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KdV hierarchies and quantum Novikov's equations.

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Abstract

The paper begins with a review of the well known KdV hierarchy, N-th Novikov equations and its finite hierarchiey in the classical commutative case. Its finite hierarchy consists of N compatible integrable polynomial dynamical systems in $\mathbb{C}^{2N}$. Then we discuss a non-commutative version of the N-th Novikov hierarchy defined on a finitely generated free associative algebra $\mathfrak{B}_N$ with $2N$ generators. Using the quantisation ideals method in $\mathfrak{B}_N$, for $N = 1, 2, 3, 4$, we have found two-sided homogeneous ideals $\mathfrak{Q}_N \subset \mathfrak{B}_N$ (quantisation ideals) which are invariant with respect to the N-th Novikov equation and such that the quotient algebra $\mathfrak{C}_N = \mathfrak{B}_N/\mathfrak{Q}_N$ has a well defined Poincare–Birkhoff–Witt basis. It enables us to define the quantum N-th Novikov equation and its hierarchy on the $\mathfrak{C}_N$. We have found N commuting quantum first integrals (Hamiltonians) and represented equations of the hierarchy in the Heisenberg form. In this paper we introduce the concept of Frobenius-Hochschild algebras and in its terms we express explicitly first integrals of the N-th Novikov hierarchy in the commutative, free and quantum cases.

1 Introduction

We dedicate this paper to the late Decio Levi, in tribute to his profound contributions to the advancement of the theory of integrable systems. His research journey began with the exploration of quantum systems. We believe that our paper, situated at the intersection of classical and quantum integrability and founded on a novel approach to the quantisation problem, will resonate with Decio's interests and legacy.

The problem of quantisation of dynamical systems has a history spanning over a century. In 1925, Heisenberg put forward a new quantum theory, suggesting that the multiplication rules, rather than the equations of classical mechanics, require modifications [16].

Almost immediately[1] Dirac reformulated Heisenberg's ideas in mathematical form, introduced quantum algebra, and noticed that in the limit $\hbar \to 0$ the Heisenberg commutators of quantum observables tend to the Poisson brackets between the corresponding observables in the classical mechanics $\hat{a}\hat{b} - \hat{b}\hat{a} \to i\hbar\{a, b\}$ [9]. In other words, non-commutative multiplication rules in the quantum theory can be regarded as a deformation of commutative multiplication of smooth functions. Nowadays there is enormous amount of papers, books and conferences devoted to deformation quantisation. In [9] Dirac stated that "The correspondence between the quantum and classical theories lies not so much in the limiting agreement when $\hbar \to 0$ as in the fact that the mathematical operations on the two theories obey in many cases the same laws" and raise the important issue of self-consistency of the quantum multiplication rules and their consistency with the equations of motion for finite value of the Plank constant $\hbar$.

Recently, AVM presented a new approach to the problem of quantisation [19]. It is suggested to commence with dynamical systems defined on a free associative algebra, i.e. with a free associative mechanics. In this theory smooth functions on a phase space (or a Poisson manifold) are replaced by elements of a free algebra, generated by the dynamical variables. Any finitely generated associative algebra, including Dirac's quantum algebra, can be regarded as a quotient of a free algebra with an equivalent number of generators over a suitable two-sided ideal. The commutation rules of a quantum theory enables one to swap positions of any two variables. In [19] by quantisation it is understood a reduction of a system defined on a free associative algebra to the dynamical system on a quotient algebra such that any two generators can be re-ordered using its multiplication rule. In order to achieve the consistency (to solve the issue raised by Dirac) the ideal (the quantisation ideal) should be invariant with respect to the derivation defined by the dynamical system. The classical commutative case corresponds to the ideal generated by the commutators of all dynamical variables. The method of quantisation proposed in [19] does not appeal to a Poisson structure of the system, and therefore it enables to define a concept of non-deformation quantisation. For example, the Volterra integrable lattice admits a deformation quantisation. Using the new method it is shown that its cubic symmetry admits two different quantisations, and one of which is non-deformation. This approach has been developed further and applied to quantisation of the Volterra hierarchy in [7]. In particular it was shown that a periodic Volterra lattice with period three admits a bi-quantum structure which is a quantum analog of the corresponding bi-Hamiltonian structure.

The aim of our paper is to apply the approach proposed in [19] to the problem of quantisation of stationary flows of the KdV hierarchy, known as the Novikov equations [5], [8], [15], [22]. Novikov discovered that the stationary flows of the KdV equation is a completely integrable dynamical system, it possess a rich family of periodic and quasiperiodic exact solutions which can be expressed in terms of Abelian functions [11],[22]. Here we would like to emphasise that we study the problem of quantisation of finite dimensional systems of ordinary differential equations and *not* of the field theory associated with partial differential equations of the KdV hierarchy.

[1]The speed of publications and Dirac's reaction was astonishing! Heisenberg's paper [16] was submitted on 29th of July, published in September. Dirac received a proof of Heisenberg's paper in August, submitted his paper containing a deep development of Heisenberg's theory on 7th of November, which was published on the 1-st of December 1925 [9].

In Section 2 we give an explicit algebraic description of N-th Novikov equation and the corresponding finite hierarchy of symmetries in the form convenient for further generalisations. The N-the Novikov equation is an ordinary differential equation of order $2N$. In Proposition 17 it is shown that a complete set of N first integrals of the N-th Novikov equations can be explicitly presented in terms of the coefficients of fractional powers $L^{\frac{2k-1}{2}}$ of the Schrödinger operator $L = D^2 - u,\ D = \frac{d}{dx}$. The KdV hierarchy defines evolutionary derivations in the graded algebra $\mathfrak{A}_0 = \mathbb{C}[u_0, u_1, u_2, \ldots]$ with the weights $|u_k| = k+2$, where $u_0 = u$, $u_{k+1} = D(u_k)$. The commuting evolutionary derivations define a representation of algebraically independent variables $u_0, u_1, u_2, \ldots$ as smooth functions $u_k = u_k(t_1, t_3, \ldots)$ of graded variables $t_{2k-1},\ k \in \mathbb{N}$ where the weight $|t_{2k-1}| = -2k+1$ and $t_1 = x$. We treat the N-th Novikov equation as a generator of a differential ideal $\mathfrak{I}_N$ in the graded ring $\mathfrak{A} = \mathcal{A}[u, u_1, \ldots]$, where $\mathcal{A}$ is a commutative algebra of graded parameters $\alpha_4, \alpha_6, \ldots, \alpha_{2N+2}$ where the weights $|\alpha_{2n}| = 2n$. The Proposition 16 shows that the KdV hierarchy induces the finite N hierarchy of integrable ordinary differential polynomial equations on the quotient ring $\mathfrak{A}\diagup\mathfrak{I}_N$ which is called N-th Novikov hierarchy. Here by integrability we understand the existence of N first integrals and N commuting symmetries, one of which is the N-th Novikov equation itself.

Let $\mathcal{B}$ be an associative $\mathbb{C}$-algebra with the unit 1 and $\mathcal{M}$ be a complex linear space. Let $\varepsilon\colon \mathcal{B} \to \mathcal{M}$ be a linear map such that $\varepsilon(1) \neq 0$. In this paper we introduce the Frobenius-Hochschild algebra $FH(\mathcal{B}, \mathcal{M})$. The name and notation are motivated by the fact that the structure of a $FH(\mathcal{B}, \mathcal{M})$-algebra $\mathcal{U}$ is given by a skew-symmetric quadratic form Φ on the $\mathcal{B}$-bimodule $\mathcal{U}$ with values in $\mathcal{M}$, and this form Φ is a 1-cocycle in the cochain Hochschild complex of the algebra $\mathcal{U}$. Partial cases of the Frobenius-Hochschild algebras are anti-Frobenius algebras. The latter was introduced and developed in connection with the associative Yang-Baxter equation, see [25]. In Section 1.2 we describe properties of the $FH(\mathcal{B}, \mathcal{M})$-algebra in the case $\mathcal{B} = \mathcal{M} = \mathfrak{A}_0$. In terms of the form $\Phi = \sigma(\cdot,\cdot)$ of this algebra, the first N-Novikov integrals are explicitly described in the case of a commutative ring of polynomials $\mathfrak{A}_0$.

