of the Q_V equation.

We can enlarge our class of transformations by including also an exchange $n \leftrightarrow m$ between the two independent variables. The subclass (45a) can be discarded because under this exchange we would get it from subclass (45b) with $\tau_2 = 0$; similarly the subclass (47a) can be discarded because under this exchange we would get it from subclass (47b) with $\tau_2 = 0$; finally the subclasses (48-50) are invariant under this transformation.

Let us include also the inversion $n \to -n$. Setting $\tilde{v}_{n,m} \coloneqq v_{-n,m}$, we have that, if $v_{n,m}$ satisfies (45b), then $\tilde{v}_{n,m}$ satisfies (47b); if $v_{n,m}$ satisfies (48) with parameters ϵ and ζ', then $\tilde{v}_{n,m} \coloneqq \operatorname{sgn}(\epsilon)\, v_{-n,m}$ satisfies (48) with parameters $1/\epsilon$ and $\zeta'/|\epsilon|$ and similarly for Eq. (49); if $v_{n,m}$ satisfies (50) with parameter ϵ, then $\tilde{v}_{n,m} \coloneqq -v_{-n,m}/\epsilon$ satisfies (50) with parameter $1/\epsilon$ (this implies that, if $v_{n,m}$ satisfies one of the four canonical forms (47b), (48-50), then also $\tilde{v}_{n,m} \coloneqq v_{-n,-m}$ does). As a consequence under this enlarged class of transformations the Eq. (47b) can be discarded and in the case of the Eqs. (48-50) we can limit the parameter ϵ to the range $-1 < \epsilon < 1$, $\epsilon \neq 0$ as the equation with parameters $1/\epsilon$ and ζ' can be obtained from the corresponding with parameters ϵ and $\zeta'\,|\epsilon|$.

Notes: In the Cases (c) and (d) of Proposition 4, when $\pi = 0$, corresponding to $\zeta' = 0$ in the cases (c′) and (d′), they reduce to the S-integrable cases analyzed in Levi-Yamilov and Ramani-Grammaticos [17].

3.4 Classification at order ε^6.

Now we perform a multiscale reduction at order ε^6 on the four canonical forms (47b), (48-50) and we find that all the so far obtained equations satisfy the A_4-integrability conditions (60). Hence we can state the following proposition

Proposition 5. Up to a restricted Möbius transformations $\tilde{v}_{n,m} \coloneqq v_{n,m}/\left(\alpha v_{n,m}+\beta\right)$, exchanges $n \leftrightarrow m$ and inversions $n \to -n$, all the A_4-asymptotically S-integrable cases in the class $\mathcal{Q}_+$ are given by

$$\begin{aligned} & v_{n,m} + v_{n+1,m+1} + 2\left(v_{n+1,m} + v_{n,m+1}\right) + v_{n+1,m}v_{n,m+1}\left(1+\tau\right) + \\ & \quad + \left(v_{n+1,m}v_{n+1,m+1} + v_{n,m}v_{n,m+1}\right)\tau + v_{n,m+1}v_{n+1,m+1} + v_{n,m}v_{n+1,m} \\ & \quad + v_{n+1,m}v_{n,m+1}\left(v_{n,m} + v_{n+1,m+1}\right)\tau = 0; \end{aligned} \tag{51a}$$

$$\begin{aligned} & v_{n,m} + v_{n+1,m+1} + \epsilon\left(v_{n+1,m} + v_{n,m+1}\right) + \\ & \quad + \delta\left[\epsilon v_{n+1,m}v_{n,m+1}\left(v_{n,m} + v_{n+1,m+1}\right) + v_{n,m}v_{n+1,m+1}\left(v_{n+1,m} + v_{n,m+1}\right)\right] \\ & \quad + \tau v_{n,m}v_{n+1,m}v_{n,m+1}v_{n+1,m+1} = 0, \quad -1<\epsilon<1, \quad \epsilon \neq 0, \quad \delta \coloneqq \pm 1, \quad \tau \geq 0; \end{aligned} \tag{51b}$$

$$\begin{aligned} & v_{n,m} + v_{n+1,m+1} + \epsilon\left(v_{n+1,m} + v_{n,m+1}\right) + \\ & \quad + \delta\left[v_{n+1,m}v_{n,m+1}\left(v_{n,m} + v_{n+1,m+1}\right) + \epsilon v_{n,m}v_{n+1,m+1}\left(v_{n+1,m} + v_{n,m+1}\right)\right] \\ & \quad + \tau v_{n,m}v_{n+1,m}v_{n,m+1}v_{n+1,m+1} = 0, \quad -1<\epsilon<1, \quad \epsilon \neq 0, \quad \delta \coloneqq \pm 1, \quad \tau \geq 0; \end{aligned} \tag{51c}$$

$$\begin{aligned} & v_{n,m} + v_{n+1,m+1} + \epsilon\left(v_{n+1,m} + v_{n,m+1}\right) + v_{n,m}v_{n+1,m+1} - v_{n+1,m}v_{n,m+1} + \\ & \quad + \left(1-\tfrac{1}{\epsilon}\right)\left[v_{n+1,m}v_{n,m+1}\left(v_{n,m} + v_{n+1,m+1}\right) - v_{n,m}v_{n+1,m+1}\left(v_{n+1,m} + v_{n,m+1}\right)\right] \\ & \quad + \left(1-\tfrac{1}{\epsilon^2}\right)v_{n,m}v_{n+1,m}v_{n,m+1}v_{n+1,m+1} = 0, \quad -1<\epsilon<1, \quad \epsilon \neq 0, \ \frac{1}{2}. \end{aligned} \tag{51d}$$

Eqs. (51a, 51d) depend on $\mathcal{N}=1$ free parameter, while (51b, 51c) depend on $\mathcal{N}=2$ free parameters (without considering the additional discrete parameter δ).

If in (51a), when $\tau=0$, we apply the (not allowed) transformation $v_{n,m} \coloneqq \sqrt{3}w_{n,m}-1$, we obtain

$$w_{n,m}w_{n+1,m} + w_{n+1,m}w_{n,m+1} + w_{n,m+1}w_{n+1,m+1} - 1 = 0, \tag{52}$$

which in the direction n satisfies two first order necessary integrability conditions given in [14] but doesn't admit the corresponding three-point generalized symmetry either autonomous or not, while in the direction m the first order integrability conditions are not satisfied. Following [8] we were able to prove the integrability of (52) constructing two five-point symmetries, one in the n direction depending on the points $(n+2,m)$, $(n+1,m)$, (n,m), $(n-1,m)$ and $(n-2,m)$ and the other one in the m direction. In [18] its integrability was finally proven providing a 3×3 Lax pair. Moreover this equation has the singularity confinement property, can be bilinearized, possesses multisoliton solutions and has a continuous limit into the $mKdV$ equation, [9].

In (51a), with $\tau=1$, applying the (not allowed) transformation $v_{n,m} \coloneqq -\left(2^{\frac{1}{3}}w_{n,m}+1\right)$ yields

$$w_{n+1,m}w_{n,m+1}\left(w_{n,m} + w_{n+1,m+1}\right) + 1 = 0, \tag{53}$$

an integrable system introduced in [15], where it was proved to satisfy the second order, but not the first order, integrability conditions, to posses a 3×3 Lax pair and to be a degeneration of the discrete integrable Tzitzeica equation proposed by Adler in [1]. Moreover this equation has the singularity confinement property, can be trilinearized and possesses multisoliton solutions, [9].

Finally, if in (51a), when $\tau \neq 0, 1$ we apply the (not allowed) transformation $v_{n,m} := \frac{1-\tau}{\tau} w_{n,m} - 1$, we obtain

$$w_{n,m}w_{n+1,m} + w_{n,m+1}w_{n+1,m+1} + w_{n+1,m}w_{n,m+1}\left(1 + w_{n,m} + w_{n+1,m+1}\right) + \chi = 0, \quad (54)$$

where $\chi := \frac{(\tau-3)\tau^2}{(1-\tau)^3}$, which doesn't satisfy the first order integrability conditions for three-point generalized symmetries either autonomous or not, either in direction n or m. In [18] we showed the integrability of the subcase $\chi = 0$ constructing two five-point generalized symmetries, one in the n direction and the other one in the m direction, and a 3×3 Lax pair. An indication of the integrability of the general case (54) for arbitrary χ was provided showing its algebraic entropy vanishes. Other strong indications of integrability for arbitrary χ, such as the singularity confinement property, bilinear form, multisoliton solutions and a continuous limit into the $mKdV$ equation when $\chi = -1$, were established in [9]. In the case $\chi = -1$ we can provide the following five-point symmetry in the n direction depending on the points $(n + 2, m)$, $(n + 1, m)$, (n, m), $(n - 1, m)$ and $(n - 2, m)$:

$$w_{n,m,t} = \frac{w_{n,m}\left(w_{n,m}+1\right)\left(w_{n,m}w_{n-1,m}-1\right)\left(w_{n+1,m}w_{n,m}-1\right)}{\left(w_{n,m}w_{n-1,m}w_{n-2,m}+1\right)\left(w_{n+1,m}w_{n,m}w_{n-1,m}+1\right)}, + \frac{w_{n+2,m}w_{n+1,m} - w_{n-1,m}w_{n-2,m}}{w_{n+1,m}w_{n+1,m}w_{n,m}+1},$$

where t is a group parameter. The last generalized symmetry is invariant under $\tilde{w}_{n,m} := 1/w_{n,m}$ and under the following Miura transformation

$$z_{n,m} := \frac{w_{n+1,m}w_{n,m} - 1}{w_{n+1,m}w_{n,m}w_{n-1,m} + 1} \quad (55)$$

it can be transformed into a Bogoyavlenskyi lattice

$$z_{n,m,t} = z_{n,m}\left(z_{n,m}+1\right)\left(z_{n+2,m}z_{n+1,m} - z_{n-1,m}z_{n-2,m}\right).$$

Proposition 6. If (54) with $\chi = -1$ is satisfied, given (55), then

$$w_{n+1,m} = -\frac{z_{n,m}+1}{\left(z_{n,m}w_{n-1,m}-1\right)w_{n,m}}, \quad (56a)$$

$$w_{n,m+1} = -\frac{\left(z_{n,m+1}+z_{n,m}+1\right)\left(w_{n-1,m}+1\right)w_{n,m}}{\left(z_{n,m}+1\right)\left(w_{n,m}+1\right)}, \quad (56b)$$

$$w_{n-1,m+1} = \frac{\left(z_{n,m}+1\right)\left(w_{n,m}w_{n-1,m}-1\right)}{z_{n,m+1}\left(w_{n-1,m}+1\right)w_{n,m}}, \quad (56c)$$

and $z_{n,m}$ satisfies

$$z_{n,m}\left(z_{n+1,m}+1\right) + z_{n+1,m+1}\left(z_{n,m+1}+z_{n,m}+1\right) = 0. \quad (57)$$

To prove (56a), just solve (55) with respect to $w_{n+1,m}$; to obtain (56b), substitute (56a) and its shifted once along direction m into the equation (54) with $\chi = -1$, solve with respect to $w_{n-1,m+1}$, substitute this result into the equation (54) with $\chi = -1$ shifted back once along direction n and solve with respect to $w_{n,m+1}$; (56c) follows inserting (56b) into the previous result for $w_{n-1,m+1}$. The three relations (56) provide a Miura transformation between equation (54) with $\chi = -1$ and (57): the compatibility between the $z-$variables implies (54) with $\chi = -1$, while the compatibility between the $w-$variables implies (57).