The KdV equations with non-commutative matrix variables were introduced in [26], [6]. The KdV hierarchies on free associative algebras were studied in [10], [12], [23], [24], [25]. In Section 3 we give a description of the integrable KdV hierarchy on a differential graded free associative algebra $\mathfrak{B}_0 = \mathbb{C}\langle u, u_1, \ldots\rangle,\ D(u_k) = u_{k+1}$. Here, by integrability we understand the existence of an infinite hierarchy of commuting symmetries, which are generators of symmetries of the non-commutative KdV equation. There is a complete classification of integrable hierarchies of evolutionary non-commutative equations [24]. In particular, it was shown that the hierarchy of the KdV equation can be generated by a (non-local) recursion operator. In the non-commutative case in order to define local conservation laws we need to introduce a linear space of functionals with the values in the quotient linear space $\mathfrak{B}_0\diagup\big(\mathrm{Span}([\mathfrak{B}_0, \mathfrak{B}_0]) \oplus D(\mathfrak{B}_0)\big)$, see [10], [23], [24]. Formal definitions of the N-th Novikov equation and its hierarchy of symmetries are the same as in the commutative case. Namely, we take a stationary flow of a linear combination of the first N members of the KdV hierarchy with commuting constant coefficients $\alpha_{2n} \in \mathcal{A}$, as a generator of the two-sided ideal $\mathfrak{I}_N \subset \mathfrak{B} = \mathcal{A}\langle u, u_1 \ldots\rangle$. The N-Novikov hierarchy is defined as the canonical projection of the KdV hierarchy to the quotient ring $\mathfrak{B}_N = \mathfrak{B}\diagup\mathfrak{I}_N$ which is free over $\mathcal{A}$ and finitely generated. The first system of the hierarchy $\partial_{t_1} u_k = \mathcal{D}(u_k),\ k = 0, \ldots 2N-1$ is the N-th Novikov equation itself, written in the form of a first order system where $\mathcal{D}$ is the derivation of $\mathfrak{B}_N$ induced by D. In contrast to the commutative case, the hierarchy of linearly independent symmetries is infinite. The case $N = 1$ is already nontrivial. For $N = 1$ the Novikov equation coincides with the (non-commutative) Newton equation $u_2 = 3u^2 + 8\alpha_4$ and in $\mathfrak{B}_1$ is represented by the first order system

$$\partial_{t_1} u = u_1, \quad \partial_{t_1} u_1 = 3u^2 + 8\alpha_4. \tag{1}$$

Equation (1) admits an infinite hierarchy of commuting symmetries. First four of them are presented in Section 2.

A general definition of first integrals for equations on free associative algebra was discussed in [20]. First integrals for the non-commutative N-th Novikov equation and its hierarchy are introduced in Definition 25. In Section 2.2 we describe the properties of the $FH(\mathcal{B},\mathcal{M})$-algebra, where $\mathcal{B} = \mathfrak{B}_0$, and $\mathcal{M} = \mathfrak{B}_0/\mathrm{Span}([\mathfrak{B}_0, \mathfrak{B}_0])$. First integrals of the non-commutative N-th Novikov hierarchy are explicitly represented in terms of the form $\Phi = \sigma(\cdot,\cdot)$ of this algebra. Using Lemma 19, we constructed infinitely many algebraically independent first integrals for the non-commutative N-th Novikov equation and its hierarchy.

In Section 4 we consider the quantisation problem for N-th Novikov equation following the method proposed in [19]. Let $\mathcal{Q}_N$ be a commutative graded ring of parameters

$$\mathcal{Q}_N = \mathbb{C}[\alpha_{2j+2}, q_{i,j}, q^{\omega}_{i,j} \mid 0 \leqslant i < j \leqslant 2N-1,\ 0 \leqslant |\omega| < i+j+4]$$

where $|q_{i,j}| = 0,\ \omega = (i_{2N-1}, \ldots, i_1, i_0) \in \mathbb{Z}^{2N}_{\geqslant},\ |\omega| = (2N+1)i_{2N-1} + \cdots + 3i_1 + 2i_0$, $|q^{\omega}_{i,j}| = i + j + 4 - |\omega|$, and $\mathfrak{B}_N(q)$ denotes the graded free associative ring $\mathfrak{B}_N(q) = \mathcal{Q}_N\langle u_0, \ldots, u_{2N-1}\rangle$. Having N-th Novikov equation on $\mathfrak{B}_N(q)$, we introduce a differential homogeneous two-sided ideal $\mathfrak{Q}_N \subset \mathfrak{B}_N(q)$ generated by the polynomials

$$p_{i,j} = u_i u_j - q_{i,j} u_j u_i + \sum_{0 \leqslant |\omega| < i+j+4} q^{\omega}_{i,j} u^{\omega}, \quad 0 \leqslant i < j \leqslant 2N-1,\ q_{i,j} \neq 0. \tag{2}$$

where $u^{\omega} = u_{2N-1}^{i_{2N-1}} \cdots u_1^{i_1} u_0^{i_0}$ are normally ordered monomials. The ideal $\mathfrak{Q}_N$ is a *quantisation ideal* of the N-th Novikov equation if the quotient algebra $\mathfrak{C}_N = \mathfrak{B}_N(q)/\mathfrak{Q}_N$ has a Poincaré–Birkhoff–Witt additive $\mathcal{Q}_N$-basis of *normally ordered monomials* $u^{\omega},\ \omega \in \mathbb{Z}^{2N}_{\geqslant}$, and $\mathfrak{Q}_N$ is invariant with respect to the derivation $\mathcal{D}$. It follows from $\mathcal{D}(\mathfrak{Q}_N) \subset \mathfrak{Q}_N$ that the coefficients $q_{i,j}, q^{\omega}_{i,j}$ satisfy a system of algebraic equations. In particular, these equation imply that $q_{i,j} = 1$ for all $0 \leqslant i < j \leqslant 2N-1$ (Lemma 33). In the cases $N = 1, 2, 3$ and 4 we have found out that the all structure constants $q^{\omega}_{i,j}$ of the quantisation ideals $\mathfrak{Q}_N$ can be parameterised by one parameter which we denote $\hbar$. In the case $N = 1$ the computations are presented in full detail in Section 4.2. In this case we have shown that the quantisation ideal for equation (1) is generated by the commutation relation $[u_1, u] = i\hbar$, which coincides with Heisenberg's commutation relation in quantum mechanics [16], [9]. In the case $N = 2$ we have shown (Proposition 40) that the quantisation ideal $\mathfrak{Q}_2$ is generated by six commutation relations

$$[u_i, u_j] = 0 \text{ for } \ i+j < 3 \text{ or } \ i+j = 4;$$
$$[u_3, u] = [u_1, u_2] = i\hbar, \quad [u_3, u_2] = 10 i\hbar u_0,$$

The quantum $N = 2$ KdV hierarchy can be written in the Heisenberg form (Theorem 42)

$$\partial_{t_1} u_k = \mathfrak{D}(u_k) = \frac{i}{\hbar}[\mathfrak{H}_{5,3}, u_k] = \begin{cases} u_{k+1}, \ 0 \leqslant k \leqslant 2, \\ 32\alpha_6 - 16\alpha_4 u + 5u_1^2 + 10u_2 u - 10u^3, \ k = 3; \end{cases}$$
$$4\partial_{t_3} u_k = \frac{i}{\hbar}[\mathfrak{H}_{5,5}, u_k] = \mathcal{D}^{k+1}(u_2 - 3u^2).$$

Here the Hamiltonian $\mathfrak{H}_{5,3} \in \mathfrak{C}_2$ for the Novikov equation coincides with the first integral of weight 8 in the commutative case, assuming that all monomials are normally ordered, while the Hamiltonian $\mathfrak{H}_{5,5} \in \mathfrak{C}_2$ requires a quantum correction (Proposition 41). These Hamiltonians commute with each other $[\mathfrak{H}_{5,3}, \mathfrak{H}_{5,5}] = 0$. We conclude Section 4 by discussion of quantum ideals for $N = 3$ and $N = 4$ and the hierarchy of quantum KdV equations in the Heisenberg form in the case $N = 3$.

We emphasize, that the method of quantisation proposed in [19] does not assume any Hamiltonian structure of the noncomutative dynamical system, nevertheless we present the quantum equations in the Heisenberg form $\partial_t u_k = \frac{i}{\hbar}[\mathfrak{H}, u_k]$ in Section 4.