Equation (57) is an integrable lattice possessing a 3×3 Lax representation, [18]. When $z_{n,m} \neq 0$, under the inversion $\tilde{w}_{n,m} := 1/z_{n,m}$ the equation (57) is mapped into the equation (54) with $\chi = 0$, so (55) provides also a Miura mapping from $w_{n,m}$ solving (54) with $\chi = -1$ to $\tilde{w}_{n,m}$ solving the same equation but with $\chi = 0$. This Miura transformation induces, through the mapping $v_{n,m} := \frac{1-\tau}{\tau} w_{n,m} - 1$, a corresponding Miura transformation from a solution $v_{n,m}$ of (51a) with $\tau = 1/3$ to a solution $\tilde{v}_{n,m}$ of (51a) with $\tau = 3$. Another set of Miura transformations between the equations (52), (53) and (54) was derived in [9].

Summing up, we have very strong indications of integrability for the master equation (51a) which, when $\tau = 3,\ 1/3$, has a continuous limit into the $mKdV$ equation, [9].

If in (51b), (51c) we apply the (not allowed) transformations $w_{n,m} := \delta \operatorname{sgn}(\epsilon) / v_{n,m}$ and $\tilde{w}_{n,m} := \delta / v_{n,m}$ respectively, we obtain

$$\begin{aligned} &\frac{\tau}{|\epsilon|} + w_{n,m} + w_{n+1,m+1} + \frac{1}{\epsilon}\left(w_{n+1,m} + w_{n,m+1}\right) \\ &\quad + \delta\left[\frac{1}{\epsilon} w_{n+1,m} w_{n,m+1}\left(w_{n,m} + w_{n+1,m+1}\right) + w_{n,m} w_{n+1,m+1}\left(w_{n+1,m} + w_{n,m+1}\right)\right] = 0, \end{aligned} \tag{58a}$$

$$\begin{aligned} &\tau + \tilde{w}_{n,m} + \tilde{w}_{n+1,m+1} + \epsilon\left(\tilde{w}_{n+1,m} + \tilde{w}_{n,m+1}\right) \\ &\quad + \delta\left[\tilde{w}_{n+1,m}\tilde{w}_{n,m+1}\left(\tilde{w}_{n,m} + \tilde{w}_{n+1,m+1}\right) + \epsilon \tilde{w}_{n,m}\tilde{w}_{n+1,m+1}\left(\tilde{w}_{n+1,m} + \tilde{w}_{n,m+1}\right)\right] = 0. \end{aligned} \tag{58b}$$

Eqs. (58a, 58b) are just an almost trivial looking modification of the two integrable systems discussed in [17], which are recovered when $\tau = 0$. In that paper it was shown that, when $\tau = 0$, Eqs. (58b, 58a) are mapped through a Möbius transformation respectively to the Hirota discrete sine-Gordon equation and to its potential form. After in (58a) we replace $\epsilon \to 1/\epsilon$ and in (58b) $\delta \to s\delta$, with $s := \operatorname{sgn}(\epsilon)$, the precise form of the potentiation induced between them is

$$w_{n,m} = |\epsilon|^{1/2} \frac{\tilde{w}_{n+1,m} + \tilde{w}_{n,m+1}}{1 + s\delta \tilde{w}_{n+1,m}\tilde{w}_{n,m+1}}.$$

These equations satisfy the first order integrability conditions for three-point generalized symmetries either autonomous or not if and only if $\tau = 0$, which in this limit, in the n direction, are respectively given by

$$\begin{aligned} w_{n,m,t} &= \frac{\left(\delta w_{n,m}^2 - \epsilon\right)\left(\delta\epsilon w_{n,m}^2 - 1\right)\left(w_{n+1,m} - w_{n-1,m}\right)}{\left(1 + \delta w_{n,m} w_{n+1,m}\right)\left(1 + \delta w_{n,m} w_{n-1,m}\right)}, \\ \tilde{w}_{n,m,\tilde{t}} &= Y \frac{\left(\delta \tilde{w}_{n,m}^2 - 1\right)\left(\tilde{w}_{n+1,m} - \tilde{w}_{n-1,m}\right)}{\delta \tilde{w}_{n+1,m}\tilde{w}_{n-1,m} - 1} + \left[(-1)^n \kappa + (-1)^m \theta\right]\left(\delta \tilde{w}_{n,m}^2 - 1\right), \end{aligned}$$

where t and $\tilde{t}$ are two group parameters, and, in the m direction, by similar expressions obtained changing $w_{n+1,m} \to w_{n,m+1}$ and $w_{n-1,m} \to w_{n,m-1}$. The second integrable system shows a two parameters non autonomous point symmetry tail too. We note that both the integrable systems are invariant under $w_{n,m} := -v_{n,m}$; the first integrable system is covariant under the inversion $w_{n,m} := 1/v_{n,m}$ as ϵ is changed into $1/\epsilon$, while the second one is invariant; under the non autonomous transformation $w_{n,m} := (-1)^{n+m} v_{n,m}$ both the integrable systems are covariant as in the first case ϵ is changed into $-\epsilon$ and δ into $-\delta$, while in the second one ϵ is changed into $-\epsilon$. This implies that in those systems we can limit ourselves to the range $0 < \epsilon < 1$. Moreover the second integrable system under the non autonomous transformation $\tilde{w}_{n,m} := (v_{n,m})^{(-1)^{n+m}}$ is invariant when $\delta = 1$ and covariant when $\delta = -1$ as ϵ is changed into $\delta\epsilon$. Finally both the integrable systems are covariant under the transformation $w_{n,m} := \mathrm{i} v_{n,m}$ as δ is changed into $-\delta$. This implies that in those systems we can always take $\delta = 1$ but in general, if we allow such a transformation, the solution will be no more a real field but a complex one. Let's also note that the non autonomous transformation $w_{n,m} := (-1)^n v_{n,m}$ or $w_{n,m} := (-1)^m v_{n,m}$ brings both the integrable systems from class $\mathcal{Q}^+$ into class $\mathcal{Q}^-$.

An indication that the general cases (58a, 58b) are not integrable when $\tau \neq 0$ can be obtained showing their algebraic entropy [3] doesn't vanish.

If in (51d) we apply the (not allowed) transformation $v_{n,m} := \frac{|\rho|^{1/2} w_{0,0}+1}{|\rho|^{1/2} w_{0,0}-1/\epsilon}$, with $\rho := \frac{-1+2\epsilon}{\epsilon(\epsilon-2)} \neq 0$, we obtain

$$w_{n,m} + w_{n+1,m+1} + w_{n,m} w_{n+1,m+1} \left[\delta \left(w_{n+1,m} + w_{n,m+1}\right) + \epsilon |\rho|^{3/2} w_{n+1,m} w_{n,m+1}\right] + \frac{\delta}{\epsilon |\rho|^{3/2}} = 0, \quad (59)$$

where $\delta := \operatorname{sgn}(\rho) = \operatorname{sgn}(1/\epsilon - 2)$, which, for $\delta = -1$, is a real discrete Tzitzeica equation with coefficient $c = 1/\left(\epsilon |\rho|^{3/2}\right)$ and for $\delta = 1$, through the (not allowed) transformation $w_{n,m} \to \mathrm{i} w_{n,m}$ becomes a complex Tzitzeica equation with coefficient $c = \mathrm{i}/\left(\epsilon |\rho|^{3/2}\right)$. We remember that the Tzitzeica equation possesses a 3×3 Lax representation and satisfies [15] the second order, but not the first order, integrability conditions.

We note that, besides not being subcases of the Q_V equation, our systems (51), except for (51b, 51c) with $\tau = 0$, where a five-point generalized symmetry depending on the points $(n+1, m)$, $(n, m+1)$, (n, m), $(n-1, m)$ and $(n, m-1)$ exists, are not included into the Garifullin-Yamilov class [10] too.

4 Concluding remarks

In this paper we have considered the application of a multiple scale expansion to a class of dispersive multilinear partial difference equation on the square lattice, $\mathcal{Q}^+$. The integrability conditions we obtain when we require that the multiple scale expansion of the discrete class of equations is equivalent to the equations of the *NLS* hierarchy reduce the $\mathcal{N} = 13$ initial parameters defining the $\mathcal{Q}^+$ class to a maximum of $\mathcal{N} = 2$ free (continuous) parameters defining four equations. A great effort has been directed to extend the expansion up to order ε^6, the related integrability conditions appearing in this paper for the first time. As a result of our efforts we have been able to compare the A_3 integrable equations to the A_4 integrable equations. They turn out to be the same, so that one could presume we might be already in the asymptotic regime and that the obtained equations might be integrable.

However a non vanishing algebraic entropy is an indication that the general cases (58a, 58b) are not integrable when $\tau \neq 0$.

An open problem seems of major importance now: the consideration of the second class of dispersive multilinear partial difference equations on the square lattice, $\mathcal{Q}^-$ is a major task which will surely provide new classes of integrable equations. However in this case the lowest order integrability conditions appear already at order ε^2 and will not produce an equation of the *NLS* type, but more likely a coupled wave equation. Work on it is in progress.

Acknowledgements

DL and CS were partly supported by the Italian Ministry of Education and Research, PRIN “Nonlinear waves: integrable fine dimensional reductions and discretizations” from 2007 to 2009 and PRIN “Continuous and discrete nonlinear integrable evolutions: from water waves to symplectic maps” from 2010. RHH is supported by projects PID2019-106802GB-I00 and PID2021-124473NB-I00 from the Ministry of Science and Innovation of Spain, and would like to thank the INFN, Sezione Roma Tre and the UPM for their support during his visits to Rome. We thank Matteo Petrera for many enlightening discussions in the first stages of this paper.

Before the publication of this work, Professor Decio Levi untimely passed away. Christian Scimiterna and Rafael Hernandez Heredero would like to express their deep admiration to their Professor, to whom they owe so many things in their personal and academic lives, and whom they miss with deep emotion.

Appendix

We present explicitly the 48 conditions for ε^6 S-asymptotic integrability (A_4-integrability) involving the real (S_j) and imaginary parts (T_j) of the coefficients η_j, $j = 1,\ldots, 34$ of the differential polynomial (27a). The expressions of the coefficients κ_m , $m = 1,\ldots, 77$ of the differential polynomial (27b) as functions of the η_j, $j = 1,\ldots, 34$ are complicated, so we will omit them.