2 Novikov's equations and the corresponding finite KdV hierarchies.

2.1 Lie algebra of evolutionary differentiations.

Consider a graded commutative differential polynomial algebra

$$\mathfrak{A}_0 = (\mathbb{C}[u_0, u_1, \ldots], \ D), \tag{3}$$

where D is a derivation of $\mathbb{C}[u_0, u_1, \ldots]$ such that $D(u_k) = u_{k+1}, \ k = 0, 1, \ldots$ In terms of grading we assume that the variables u_k have weight $|u_k| = k + 2$ and operator D have weight $|D| = 1$. The variable u_0 will be often denoted as u. The derivation D can be represented in the form

$$D = X_{u_1} = \sum_{k=0}^{\infty} u_{k+1} \frac{\partial}{\partial u_k}.$$

Derivations of $\mathfrak{A}_0$ form a Lie algebra $\operatorname{Der}\mathfrak{A}_0$ over $\mathbb{C}$. A formal sum

$$X = \sum_{n=0}^{\infty} f_n \frac{\partial}{\partial u_n}, \qquad f_n \in \mathfrak{A}_0, \tag{4}$$

is a derivation in $\mathfrak{A}_0$. Its action $X : \mathfrak{A}_0 \mapsto \mathfrak{A}_0$ is well defined, since any element $a \in \mathfrak{A}_0$ depends on a finite subset of variables, and therefore the sum $X(a)$ contains only a finite number of non-vanishing terms. The $\mathbb{C}$ linearity and the Leibniz rule are obviously satisfied. For example, partial derivatives $\frac{\partial}{\partial u_i}, \ i = 0, 1, \ldots$, are commuting derivations in $\mathfrak{A}_0$.

A derivation X is said to be *evolutionary* if it commutes with the derivation D. For an evolutionary derivation it follows from the condition $XD = DX$ that all coefficients f_n in

(4) can be expressed as $f_n = D^n(f)$ in terms of one element $f \in \mathfrak{A}_0$, which is called the *characteristic* of the evolutionary derivation. We will use notation

$$X_f = \sum_{i=0}^{\infty} D^i(f) \frac{\partial}{\partial u_i} \tag{5}$$

for the evolutionary derivation corresponding to the characteristic f. The derivation D is also evolutionary $D = X_{u_1}$ with the characteristic u_1.

Evolutionary derivations form a Lie subalgebra of the Lie algebra $\operatorname{Der} \mathfrak{A}_0$. Indeed,

$$\alpha X_f + \beta X_g = X_{\alpha f + \beta g}, \quad \alpha, \beta \in \mathbb{C},$$
$$[X_f, X_g] = X_{[f,g]},$$

where $[f, g] \in \mathfrak{A}_0$ denotes the Lie bracket

$$[f, g] = X_f(g) - X_g(f), \tag{6}$$

which is bi-linear, skew-symmetric and satisfying the Jacobi identity. Thus $\mathfrak{A}_0$ is a Lie algebra with Lie bracket defined by (6).

Let $a(u, \ldots, u_n)$ be a non-constant element of $\mathfrak{A}_0$. Then $X_f(a)$ can be represented by a finite sum

$$X_f(a) = \sum_{i=0}^{n} \frac{\partial a}{\partial u_i} D^i(f) = a_*(f), \tag{7}$$

where

$$a_* = \sum_{i=0}^{n} \frac{\partial a}{\partial u_i} D^i \tag{8}$$

is the *Fréchet derivative* of $a(u, \ldots, u_n)$ and $a_*(f)$ is the Fréchet derivative of a in the direction f. Using the Fréchet derivative we can represent the Lie bracket (6) in the form

$$[f\ ,\ g] = g_*(f) - f_*(g). \tag{9}$$

An evolutionary derivation X_f we identify with the partial differential equation

$$\partial_t(u) = f, \qquad f \in \mathfrak{A}_0. \tag{10}$$

Following [25] we define symmetries of (10).

Definition 1. A dynamical system

$$\partial_\tau(u) = g, \qquad g \in \mathfrak{A}_0 \tag{11}$$

is called an infinitesimal symmetry (or just symmetry for brevity) for (10) if (10) and (11) are compatible.

It is clear that equation (11) is a symmetry of equation (10) iff $[X_f, X_g] = 0$. By a symmetry we will also call the evolutionary derivation ∂_τ which commutes with ∂_t.

2.2 Frobenius–Hochschild algebras.

We shall assume that u is a smooth function $u = u(t_1, t_3, \ldots, t_{2k-1}, \ldots)$ of graded variables t_{2k-1}, $k = 1, 2, \ldots$, where $|t_{2k-1}| = 1-2k$. The variable t_1 we will identity with x. We use abbreviated notations for partial derivatives $\frac{\partial u}{\partial t_{2k-1}} = \partial_{t_{2k-1}}(u)$ and $\partial_x = \partial_{t_1} = D$. The grading weights $|\partial_{t_{2k-1}}| = 2k-1$.

Let us define a differential operator of order m as a finite sum of the form

$$A = \sum_{i=0}^{m} a_i D^i,\ a_i \in \mathfrak{A}_0,\ a_m \neq 0$$

where $D^0 = 1$ is the identity operator.

An operator A is called *homogeneous* of weight k, if $|a_i| + i = k$ for all i. Differential operators act naturally on the algebra $\mathfrak{A}_0$.

The set of differental graded operators

$$\mathfrak{A}_0[D] = \Big\{ \sum_{i=0}^{m} a_i D^i \,|\, a_i \in \mathfrak{A}_0,\ a_m \neq 0,\ m \in \mathbb{Z}_{\geqslant 0} \Big\}$$

and the set of graded formal differential series

$$\mathfrak{A}_0^D = \mathfrak{A}_0[D][[D^{-1}]] = \Big\{ \sum_{i \leqslant m} a_i D^i \,|\, a_i \in \mathfrak{A}_0,\ a_m \neq 0,\ m \in \mathbb{Z} \Big\} \tag{12}$$

are non-commutative associative algebras. In this algebra, multiplication is defined by the composition of series using the formula

$$bD^k a\, D^l = \sum_{i \geqslant 0} \binom{k}{i} b a^{(i)} D^{k+l-i} \tag{13}$$

reflecting the Leibniz rule. Here $a^{(k)} = D^k(a) \in \mathfrak{A}_0$ and

$$\binom{k}{0} = 1, \quad \binom{k}{i} = \frac{k(k-1)\cdots(k-i+1)}{i!} = (-1)^i \binom{-k+i-1}{i},\ i > 0. \tag{14}$$

Obviously $\mathfrak{A}_0[D] \subset \mathfrak{A}_0^D$ and $\mathfrak{A}_0[D]$ is a subalgebra in $\mathfrak{A}_0^D$.

For example,

$$D^k a = \sum_{i=0}^{k} \binom{k}{i} a^{(i)} D^{k-i},\ k \geqslant 0;$$
$$D^{-1} a = \sum_{i \geqslant 0} (-1)^i a^{(i)} D^{-(i+1)} = aD^{-1} - D(a)D^{-2} + D^2(a)D^{-3} - D^3(a)D^{-4} + \cdots$$

For any two elements $A, B \in \mathfrak{A}_0^D$ we have the commutator $[A, B] = AB - BA$. For instance, for any $a \in \mathfrak{A}_0$, the formulas are fulfilled:
$[D, a] = D(a);\ \ [D^{-1}, a] = -D(a)D^{-2} + D^2(a)D^{-3} - \cdots$

Definition 2. For a formal series $A \in \mathfrak{A}_0^D$ the coefficient a_{-1} of the term $a_{-1}D^{-1}$ is called the residue of this series A and denoted by $\operatorname{res} A$.

Lemma 3.

1) For any $B \in \mathfrak{A}_0^D$ and $a \in \mathfrak{A}_0$ we have $\operatorname{res}\,[a, B] = 0$.

2) For any $a \in \mathfrak{A}_0$ and $B, C \in \mathfrak{A}_0^D$

$$\operatorname{res}\,[aB, C] = \operatorname{res}\,[B, Ca]. \tag{15}$$

Proof. 1) Let $B = \sum_{k \leqslant m} b_k D^k$. Then

$$[a, B] = -\sum_{k \leqslant m} \sum_{i > 0} \binom{k}{i} b_k a^{(i)} D^{k-i}.$$

Therefore

$$\operatorname{res}\,[a, B] = \binom{k}{k+1} b_k a^{(k+1)} = 0, \quad k + 1 > 0.$$

2) For any elements A, B, C of any associative algebra, the identity

$$[A, BC] + [B, CA] + [C, AB] = 0 \tag{16}$$

holds. Therefore, for $a \in \mathfrak{A}_0$ and $B, C \in \mathfrak{A}_0^D$

$$[aB, C] = [B, Ca] + [a, BC]. \tag{17}$$

Applying the operator "res" to (17) and using already proved statement 1), we obtain the proof of statement 2). ■

Let $\mathcal{B}$ be some associative $\mathbb{C}$-algebra with the unit 1 and $\mathcal{M}$ some complex linear space. Let a linear mapping $\varepsilon\colon \mathcal{B} \to \mathcal{M}$ be given such that $\varepsilon(1) \neq 0$.

Definition 4. An associative $\mathbb{C}$-algebra $\mathcal{U}$ with unit 1 will be called a Frobenius–Hochschild algebra over $(\mathcal{B}, \mathcal{M})$ (briefly $FH(\mathcal{B}, \mathcal{M})$-algebra) if:

i) The algebra $\mathcal{B}$ is a subalgebra of $\mathcal{U}$, and hence $\mathcal{U}$ is a two-sided $\mathcal{B}$-module.

ii) The bilinear mapping $\Phi(\cdot, \cdot)\colon \mathcal{U} \otimes_{\mathbb{C}} \mathcal{U} \to \mathcal{M}$ is defined such that:
1) for any $A \in \mathcal{U}$ and $b \in \mathcal{B}$ we have $\Phi(A, b) = 0$;
2) for any $A, B, C \in \mathcal{U}$ the relation

$$\Phi(A, BC) + \Phi(B, CA) + \Phi(C, AB) = 0 \tag{18}$$

is satisfied.

Lemma 5. Let $\mathcal{U}$ be some $FH(\mathcal{B}, \mathcal{M})$-algebra. Then the bilinear mapping
$\Phi(\cdot, \cdot)\colon \mathcal{U} \otimes_{\mathbb{C}} \mathcal{U} \to \mathcal{M}$

a) is skew-symmetric, i.e. for any $A, B \in \mathcal{U}$ the equality $\Phi(A, B) = -\Phi(B, A)$ is true;

b) defines a bilinear mapping $\mathcal{U} \otimes_{\mathcal{B}} \mathcal{U} \to \mathcal{M}$, i.e., for any $A, B \in \mathcal{U}$ and $a \in \mathcal{B}$, the equality $\Phi(aA, B) = \Phi(A, Ba)$ is true.

Proof. Let us substitute $C = 1$ in (18). Then, according to item 1) of the Definition 4, we obtain a proof of assertion a). If we substitute $C = a$ in (18) then, according to item 1) of the Definition 4, obtain a proof of assertion b). ■

Theorem 6. The algebra $\mathfrak{A}_0^D$ is a $FH(\mathfrak{A}_0, \mathfrak{A}_0)$-algebra in which the bilinear form $\Phi = \sigma\colon \mathfrak{A}_0^D \otimes_{\mathfrak{A}_0} \mathfrak{A}_0^D \to \mathfrak{A}_0$ is uniquely given by the formula

$$\sigma(D^n, bD^m) = \begin{cases} \binom{n}{n+m+1} b^{(n+m)}, & \text{if } n+m \geqslant 0,\, nm < 0, \\ 0, & \textit{otherwise}. \end{cases} \tag{19}$$

Proof. Let $A = \sum\limits_{k \leqslant m} a_k D^k$. Then $aA \in \mathfrak{A}_0^D$ for any $a \in \mathfrak{A}_0$ and therefore the algebra $\mathfrak{A}_0^D$ is a left $\mathfrak{A}_0$-module with respect to the embedding $\varepsilon\colon \mathfrak{A}_0 \to \mathfrak{A}_0^D : a \to aD^0$. According to (13), the structure of the right $\mathfrak{A}_0$-module is given by the formula

$$Aa = \sum_{j \leqslant m} \left(\sum_{i=0}^{m-j} \binom{j+i}{i} a_{j+i} a^{(i)} \right) D^j.$$

According to (19), we obtain $\sigma(\varepsilon(a), A) = \sigma(aD^0, A) = 0$. Thus item 1) of condition ii) of the Definition 4 has been verified.

The proof of item 2) of condition ii) is based on two lemmas, which are of independent interest:

Lemma 7. For any $A, B \in \mathfrak{A}_0^D$

$$D\big(\sigma(A, B)\big) = \operatorname{res}[A, B]. \tag{20}$$

Proof. Forms $D\big(\sigma(\cdot,\cdot)\big)$ and $\operatorname{res}[\cdot,\cdot]$ are bilinear so it suffices to proof the relation:

$$D\big(\sigma(aD^n, bD^m)\big) = \operatorname{res}[aD^n, bD^m],\ a, b \in \mathfrak{A}_0,\ n, m \in \mathbb{Z}. \tag{21}$$

According to the condition of Theorem 6, for any $a, b \in \mathfrak{A}_0$ we have

$$\sigma(aD^n, bD^m) = \sigma(D^n, bD^m a).$$

But according to item 2) of Lemma 3

$$\operatorname{res}[aD^n, bD^m] = \operatorname{res}[D^n, bD^m a].$$

Therefore, it suffices to proof the relation (21) in the case $a = 1$. But in this case we have:

$$\operatorname{res}[D^n, bD^m] = \binom{n}{n+m+1} b^{(n+m+1)} = D\big(\sigma(D^n, bD^m)\big).$$

Lemma 7 is proved. ■

The monomials

$$u^{\xi} = u_n^{i_n} \cdots u_0^{i_0},\ \xi = (i_n, \ldots, i_0),\ i_n > 0,\ i_k \geqslant 0,\ k = 0, \ldots, n-1,\ |u^{\xi}| = \sum_{k=0}^{n} (k+2) i_k,$$

form an additive basis of the graded algebra $\mathfrak{A}_0 = \mathbb{C}[u_0, u_1, \ldots]$. We will consider $\mathfrak{A}_0$ as a graded algebra $\mathfrak{A}_0 = \mathbb{C} \oplus \widetilde{\mathfrak{A}}_0$, where $\widetilde{\mathfrak{A}}_0 = \oplus_m \mathfrak{A}_0^m$ and $\mathfrak{A}_0^m$ is a graded finite-dimensional $\mathbb{C}$-linear subspace in $\mathfrak{A}_0$ with an additive basis $\{u^{\xi},\ |u^{\xi}| = m\}$.

For example, $\{u\}$, $\{u_1\}$, and $\{u_2, u^2\}$ are the basises of the spaces $\mathfrak{A}_0^2$, $\mathfrak{A}_0^3$, and $\mathfrak{A}_0^4$, respectively.

The vectors of the space $\mathfrak{A}_0^m$ are called homogeneous polynomials of weight m. Let us introduce an ordering of the multiplicative generators of the algebra $\mathfrak{A}_0$:

$$u = u_0 < u_1 < \cdots < u_k < u_{k+1} < \cdots$$

Then a strict order is defined in the monomial basis $\{u^{\xi}\}$ of the space $\mathfrak{A}_0^m$ for each $m > 0$. This order is induced by the lexicographic order of the sequences ξ.

Lemma 8. For any $m > 0$ the homomorphism $D\colon \mathfrak{A}_0 \to \mathfrak{A}_0$ defines a monomorphism $\mathfrak{A}_0^m \to \mathfrak{A}_0^{m+1}$.

Proof. By definition, $D(1) = 0$ and $D(u_k) = u_{k+1}$, $k = 0, 1, \ldots$ Therefore, the differentiation operator D takes $\mathfrak{A}_0^m$ to $\mathfrak{A}_0^{m+1}$. Let $u^{\xi} \in \mathfrak{A}_0^m$ where $\xi = (i_n, \ldots, i_0)$, $i_n \neq 0$. Then $D(u^{\xi}) = i_n u_{n+1} u_n^{i_n - 1} u^{\hat{\xi}} + u_n^{i_n} D(u^{\hat{\xi}})$. The composition of linear homomorphisms

$$\overline{D}\colon \mathfrak{A}_0^m \to \mathfrak{A}_0^{m+1} \to \mathfrak{A}_0^{m+1} \ :\ \overline{D}(u^{\xi}) = u^{\xi'} = i_n u_{n+1} u_n^{i_n - 1} u^{\hat{\xi}}$$

maps the ordered set of monomials $u^{\xi} \in \mathfrak{A}_0^m$ into the ordered set of monomials $u^{\xi'} \in \mathfrak{A}_0^{m+1}$. We will show that this mapping is monotone and thus we obtain that the homomorphism D is a monomorphism for $m > 0$.

Let $\xi_1 = (i_{n_1}, \ldots, i_{0_1}) > \xi_2 = (i_{n_2}, \ldots, i_{0_2})$ where $i_{n_1} \neq 0$ and $i_{n_2} \neq 0$. Then $n_1 \geqslant n_2$. If $n_1 > n_2$ then $\xi_1' > \xi_2'$. If $n_1 = n_2$ and $i_{n_1} > i_{n_2}$ then in this case it is also obvious that $\xi_1' > \xi_2'$. Finally, let $n_1 = n_2 = n$ and $i_{n_1} = i_{n_2} = i_n$. Then there is a sequence $\zeta = (i_n, \ldots, i_k)$, $0 < k \leqslant n$, such that $\xi_1 = (\zeta, \eta_1)$ and $\xi_2 = (\zeta, \eta_2)$ where $\eta_1 > \eta_2$. In this case $u^{\xi_1'} = \overline{D}(u^{\xi_1}) = \overline{D}(u^{\zeta}) u^{\eta_1}$ and $u^{\xi_2'} = \overline{D}(u^{\xi_2}) = \overline{D}(u^{\zeta}) u^{\eta_2}$ and therefore $u^{\xi_1'} > u^{\xi_2'}$. Lemma 8 is proved. ∎

We now continue the proof of Theorem 6. It remains to prove that item 2) of condition ii) of Definition 4 is satisfied, i.e. that relation (18) is true.