$$
\begin{aligned}
T_2 &= \left(\frac{a}{11}+\frac{3b}{4}\right)\frac{S_{10}}{\rho_2}+\left(\frac{13a}{11}+\frac{b}{2}\right)\frac{S_{18}-S_{15}}{2\rho_1}+\left(\frac{37a}{11}+b\right)\frac{I_6S_{27}}{8\rho_1\rho_2}+T_1+\left(\frac{a^2}{2}+\frac{13ab}{11}+\frac{b^2}{2}\right)\frac{T_{22}}{4\rho_1\rho_2} \qquad (60)\\
&+\left[I_1+\left(37a^2-15ab-11b^2\right)\frac{I_8}{44\rho_1\rho_2}\right]\frac{T_{27}}{2\rho_2}+\left[\left(\frac{13a}{11}+\frac{b}{2}\right)\frac{R_2}{8}-\left(32a^2+27ab-189b^2\right)\frac{I_6}{352}+\left(\frac{a}{11}+\frac{b}{4}\right)\frac{aI_{12}}{8}\right]\frac{T_{32}}{\rho_1\rho_2}-\\
&-\left[\left(9a-\frac{7b}{2}\right)\frac{R_{12}}{22\rho_1\rho_2}+\frac{I_1}{\rho_2}+\frac{7I_2-29I_3-14I_5}{44\rho_1}+\left(39a^2-53ab+\frac{67b^2}{2}\right)\frac{I_8}{44\rho_1\rho_2^2}\right]\frac{T_{33}}{2},\\
T_3 &= \frac{a}{2\rho_2}\left(\frac{\rho_1}{\rho_2}S_{10}+S_{15}-S_{18}+S_{20}\right)-\left(a-\frac{b}{2}\right)\frac{I_6S_{27}}{2\rho_2^2}-\frac{abT_{22}}{4\rho_2^2}+\left[\frac{aR_{12}}{2\rho_2}-I_3-(a-b)\frac{aI_8}{2\rho_2^2}\right]\frac{T_{27}}{2\rho_2}-\\
&-\left[R_2+\frac{(3a+b)\,I_6+aI_{12}}{2\rho_2}\right]\frac{aT_{32}}{8\rho_2^2}-\left[(3a+b)\frac{R_{12}}{4\rho_2}+\frac{\rho_1I_1}{\rho_2}-\frac{I_2+3I_3+2I_5}{4}-\left(a^2-3ab+2b^2\right)\frac{I_8}{4\rho_2^2}\right]\frac{T_{33}}{2\rho_2},\\
T_4 &= \left(\frac{35a}{33}+\frac{b}{2}\right)\frac{\rho_1S_{10}}{2\rho_2^2}-\left(\frac{35a}{33}-\frac{b}{2}\right)\frac{S_{15}}{2\rho_2}+\left(\frac{34a}{33}+\frac{b}{4}\right)\frac{S_{18}}{\rho_2}-\frac{aS_{20}}{2\rho_2}+\left[\frac{R_2}{2}+\frac{67aI_6}{33\rho_2}+\left(\frac{a}{2}+b\right)\frac{I_{12}}{2\rho_2}\right]\frac{S_{27}}{2\rho_2}+\frac{\rho_1T_1}{\rho_2}+\\
&+\left(\frac{a^2}{2}+\frac{101ab}{33}+\frac{b^2}{2}\right)\frac{T_{22}}{4\rho_2^2}+\left[-(a-b)\frac{R_{12}}{8\rho_2}+\frac{\rho_1I_1}{\rho_2}-\frac{I_2-I_3-2I_5}{8}+\left(\frac{17a^2}{33}-\frac{103ab}{264}+\frac{b^2}{8}\right)\frac{I_8}{\rho_2^2}\right]\frac{T_{27}}{\rho_2}+\\
&+\left[\left(\frac{101a}{33}+\frac{3b}{2}\right)\frac{R_2}{2}-\left(\frac{19a^2}{3}+\frac{97ab}{4}-\frac{591b^2}{4}\right)\frac{I_6}{22\rho_2}+\left(\frac{17a^2}{33}+\frac{3ab}{8}+\frac{b^2}{2}\right)\frac{I_{12}}{\rho_2}\right]\frac{T_{32}}{4\rho_2^2}+\left[\left(\frac{37a}{2}-35b\right)\frac{R_{12}}{66\rho_2}+\right.\\
&\left.+\frac{35I_2-2I_3-70I_5}{66}-\left(\frac{25a^2}{3}-\frac{317ab}{6}+\frac{323b^2}{4}\right)\frac{I_8}{22\rho_2}\right]\frac{T_{33}}{2\rho_2}, \quad S_5=-\frac{\rho_1}{\rho_2}\left(S_1+\frac{3S_2}{2}\right)+\frac{9}{2}\left(S_3+S_4\right)-S_6+\frac{S_7}{2}-
\end{aligned}
$$

$$-\left(a^2+\frac{7ab}{2}+\frac{b^2}{4}\right)\frac{S_{22}}{2\rho_2^2}+\left[-(5a-7b)\frac{R_{12}}{16\rho_2}-\frac{7\rho_1 I_1}{4\rho_2}-\frac{7I_2-35I_3+18I_5}{16}-\left(\frac{7a}{16}+b\right)\frac{aI_8}{\rho_2^2}\right]\frac{S_{27}}{\rho_2}+(a+7b)\frac{\rho_1 T_{10}}{4\rho_2^2}+$$

$$+\left(a-\frac{5b}{2}\right)\frac{T_{15}}{2\rho_2}+\left(a+\frac{3b}{2}\right)\frac{T_{18}}{2\rho_2}+(a+b)\frac{T_{20}}{4\rho_2}-\left[3R_2+(a-23b)\frac{I_6}{2\rho_2}+\left(\frac{a}{2}+b\right)\frac{3I_{12}}{\rho_2}\right]\frac{T_{27}}{8\rho_2}+$$

$$+\left\{\left(\frac{a^2}{2}-ab+3b^2\right)\frac{R_{12}}{8\rho_2}+\frac{\rho_1 aI_1}{\rho_2}+\frac{(3a-5b)I_2+(a+39b)I_3}{16}+\left(\frac{a}{4}-b\right)\frac{I_5}{2}+\left[\frac{a(5a+13b)}{2}-11b^2\right]\frac{aI_8}{8\rho_2^2}\right\}\frac{T_{32}}{\rho_2^2}-$$

$$-\left[7R_2+(7a+15b)\frac{I_6}{2\rho_2}+\left(\frac{a}{2}+b\right)\frac{7I_{12}}{\rho_2}\right]\frac{T_{33}}{8\rho_2},\quad T_5=-\left(\frac{31a}{11}-3b\right)\frac{\rho_1 S_{10}}{4\rho_2^2}+\left(\frac{31a}{11}-5b\right)\frac{S_{15}}{4\rho_2}-\left(\frac{53a}{11}-3b\right)\frac{S_{18}}{4\rho_2}+$$