Let $A, B, C \in \mathfrak{A}_0^D$. Take the residue res of the left side of equality (16). Then, according to Lemma 7, we obtain:

$$D\big(\sigma(A, BC) + \sigma(B, CA) + \sigma(C, AB)\big) = 0.$$

Since according to Lemma 8 the operator D is the monomorphism on non-constant series, we obtain that relation (18) is true. Theorem 6 is proved. ∎

Corollary 9.

1) For any $a, b \in \mathfrak{A}_0$ we have

$$\sigma(aD^n, bD^m) = \begin{cases} \binom{n}{n+m+1} \sum\limits_{s=0}^{n+m} (-1)^s a^{(s)} b^{(n+m-s)}, & \text{if } n+m \geqslant 0,\ nm < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

2) For any $A \in \mathfrak{A}_0^D$ we have

$$\sigma(D, A) = \operatorname{res} A. \tag{23}$$

3) For any $A, B \in \mathfrak{A}_0^D$ we have

$$\sigma\big(D, [A, B]\big) = D\big(\sigma(A, B)\big). \tag{24}$$

Proof. Assertion 1) follows from Lemma 5 and formulas (19) and (13). Assertion 2) follows from formula (22). Assertion 3) follows from formula (23) and Lemma 7. ∎

For $A = \sum\limits_{i \leqslant m} a_i D^i$, $a_m \neq 0$, we put $A = A_+ + A_-$ where $A_+ = 0$ if $m < 0$, and $A_+ = \sum\limits_{i=0}^{m} a_i D^i$ if $m \geqslant 0$.

Corollary 10.

1) $\sigma(A, B) = \sigma(A_+, B_-) + \sigma(A_-, B_+)$.

2) Let $[A, B] = 0$. Then $\sigma(A, B) = 0$.

Proof. Assertion 1) follows from formula (22). Assertion 2) follows from item 3) of Corollary 9 and Lemma 8. ∎

Theorem 11. The form $\sigma(\cdot, \cdot)$ is given in terms of the operation res by the recursive formula

$$\sigma(A, BD) = \sigma(DA, B) - \operatorname{res} AB \tag{25}$$

with the initial condition $\sigma(A, bD) = -\operatorname{res} Ab$ for any $A \in \mathfrak{A}_0^D$ and $b \in \mathfrak{A}_0$.

Proof. As was noted above, $\operatorname{res} A = \sigma(D, A)$. For the triple (A, B, D), according to identity (18), we obtain formula (25). ∎

For example:

$$\sigma(A, bD^2) = \sigma(DA, bD) - \operatorname{res} AbD = -\operatorname{res}(DAb + AbD)$$

for any $A \in \mathfrak{A}_0^D$ and $b \in \mathfrak{A}_0$.

2.3 KdV hierarchy.

Let us consider a homogeneous operator $L = D^2 - u$, $|L| = 2$.

Lemma 12. A homogeneous formal series

$$\mathcal{L} = D + \sum_{n\geqslant 1} I_{1,n} D^{-n},\ |\mathcal{L}| = 1,$$

where $I_{1,n} \in \mathfrak{A}_0$ are homogeneous polynomials of the weight $n+1$, satisfies the equation $\mathcal{L}^2 = L$ if and only if $I_{1,1} = -\frac{1}{2}u,\ I_{1,2} = \frac{1}{4}u_1$ and

$$2I_{1,n} + I'_{1,n-1} + \sum_{k=1}^{n-2} I_{1,k} \sum_{i=0}^{n-k-2} (-1)^i \binom{k+i-1}{i} I^{(i)}_{1,n-k-i-1} = 0,\ n \geqslant 3. \tag{26}$$

Proof. Consider the equation

$$\left(D + \sum_{k\geqslant 1} I_{1,k} D^{-k}\right)\left(D + \sum_{q\geqslant 1} I_{1,q} D^{-q}\right) = D^2 - u.$$

We obtain

$$\sum_{q\geqslant 1} D I_{1,q} D^{-q} + \sum_{k\geqslant 1} I_{1,k} D^{-k+1} + \sum_{k\geqslant 1, q\geqslant 1} I_{1,k} D^{-k} I_{1,q} D^{-q} = -u.$$

Using (13), we get (26). ■

Formula (26) allows to calculate the polynomials $I_{1,n}$, $n \geqslant 3$, recursively

$$\mathcal{L} = D - \frac{1}{2}uD^{-1} + \frac{1}{4}u_1 D^{-2} - \frac{1}{8}(u_2 + u^2)D^{-3} + \frac{1}{16}(u_3 + 6uu_1)D^{-4} - \\ - \frac{1}{32}(u_4 + 14u_2 u + 11u_1^2 + 2u^3)D^{-5} + \dots$$

Let us define a sequence of differential operators

$$A_{2k-1} = \mathcal{L}_+^{2k-1} = D^{2k-1} - \frac{1}{2}(2k-1)uD^{2k-3} + \cdots + a_{2k-1},\ k = 1, 2, \dots, \tag{27}$$

where $a_{2k-1} = A_{2k-1}(1) \in \mathfrak{A}_0$, $|a_{2k-1}| = 2k-1$, and homogeneous differential polynomials $\rho_{2k} \in \mathfrak{A}_0$, $|\rho_{2k}| = 2k$,

$$\rho_0 = 1, \quad \rho_{2k} = \operatorname{res} \mathcal{L}^{2k-1},\ k = 1, 2, \dots \tag{28}$$

Thus

$$\mathcal{L}^{2k-1} = A_{2k-1} + \rho_{2k} D^{-1} + \dots,\ k > 0.$$

We have $a_1 = 0$ and $\rho_2 = -\frac{1}{2}u$.

Let be

$$\mathcal{L}^{2k-1} = A_{2k-1} + \sum_{n\geqslant 1} I_{2k-1,n}D^{-n},\ k>0.$$

From the relation $\mathcal{L}^{2k+1} = L\mathcal{L}^{2k-1}$ we obtain

$$A_{2k+1} = (D^2-u)A_{2k-1} + I_{2k-1,1}D + (I_{2k-1,2} + 2I'_{2k-1,1}), \tag{29}$$

$$I_{2k+1,n} = I_{2k-1,n+2} + 2I'_{2k-1,n+1} + I''_{2k-1,n} - uI_{2k-1,n}. \tag{30}$$

Corollary 13.

$$a_{2k+1} = (D^2-u)(a_{2k-1}) + I_{2k-1,2} + 2\rho'_{2k},\ k\geqslant 1, \tag{31}$$

$$\rho_{2k+2} = I_{2k-1,3} + 2I'_{2k-1,2} + \rho''_{2k} - u\rho_{2k}. \tag{32}$$

Let $J = \langle u, u_1, \ldots\rangle \subset \mathfrak{A}_0$ be the two-sided maximal ideal generated by $u, u_1, \ldots$

Proposition 14. For $k\in\mathbb{N}$ the following formula holds:

$$\rho_{2k+2} = -\frac{1}{2^{2k+1}}\left(u_{2k} + \ldots + (-1)^k\binom{2k+1}{k}u^{k+1}\right) = -\frac{1}{2^{2k+1}}(u_{2k} - \widehat{\rho}_{2k+2}) \tag{33}$$

where $\widehat{\rho}_{2k+2}\in J^2$.

Proof. By definition, $\rho_{2n} = \operatorname{res} L^{\frac{2n-1}{2}}$, $n\geqslant 1$. Since $[D,u]=u_1$, then to calculate the coefficient at u^n, it is sufficient to calculate the coefficient at x^{-1} of the series $f(x) = (x^2-a)^{\frac{2n-1}{2}}$ where $[x,a]=0$. We have

$$f(x) = x^{2n-1}\left(1-x^{-2}a\right)^{\frac{2n-1}{2}} = x^{2n-1}\left(1+\sum_{k\geqslant 1}(-1)^k\binom{\frac{2n-1}{2}}{k}x^{-2k}a^k\right).$$

Therefore, the desired coefficient is

$$(-1)^n\binom{\frac{2n-1}{2}}{n}a^n = (-1)^n\frac{1}{2^{2n-1}}\binom{2n-1}{n}a^n.$$

Formula (26) implies that

$$I_{1,n} = (-1)^n\frac{1}{2^n}u_{n-1} \mod J^2,\ n>0.$$

Using formulas (30) and (32), we obtain by induction that

$$\rho_{2k+2} = -\frac{1}{2^{2k+1}}u_{2k} \mod J^2,\ k\geqslant 0.$$

■

Examples:

$$\rho_4 = -\frac{1}{8}(u_2 - 3u^2), \qquad \rho_6 = -\frac{1}{32}(u_4 - 10u_2u - 5u_1^2 + 10u^3),$$
$$\rho_8 = -\frac{1}{128}(u_6 - 28u_3u_1 - 14u_4u - 21u_2^2 + 70u_2u^2 + 70u_1^2u - 35u^4).$$

It is easy to show that $[\mathcal{L}^{2k-1}, L] = 0$ and therefore the commutator

$$[A_{2k-1}, L] = [\mathcal{L}^{2k-1} - (\mathcal{L}^{2k-1})_-, L] = 2D(\rho_{2k}) \in \mathfrak{A}_0,$$

is the operator of multiplication on the function $2D(\rho_{2k})$.

Definition 15. The KdV hierarchy is defined as an infinite sequence of differential equations

$$\partial_{t_{2k-1}}(u) = -2D(\rho_{2k}),\ k \in \mathbb{N}. \tag{34}$$

Examples:

$$\begin{aligned} \partial_{t_1}(u) &= u_1, \\ 4\partial_{t_3}(u) &= u_3 - 6uu_1, \\ 16\partial_{t_5}(u) &= u_5 - 10uu_3 - 20u_1u_2 + 30u^2u_1, \end{aligned}$$

and so on.

The partial derivatives $\partial_{t_{2k-1}}$ can be extended to derivations of the algebra $\mathfrak{A}_0^D$

$$\partial_{t_{2k-1}}(A) = \sum_{i\leqslant m} \partial_{t_{2k-1}}(a_i)D^i, \quad \text{where } A = \sum_{i\leqslant m} a_i D^i.$$

Therefore the KdV hierarchy can be written in the form of Lax's equations

$$\partial_{t_{2k-1}}(L) = [A_{2k-1}, L]. \tag{35}$$

It can be shown, that the derivations $\partial_{t_{2k-1}}$ commute with each other [25], and thus the KdV hierarchy is a system of compatible equations.

It follows from $\partial_{t_{2k-1}}(L) = \partial_{t_{2k-1}}(\mathcal{L}^2) = \partial_{t_{2k-1}}(\mathcal{L})\mathcal{L} + \mathcal{L}\partial_{t_{2k-1}}(\mathcal{L})$ and (35) that

$$\partial_{t_{2k-1}}(\mathcal{L}) = [A_{2k-1}, \mathcal{L}],$$

and therefore,

$$\partial_{t_{2k-1}}(\mathcal{L}^{2n-1}) = [A_{2k-1}, \mathcal{L}^{2n-1}],\ n, k \in \mathbb{N}. \tag{36}$$

Let's put

$$\sigma_{2k-1,2n-1} = \sigma(\mathcal{L}_+^{2k-1}, \mathcal{L}_-^{2n-1}) \in \mathfrak{A}_0. \tag{37}$$

According to Corollary 10, we obtain:

$$\sigma_{2k-1,2n-1} = \sigma_{2n-1,2k-1}. \tag{38}$$

Taking the residue from the equation 36, we get

$$\partial_{t_{2k-1}}(\rho_{2n}) = D(\sigma_{2k-1,2n-1}). \tag{39}$$

Thus $\{\rho_{2n},\ n \in \mathbb{N}\}$ is a sequence of common conserved densities for the infinite KdV hierarchy (34), and $\sigma_{2k-1,2n-1}$ are homogeneous differential polynomials, $|\sigma_{2k-1,2n-1}| = 2n + 2k - 2$.

On the algebra $\mathfrak{A}_0$ the evolutionary derivations $\partial_{t_{2k-1}}$ are represented by commuting derivations

$$D_{2k-1} = -2\sum_{\ell=0}^{\infty} D^{\ell+1}(\rho_{2k})\frac{\partial}{\partial u_\ell}. \tag{40}$$

In particular $D_1 = D$,

$$D_3 = \frac{1}{4}(u_3 - 6uu_1)\frac{\partial}{\partial u} + \frac{1}{4}(u_4 - 6uu_2 - 6u_1^2)\frac{\partial}{\partial u_1} + \cdots$$

2.4 The N-th Novikov hierarchy.

Let us choose a positive integer N. Let $\mathcal{A} = \mathbb{C}[\alpha_4, \ldots, \alpha_{2N+2}]$ be a graded algebra of parameters and $\mathfrak{A} = \mathcal{A}[u_0, u_1, \ldots]$. We assume that $|\alpha_{2n}| = 2n$, $n \geqslant 2$, and α_{2n} are constants, meaning that $D_{2k-1}(\alpha_{2n}) = 0$ for all $k \geqslant 1$ and $n \geqslant 2$.

Let us define a symmetry ∂_τ of the KdV equation taking a linear combination with constant coefficients of the first N members of the KdV hierarchy (34)

$$\partial_\tau(u) = \partial_{t_{2N+1}}(u) + \sum_{k=1}^{N-1} \alpha_{2(N-k+1)}\partial_{t_{2k-1}}(u). \tag{41}$$

Let us define a polynomial

$$F_{2N+2} = \rho_{2N+2} + \sum_{k=0}^{N-1} \alpha_{2(N-k+1)}\rho_{2k}. \tag{42}$$

In (42) we assume that $\rho_0 = 1$ and α_{2N+2} is a constant parameter of weight $|\alpha_{2N+2}| = 2N + 2$. The polynomial F_{2N+2} (see (42)) is homogeneous of weight $2N + 2$. Let us restrict ourselves with solutions of the KdV hierarchy which are invariant with respect to the symmetry (41). It implies that

$$\rho_{2N+2} + \sum_{k=0}^{N-1} \alpha_{2(N-k+1)}\rho_{2k} = 0. \tag{43}$$

It follows from (33) that equation (43) can be resolved with respect to the variable u_{2N} and written in the form

$$u_{2N} = f_{2N+2}(u_0, u_1, \ldots, u_{2N-2}) \tag{44}$$

where $f_{2N+2} = \widehat{\rho}_{2N+2} + 2^{2N+1}\sum_{k=0}^{N-1}\alpha_{2(N-k+1)}\rho_{2k} \in \mathfrak{A}$ is a homogeneous polynomial, $|f_{2N+2}| = 2N+2$. Equation (44) is called N-th Novikov equation. Since $\rho_{2n} \in \mathfrak{A}_0$, these equations depend linearly on $\alpha_4, \dots, \alpha_{2N+2}$.

For example:

$$
\begin{aligned}
N=1: \quad u_2 &= 3u^2 + 8\alpha_4,\\
N=2: \quad u_4 &= 10(u_2 - u^2)u + 5u_1^2 - 16\alpha_4 u + 32\alpha_6,\\
N=3: \quad u_6 &= 14(u_4 - 5u_2 u + 5u_1^2)u + 28u_1u_3 + 21u_2^2 + 35u^4 - 16\alpha_4(u_2 - 3u^2)\\
&- 64\alpha_6 u + 128\alpha_8.
\end{aligned}
$$

Since $u_k = D^k(u) = u^{(k)}$, the N-th Novikov equation is an ordinary differential equation of the $2N$-th order for the function $u = u(x)$.

Let $\mathfrak{I}_N = (F_{2N+2}) \subset \mathfrak{A}$ be a differential ideal generated by the polynomial F_{2N+2} and the D derivatives. For any element of $\mathfrak{A}$ the canonical projection

$$\pi_N \;:\; \mathfrak{A} \mapsto \mathfrak{A}/\mathfrak{I}_N$$

is the result of the elimination of variables $u_k,\ k \geqslant 2N$, using equation (44) and equation $u_{2N+k} = D^k(f_{2N+2})$ recursively.

Proposition 16. The ideal $\mathfrak{I}_N$ is invariant with respect to evolutionary derivations $\partial_{t_{2k-1}}$, $k \in \mathbb{N}$.