$$+\left(a-\frac{b}{2}\right)\frac{S_{20}}{\rho_2}-\left[\frac{R_2}{2}+\left(\frac{97a}{44}-b\right)\frac{I_6}{\rho_2}+\left(\frac{a}{2}+b\right)\frac{I_{12}}{2\rho_2}\right]\frac{S_{27}}{2\rho_2}-\frac{\rho_1 T_1}{\rho_2}+T_6-\left(3a^2+\frac{31ab}{11}-b^2\right)\frac{T_{22}}{8\rho_2^2}+$$

$$+\left\{\left(a-\frac{b}{2}\right)\frac{R_{12}}{2\rho_2}-\frac{\rho_1 I_1}{2\rho_2}+\frac{I_2-I_3-4I_5}{4}-\left[\frac{a(8a-19b)}{11}+\frac{9b^2}{8}\right]\frac{I_8}{\rho_2^2}\right\}\frac{T_{27}}{\rho_2}$$

$$-\left[\left(\frac{75a}{11}-b\right)R_2+\left(\frac{49a^2}{2}-205ab+\frac{615b^2}{2}\right)\frac{I_6}{11\rho_2}+\left(\frac{9a}{11}+7b\right)\frac{aI_{12}}{2\rho_2}\right]\frac{T_{32}}{16\rho_2^2}$$

$$-\left\{\left(\frac{9a}{2}+b\right)\frac{R_{12}}{44\rho_2}+\frac{\rho_1 I_1}{2\rho_2}-\frac{I_2-12I_3+20I_5}{44}-\left[13a^2-\frac{(585a-533b)b}{8}\right]\frac{I_8}{22\rho_2^2}\right\}\frac{T_{33}}{\rho_2},$$

$$T_7=-\left(\frac{23a}{11}+\frac{b}{2}\right)\frac{\rho_1 S_{10}}{\rho_2^2}+\left(\frac{23a}{11}-\frac{b}{2}\right)\frac{S_{15}}{\rho_2}-\left(\frac{34a}{11}+\frac{b}{2}\right)\frac{S_{18}}{\rho_2}+\frac{3aS_{20}}{2\rho_2}-\left[R_2+\left(\frac{123a}{11}-b\right)\frac{I_6}{2\rho_2}+\left(\frac{a}{2}+b\right)\frac{I_{12}}{\rho_2}\right]\frac{S_{27}}{2\rho_2}-$$

$$-\frac{2\rho_1 T_1}{\rho_2}+T_6-\left(\frac{a^2}{2}+\frac{45ab}{11}+\frac{b^2}{2}\right)\frac{T_{22}}{2\rho_2^2}+\left[\left(a-\frac{b}{2}\right)\frac{R_{12}}{2\rho_2}-\frac{2\rho_1 I_1}{\rho_2}+\frac{I_2-3I_3-2I_5}{4}-\left(\frac{17a^2}{11}-\frac{57ab}{44}+\frac{b^2}{4}\right)\frac{I_8}{\rho_2^2}\right]\frac{T_{27}}{\rho_2}-$$

$$-\left\{3\left(\frac{15a}{11}+\frac{b}{2}\right)R_2+\left[47a^2-\frac{(107a-815b)b}{2}\right]\frac{I_6}{22\rho_2}+\left(\frac{17a^2}{11}+\frac{3ab}{4}+b^2\right)\frac{I_{12}}{\rho_2}\right\}\frac{T_{32}}{4\rho_2^2}-\left[(81a-59b)\frac{R_{12}}{44\rho_2}+\frac{\rho_1 I_1}{\rho_2}+\right.$$

$$\left.+\frac{59(I_2-I_3)-162I_5}{44}-\left(\frac{105a^2}{4}-71ab+\frac{457b^2}{4}\right)\frac{I_8}{11\rho_2^2}\right]\frac{T_{33}}{2\rho_2},\quad S_8=-\frac{2\rho_1 S_2}{\rho_2}+3S_3+6S_4-2S_6+S_7-$$

$$-\left(\frac{a^2}{2}+ab+\frac{b^2}{2}\right)\frac{S_{22}}{\rho_2^2}+\left[(a+b)\frac{R_{12}}{4\rho_2}-\frac{\rho_1 I_1}{\rho_2}-\frac{I_2-5I_3+6I_5}{4}-\left(\frac{a}{4}+b\right)\frac{aI_8}{\rho_2^2}\right]\frac{S_{27}}{\rho_2}+(a+b)\frac{\rho_1 T_{10}}{\rho_2^2}-\frac{b}{\rho_2}(T_{15}-T_{18})-$$

$$-\left[R_2+(a-5b)\frac{I_6}{2\rho_2}+\left(\frac{a}{2}+b\right)\frac{I_{12}}{\rho_2}\right]\frac{T_{27}}{2\rho_2}+\left[a(a+b)\frac{R_{12}}{4\rho_2}+\frac{\rho_1 aI_1}{\rho_2}+\frac{b(I_2+5I_3-4I_5)}{4}+a\left(\frac{a^2}{2}+\frac{ab}{4}-b^2\right)\frac{I_8}{\rho_2^2}\right]\frac{T_{32}}{\rho_2^2}-$$

$$-\left[R_2+(a+3b)\frac{I_6}{2\rho_2}+\left(\frac{a}{2}+b\right)\frac{I_{12}}{\rho_2}\right]\frac{T_{33}}{2\rho_2},\quad T_8=-\left(\frac{31a}{11}-b\right)\frac{\rho_1 S_{10}}{2\rho_2^2}+\left(\frac{31a}{11}-3b\right)\frac{S_{15}}{2\rho_2}-\left(\frac{53a}{11}-b\right)\frac{S_{18}}{2\rho_2}+$$