Proof. Indeed, it follows from (7), (38), and (40) that

$$
\begin{aligned}
\partial_{t_{2k-1}}(F_{2N+2}) &= \partial_{t_{2k-1}}\Big(\rho_{2N+2} + \sum_{\ell=0}^{N-1}\alpha_{2(N-\ell+1)}\rho_{2\ell}\Big) =\\
&= \Big(\partial_{2N+1} + \sum_{\ell=1}^{N-1}\alpha_{2(N-\ell+1)}\partial_{2\ell-1}\Big)(\rho_{2k}) = -2(\rho_{2k})_*(D(F_{2N+2})) \subset \mathfrak{I}_N. \quad (45)
\end{aligned}
$$

■

Commuting derivations D_{2k-1}, $1 \leqslant k \leqslant N$ (see (40)) induce on $\mathfrak{A}/\mathfrak{I}_N$ the derivations

$$
\begin{aligned}
\mathcal{D} = \mathcal{D}_1 &= \sum_{\ell=0}^{2N-2} u_{\ell+1}\frac{\partial}{\partial u_\ell} + f_{2N}\frac{\partial}{\partial u_{2N-1}},\\
\mathcal{D}_{2k-1} &= -\,2\sum_{\ell=0}^{2N-1}\mathcal{D}^{\ell+1}(\rho_{2k})\frac{\partial}{\partial u_\ell}, \quad 1 \leqslant k \leqslant N.
\end{aligned}
$$

In $\mathbb{C}^{2N}$ there are N compatible systems of N ordinary differential equations

$$\partial_{t_{2k-1}}(u_s) = \mathcal{D}_{2k-1}(u_s) = -2\mathcal{D}^{s+1}(\rho_{2k}), \quad s = 0, \dots, N-1,\ k = 1, \dots, N, \qquad (46)$$

which we will call N-th Novikov hierarchy. In this case, the parameters α_{2k} are assumed to be fixed complex numbers. In the hierarchy (46), system with $k=1$, $s = 0, \dots, N-1$ represents the N-th Novikov equation (44) as a first order system of $2N$ ordinary differential equations.

Proposition 17. The N-th Novikov equation possesses N first integrals

$$H_{2n+1,2N+1} = \sigma_{2n+1,2N+1} + \sum_{k=1}^{N-1} \alpha_{2N-2k+2}\sigma_{2n+1,2k-1}, \quad n = 1, \dots, N. \tag{47}$$

The polynomials $H_{2n+1,2N+1}$ are homogeneous of weight $|H_{2n+1,2N+1}| = 2N + 2n + 2$.

Proof. It follows from (38) that

$$\partial_{t_{2n-1}}(F_{2N+2}) = D(H_{2n+1,2N+1})$$

where

$$H_{2n+1,2N+1} = \sigma_{2n+1,2N+1} + \sum_{k=1}^{N-1} \alpha_{2N-2k+2}\sigma_{2n+1,2k-1}. \tag{48}$$

Thus, it follows from (45) that $D(H_{2n+1,2N+1}) = \partial_{t_{2n+1}}(F_{2N+2}) \in \mathfrak{I}_N$ and thus vanish in $\mathfrak{A}/\mathfrak{I}_N$. For $n = 1, \dots, N$ one can check that $H_{2n+1,2N+1} \notin \mathfrak{I}_N$, and thus $H_{2n+1,2N+1}$ is a first integral of Novikov's equation. Moreover the first integrals $H_{2n+1,2N+1},\ n = 1, \dots, N$, are algebraically independent. ■

It is known that N-th Novikov equation and equations of the N-th Novikov hierarchy can be reduced to integrable Hamiltonian systems [5]. It follows from the general theory of integrable Hamiltonian systems that the derivations $\mathcal{D}_{t_{2s-1}}$ with $s > N$ in $\mathfrak{A}/\mathfrak{I}_N$ are dependent, since they are linear combinations of $\mathcal{D}_{t_{2k-1}},\ k = 1, \dots, N$, i.e.

$$\mathcal{D}_{t_{2s-1}} = \sum_{k=1}^{N} a_k \mathcal{D}_{t_{2k-1}} \tag{49}$$

with coefficients $a_k \in \mathbb{C}[\alpha_4, \dots, \alpha_{2N+2}, H_{2N+1,3}, \dots, H_{2N+1,2N+1}]$ where $H_{2n+1,2N+1},\ n = 1, \dots, N$, are first integrals of the N-th Novikov equation. Using (47) one can find the polynomials $H_{2n+1,2N+1},\ n > N$, which are also first integrals (it follows from the proof of Proposition 17), but they are algebraically dependent with $H_{3,2N+1}, \dots, H_{2N+1,2N+1}$.

For example:

$\mathbf{N=1}$: The $N = 1$ Novikov equation coincides with the Newton equation $\partial_{t_1}^2 u = 3u^2 + 8\alpha_4$.

$$\begin{aligned} \partial_{t_1}(u) &= u_1; \\ \partial_{t_1}(u_1) &= 3u^2 + 8\alpha_4, \end{aligned}$$

and according Proposition 17 we get one first integral

$$H_{3,3} = -\frac{3}{16}\left(\frac{1}{2}u_1^2 - u^3 - 8\alpha_4 u\right). \tag{50}$$

$\mathbf{N=2}$: The hierarchy consists of two compatible systems in which the first one is the $N = 2$ Novikov equation

$$\begin{aligned} \partial_{t_1}(u_s) &= u_{s+1}, \quad s = 0, 1, 2; \\ \partial_{t_1}(u_3) &= 32\alpha_6 - 10u^3 - 16\alpha_4 u + 10u_2 u + 5u_1^2; \\ 4\partial_{t_3}(u_s) &= \mathcal{D}^{s+1}(u_2 - 3u^2), \quad s = 0, 1, 2, 3\,. \end{aligned}$$

Proposition 17 give us two first integrals

$$
\begin{aligned}
H_{3,5} &= -\frac{3}{128}\left(5u^4+16\alpha_4u^2-64\alpha_6u-10u_1^2u-u_2^2+2u_3u_1\right), && (51)\\
H_{5,5} &= \frac{5}{512}(24u^5+64\alpha_4u^3-20u_2u^3-192\alpha_6u^2-30u_1^2u^2-32\alpha_4u_2u+ && (52)\\
&+ 16\alpha_4u_1^2+64\alpha_6u_2+4u_2^2u+12u_3u_1u-u_3^2-2u_2u_1^2).
\end{aligned}
$$

Obviously $H_{2n+1,2N+1}$ (see (47) in case $N=2$) are first integrals for any n, but they are algebraically dependent with $H_{3,5}, H_{5,5}$

$$
\begin{aligned}
H_{7,5} &= \frac{7}{6}(3\alpha_6^2-2\alpha_4H_{3,5}),\\
H_{9,5} &= -3\alpha_6H_{3,5}-\frac{9\alpha_4}{5}H_{5,5},\\
H_{11,5} &= \frac{11}{90}\Big(-45\alpha_4\alpha_6^2-18\alpha_6H_{5,5}+30\alpha_4^2H_{3,5}+5H_{3,5}^2\Big),
\end{aligned}
$$

and so on.

N = 3: The hierarchy consists of three compatible systems. The first one is the $N=3$ Novikov equation, the rest are its commuting symmetries:

$$
\begin{aligned}
\partial_{t_1}(u_s) &= u_{s+1}, && s=0,\dots,4;\\
\partial_{t_1}(u_5) &= 128\alpha_8+35u^4+48\alpha_4u^2-70u_2u^2-64\alpha_6u\\
&- 16\alpha_4u_2-70u_1^2u+14u_4u+21u_2^2+28u_1u_3;\\
4\partial_{t_3}(u_s) &= \mathcal{D}^{s+1}(u_2-3u^2), && s=0,\dots,5;\\
16\partial_{t_5}(u_s) &= \mathcal{D}^{s+1}(u_4-5u_1^2-10uu_2+10u^3+16\alpha_4u), && s=0,\dots,5.
\end{aligned}
$$

It follows from Proposition 17 that there are three common first integrals of this systems

$$
\begin{aligned}
H_{3,7} = \frac{3}{2^9}\big(&14u^5+32\alpha_4u^3-64\alpha_6u^2-70u_1^2u^2+256\alpha_8u-16\alpha_4u_1^2-14u_2^2u+\\
&+28u_3u_1u-u_3^2+28u_2u_1^2+2u_4u_2-2u_5u_1\big); \quad (53)
\end{aligned}
$$

$$
\begin{aligned}
H_{5,7} = -\frac{5}{2^{11}}\big(&70u^6+144\alpha_4u^4-70u_2u^4-256\alpha_6u^3-280u_1^2u^3-96\alpha_4u_2u^2+\\
&+768\alpha_8u^2-14u_2^2u^2+168u_3u_1u^2+128\alpha_6u_2u+16\alpha_4u_2^2-64\alpha_6u_1^2-\\
&-256\alpha_8u_2-20u_3^2u+140u_2u_1^2u+12u_4u_2u-12u_5u_1u-35u_1^4-2u_2^3-\\
&-u_4^2-36u_3u_2u_1+12u_4u_1^2+2u_5u_3\big); \quad (54)
\end{aligned}
$$

$$
\begin{aligned}
H_{7,7} = \frac{7}{2^{13}}\big(&300u^7+576\alpha_4u^5-700u_2u^5-960\alpha_6u^4-1050u_1^2u^4+70u_4u^4-\\
&-960\alpha_4u_2u^3+2560\alpha_8u^3+420u_2^2u^3+560u_3u_1u^3+96\alpha_4u_4u^2+1280\alpha_6u_2u^2-\\
&-100u_3^2u^2+1400u_2u_1^2u^2-80u_4u_2u^2-60u_5u_1u^2+256\alpha_4u_2^2u-192\alpha_4u_3u_1u-\\
&-128\alpha_6u_4u-2560\alpha_8u_2u+16\alpha_4u_3^2+192\alpha_4u_2u_1^2-32\alpha_4u_4u_2-64\alpha_6u_2^2+\\
&+128\alpha_6u_3u_1-1280\alpha_8u_1^2+256\alpha_8u_4-20u_2^3u+4u_4^2u-360u_3u_2u_1u-20u_4u_1^2u+\\
&+20u_5u_3u-410u_2^2u_1^2-u_5^2+20u_3u_1^3+2u_4u_2^2-4u_4u_3u_1+40u_5u_2u_1\big). \quad (55)
\end{aligned}
$$