$$+(2a-b)\frac{S_{20}}{\rho_2}-\left[R_2+\left(\frac{97a}{22}-b\right)\frac{I_6}{\rho_2}+\left(\frac{a}{2}+b\right)\frac{I_{12}}{\rho_2}\right]\frac{S_{27}}{2\rho_2}-\frac{2\rho_1 T_1}{\rho_2}+T_6-\left(a^2+\frac{75ab}{44}-b^2\right)\frac{T_{22}}{4\rho_2^2}+$$

$$+\left[(a-b)\frac{3R_{12}}{4\rho_2}-\frac{2\rho_1 I_1}{\rho_2}+\frac{3(I_2-I_3-2I_5)}{4}-\left(\frac{53a^2}{44}-\frac{16ab}{11}+\frac{3b^2}{4}\right)\frac{I_8}{\rho_2^2}\right]\frac{T_{27}}{\rho_2}-\left[\left(\frac{75a}{11}+b\right)\frac{R_2}{8}-\right.$$

$$\left.-\left(\frac{39a^2}{16}+5ab-\frac{461b^2}{16}\right)\frac{I_6}{11\rho_2}+\left(\frac{53a^2}{44}+\frac{ab}{4}+b^2\right)\frac{I_{12}}{4\rho_2}\right]\frac{T_{32}}{\rho_2^2}-\left\{\left(5a-\frac{31b}{4}\right)\frac{R_{12}}{11\rho_2}+\frac{31I_2-9I_3-62I_5}{44}-\right.$$

$$\left.-\left[19a^2-\frac{(255a-313b)b}{2}\right]\frac{I_8}{44\rho_2^2}\right\}\frac{T_{33}}{\rho_2},$$

$$S_9=\frac{S_{10}}{2}-\frac{\rho_2}{2\rho_1}(S_{15}-S_{18})+\frac{I_6S_{27}+bT_{22}}{4\rho_1}+(a-b)\frac{I_8T_{27}}{4\rho_1\rho_2}+\left[R_2+\frac{(3a-7b)I_6+aI_{12}}{2\rho_2}\right]\frac{T_{32}}{8\rho_1}+\frac{3bI_8T_{33}}{4\rho_1\rho_2},$$

$$T_9=T_{10}-\frac{I_1T_{32}}{2\rho_2}-\frac{I_6T_{33}}{4\rho_1},\quad S_{11}=\frac{S_{22}}{2}-\left[\frac{R_{12}}{2}+(a-b)\frac{I_8}{\rho_2}\right]\frac{T_{32}}{2\rho_2},\quad T_{11}=T_{22}+\frac{I_8T_{27}-(I_6+I_{12})T_{32}}{2\rho_2},$$

$$S_{12}=-\frac{I_8S_{27}}{\rho_2}+\left(\frac{a}{2}-b\right)\frac{I_8T_{32}}{\rho_2^2},\quad T_{12}=-\frac{I_8(T_{27}-T_{33})}{\rho_2},\quad S_{13}=-\frac{2\rho_1S_{10}}{\rho_2}+2S_{15}-S_{18}+S_{20}-\frac{I_6S_{27}}{\rho_2}-$$

$$-\frac{aT_{22}}{\rho_2}-2(a-b)\frac{I_8T_{27}}{\rho_2^2}-\left[R_2+(5a-7b)\frac{I_6}{2\rho_2}-\left(\frac{a}{2}-b\right)\frac{I_{12}}{\rho_2}\right]\frac{T_{32}}{2\rho_2}+(3a-4b)\frac{I_8T_{33}}{\rho_2^2},\quad T_{13}=T_{18}+T_{20}+$$

$$+\left[(a+b)\frac{R_{12}}{4\rho_2}+\frac{\rho_1I_1}{\rho_2}-\frac{I_2-I_3+2I_5}{4}+\left(\frac{a^2}{2}+ab-b^2\right)\frac{I_8}{2\rho_2^2}\right]\frac{T_{32}}{\rho_2},\quad S_{14}=0,\quad T_{14}=-\frac{I_8T_{32}}{2\rho_2},$$

$$S_{16}=-\frac{I_6S_{27}+aT_{22}}{2\rho_2}+\left(a-\frac{3b}{2}\right)\frac{I_6T_{32}}{2\rho_2^2}+\frac{R_{12}T_{33}}{2\rho_2},\quad T_{16}=\frac{aS_{22}-I_6T_{27}}{2\rho_2}+\left(\frac{aR_{12}}{2\rho_2}-I_3\right)\frac{T_{32}}{2\rho_2}+(I_6+I_{12})\frac{T_{33}}{2\rho_2},$$

$$S_{17}=S_{15}+\frac{I_{12}S_{27}}{2\rho_2}-(a-b)\frac{I_8T_{27}}{2\rho_2^2}+[(2a+3b)I_6+bI_{12}]\frac{T_{32}}{4\rho_2^2}+\left(a-\frac{5b}{4}\right)\frac{I_8T_{33}}{\rho_2^2},\quad T_{17}=-\frac{aS_{22}}{2\rho_2}+$$

$$+\left(R_{12}-\frac{bI_8}{\rho_2}\right)\frac{S_{27}}{2\rho_2}+\frac{\rho_1T_{10}}{\rho_2}+\left[(a+3b)\frac{R_{12}}{4\rho_2}+\frac{\rho_1I_1}{\rho_2}-\frac{I_2-I_3-2I_5}{4}+\left(\frac{3a^2}{2}-ab-b^2\right)\frac{I_8}{2\rho_2^2}\right]\frac{T_{32}}{2\rho_2},\quad S_{19}=S_{20},$$

$$T_{19}=T_{20},\quad S_{21}=S_{22},\quad T_{21}=T_{22},\quad S_{23}=-\frac{2\rho_1S_{10}}{\rho_2}+2S_{15}-S_{18}+S_{20}-\left(I_6-\frac{I_{12}}{2}\right)\frac{S_{27}}{\rho_2}-(a+b)\frac{T_{22}}{2\rho_2}+$$

$$+\left[R_{12}-(a-b)\frac{I_8}{\rho_2}\right]\frac{T_{27}}{2\rho_2}-\left[R_2-\left(\frac{a}{2}-3b\right)\frac{I_6}{\rho_2}+(a-b)\frac{I_{12}}{2\rho_2}\right]\frac{T_{32}}{2\rho_2}-\left(R_{12}-\frac{3bI_8}{2\rho_2}\right)\frac{T_{33}}{2\rho_2},$$

$$T_{23}=\left(a-\frac{b}{2}\right)\frac{S_{22}}{\rho_2}-\left(R_{12}-\frac{bI_8}{\rho_2}\right)\frac{S_{27}}{2\rho_2}-\frac{\rho_1 T_{10}}{\rho_2}+T_{15}+T_{18}+\frac{I_{12}T_{27}}{2\rho_2}+\left[\left(\frac{a}{2}-b\right)\frac{R_{12}}{\rho_2}+\frac{I_2-I_3-4I_5}{2}\right]\frac{T_{32}}{2\rho_2}-$$
$$-\left(\frac{a}{2}-b\right)\frac{aI_8T_{32}}{2\rho_2^3}-\frac{I_{12}T_{33}}{2\rho_2},\quad S_{24}=S_{18}+S_{20}-\frac{(I_6-I_{12})\,S_{27}+bT_{22}}{2\rho_2}+\left[R_{12}+(a-b)\frac{I_8}{\rho_2}\right]\frac{T_{27}}{2\rho_2}+$$
$$+\left[\left(a-\frac{3b}{2}\right)I_6+\frac{bI_{12}}{4}\right]\frac{T_{32}}{\rho_2^2}-\left(a-\frac{5b}{4}\right)\frac{I_8T_{33}}{\rho_2^2},\quad T_{24}=(a+b)\frac{S_{22}}{2\rho_2}+\left(R_{12}+\frac{bI_8}{\rho_2}\right)\frac{S_{27}}{2\rho_2}-\frac{\rho_1T_{10}}{\rho_2}+T_{15}+T_{18}-$$
$$-(I_6-I_{12})\frac{T_{27}}{2\rho_2}-\left[\frac{I_2+I_3}{2}+\left(\frac{a}{2}-b\right)\frac{aI_8}{\rho_2^2}\right]\frac{T_{32}}{2\rho_2}+\frac{I_6T_{33}}{2\rho_2},\quad S_{25}=2S_{15}+\frac{I_{12}S_{27}}{2\rho_2}-(a-b)\frac{T_{22}}{2\rho_2}-$$
$$-\left[R_{12}+(a-b)\frac{3I_8}{\rho_2}\right]\frac{T_{27}}{2\rho_2}-\left[\left(3a-\frac{13b}{2}\right)I_6-\frac{bI_{12}}{2}\right]\frac{T_{32}}{2\rho_2^2}+\left[\frac{R_{12}}{2}+\left(3a-\frac{19b}{4}\right)\frac{I_8}{\rho_2}\right]\frac{T_{33}}{\rho_2},$$
$$T_{25}=-\left(a-\frac{b}{2}\right)\frac{S_{22}}{\rho_2}+\left(R_{12}-\frac{bI_8}{\rho_2}\right)\frac{S_{27}}{2\rho_2}+\frac{\rho_1T_{10}}{\rho_2}+T_{15}+T_{20}+\frac{I_{12}T_{27}}{2\rho_2}+\left[\frac{3bR_{12}}{4\rho_2}+\frac{\rho_1I_1}{\rho_2}-\frac{I_2-I_3-I_5}{2}\right]\frac{T_{32}}{\rho_2}+$$
$$+\left(a^2-b^2\right)\frac{I_8T_{32}}{2\rho_2^3}-\frac{I_{12}T_{33}}{2\rho_2},\quad S_{26}=-\frac{I_8S_{27}}{\rho_2}-\left(R_{12}+\frac{aI_8}{\rho_2}\right)\frac{T_{32}}{2\rho_2},\quad T_{26}=T_{22}-\frac{I_6T_{32}-I_8T_{33}}{\rho_2},$$
$$S_{28}=S_{27}-\left(\frac{a}{2}-b\right)\frac{T_{32}}{\rho_2},\quad T_{28}=T_{27}-T_{33},\quad S_{29}=S_{27}-\left(\frac{a}{2}-b\right)\frac{T_{32}}{\rho_2},\quad T_{29}=T_{27}-T_{33},$$
$$S_{30}=0,\quad T_{30}=T_{32},\quad S_{31}=0,\quad T_{31}=T_{32},\quad S_{32}=0,\quad S_{33}=-\frac{aT_{32}}{2\rho_2},\quad S_{34}=S_{27}+\frac{bT_{32}}{2\rho_2},\quad T_{34}=0,$$
$$4a\left[2\rho_2\left(\rho_1S_{10}-\rho_2S_{15}+\rho_2S_{18}\right)+\rho_2\left(I_6S_{27}+bT_{22}\right)+(a-b)\,I_8T_{27}\right]+a\left[2\rho_2R_2+(a+3b)\,I_6+aI_{12}\right]T_{32}+$$
$$+2\left[2\rho_2\,(a-b)\,R_{12}+2\rho_2^2\left(I_2-I_3-2I_5\right)+a\,(b-2a)\,I_8\right]T_{33}=0,\quad I_6T_{32}-I_8T_{33}=0.$$

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