3 KdV hierarchy and Novikov equations on free associative algebra.

3.1 KdV hierarchy on free associative algebra.

It is well known that the KdV equation and its hierarchy can be defined on a free differential algebra $\mathfrak{B}_0 = (\mathbb{C}\langle u_0, u_1, \ldots\rangle, D)$ with infinite number of noncommuting variables (see for example, [24], [25]). Algebra $\mathfrak{B}_0$ has monomial additive basis $\{u_\xi = u_{i_1}u_{i_2}\ldots u_{i_m} \,|\, i_k \in \mathbb{Z}_{\geqslant 0},\ m \in \mathbb{N}\}$. It is graded algebra

$$\mathfrak{B}_0 = \mathbb{C}\bigoplus_{n\geqslant 2}\mathfrak{B}_{0,n}, \tag{56}$$

induced by the grading of the variables u_k, $|u_k| = k+2$ for any $k \geqslant 0$ and therefore $|u_\xi| = i_1+\cdots+i_m+2m = n$. Here $\mathfrak{B}_{0,n}$ is a finite dimensional space $\mathfrak{B}_{0,n} = \mathrm{Span}_{\mathbb{C}}\langle u_\xi\,;\,|u_\xi| = n\rangle$.

The construction of the hierarchy is similar to the commutative case, although one has to take care on the order of the variables, since $u_k \cdot u_s \neq u_s \cdot u_k$ if $k \neq s$. Starting with the operator $L = D^2 - u$, one can find its square root by the formula $\mathcal{L} = D + \sum\limits_{n\geqslant 1} I_{1,n}D^{-n}$, where $I_{1,n} \in \mathfrak{B}_0$ are non-commutative polynomials. It follows from the proof of Lemma 2 that formula (26) for the recursive calculation of the polynomials $I_{1,n}$ is also applicable in the case of a free associative algebra $\mathfrak{B}_0$.

Now the initial segment of the series $\mathcal{L}$ has the form

$$\mathcal{L} = D - \frac{1}{2}uD^{-1} + \frac{1}{4}u_1D^{-2} - \frac{1}{8}(u_2 + u^2)D^{-3} + \frac{1}{16}(u_3 + 2u_1u + 4uu_1)D^{-4} + \cdots$$

Similarly to the commutative case, we introduce fractional powers $\mathcal{L}^{2k-1}$ and polynomials $\varrho_{2k} = \mathrm{res}\,\mathcal{L}^{2k-1}$. From the identity $\mathcal{L}^{2k+1} = L\mathcal{L}^{2k-1}$ follows a formula for the recursive calculation of the polynomials ϱ_{2k+2}. It follows from the proof of the formulas (30) and (32) that they are applicable in the case of a free associative algebra $\mathfrak{B}_0$. However, expressions for $\varrho_{2k} \in \mathfrak{B}_0$ are different from expressions for $\rho_{2k} \in \mathfrak{A}_0$, $k \geqslant 3$,

$$\varrho_2 = -\frac{1}{2}u; \quad \varrho_4 = \frac{1}{8}(3u^2 - u_2); \quad \varrho_6 = \frac{1}{32}\big(5(u_2u + uu_2) + 5u_1^2 - 10u^3 - u_4\big); \tag{57}$$

$$\begin{aligned}\varrho_8 = \frac{1}{128}\big(&7(u_4u + uu_4) + 14(u_3u_1 + u_1u_3) + 21u_2^2 - 21(u_2u^2 + u^2u_2) - \\ &- 28uu_2u - 28(u_1^2u + uu_1^2) - 14u_1uu_1 + 35u^4 - u_6\big);\end{aligned} \tag{58}$$

and so on.

Definition 18. The compatible system of equations on the free associative algebra $\mathfrak{B}_0$

$$\partial_{t_{2k-1}}(u) = -2D(\varrho_{2k}) \tag{59}$$

is called the KdV hierarchy (similar to the commutative case (34)).

Equations of the KdV hierarchy define the commuting evolutionary derivations D_{2k-1} of $\mathfrak{B}_0$. Their action on the variables u_n is given by

$$D_{2k-1}(u_n) = -2D^{n+1}(\varrho_{2k})$$

and it can be extended to $\mathfrak{B}_0$ by the linearity and the Leibniz rule.

We have

$$\partial_{t_{2k-1}}(\varrho_{2n}) = \partial_{t_{2n-1}}(\varrho_{2k}) = \operatorname{res}[\mathcal{L}_+^{2n-1}, \mathcal{L}_-^{2k-1}] = \operatorname{res}[\mathcal{L}_+^{2k-1}, \mathcal{L}_-^{2n-1}]. \tag{60}$$

In the non-commutative case $\operatorname{res}[A,B]$ is not any more in the image of the derivation D and Lemma 7 should be modified. Let us introduce the algebra

$$\mathfrak{B}_0^D = \Big\{A = \sum_{i\leqslant m} a_i D^i \mid a_i \in \mathfrak{B}_{0,|a_m|+m-i},\ a_m \neq 0,\ m \in \mathbb{Z}\Big\} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{B}_{0,k}^D\,.$$

Definition 19. Let us introduce the homogeneous skew-symmetric bilinear over $\mathbb{C}$ form

$$\widehat{\sigma}(\cdot,\cdot)\colon \mathfrak{B}_0^D \otimes \mathfrak{B}_0^D \to \mathfrak{B}_0, \quad |\widehat{\sigma}(A,B)| = |A| + |B|,$$

such that for $n, m \in \mathbb{Z}$

$$\widehat{\sigma}(aD^n, bD^m) = \begin{cases} \frac{1}{2}\binom{n}{n+m+1}\sum_{s=0}^{n+m}(-1)^s(a^{(s)}b^{(n+m-s)} + b^{(n+m-s)}a^{(s)}), \text{if } n+m \geqslant 0,\ nm<0,\\ 0, \qquad \text{otherwise.}\end{cases} \tag{61}$$

Lemma 20. We have $\operatorname{res}[A,B] = D\big(\widehat{\sigma}(A,B)\big) - \Delta(A,B)$ for any $A, B \in \mathfrak{B}_0^D$ where

$$\Delta(aD^n, bD^m) = \frac{1}{2}\binom{n}{n+m+1}\Big([a, b^{(n+m+1)}] + (-1)^{n+m}[b, a^{(n+m+1)}]\Big).$$

Proof. The statement of this lemma is verified by directly calculating the value of $\Delta(aD^n, bD^m)$. ∎

Corollary 21. Let

$$A_{2k-1} = \sum_{i=0}^{2k-1} a_{2k-1-i}D^i, \qquad \mathcal{L}_-^{2n-1} = \sum_{j\geqslant 1} I_{2n-1,j}D^{-j},\ n \geqslant 1.$$

Then

$$\operatorname{res}[A_{2k-1}, \mathcal{L}_-^{2n-1}] = D(\widehat{\sigma}_{2k-1,2n-1}) - \Delta_{2k,2n-1}$$

where

$$\widehat{\sigma}_{2k-1,2n-1} = \frac{1}{2}\sum_{i=1}^{2k-1}\sum_{j=1}^{i}\binom{i}{i-j+1}\sum_{s=0}^{i-j}(-1)^s\left(a_{2k-1-i}^{(s)}I_{2n-1,j}^{(i-j-s)} + I_{2n-1,j}^{(i-j-s)}a_{2k-1-i}^{(s)}\right), \tag{62}$$

$$\Delta_{2k,2n-1} = \frac{1}{2}\sum_{i=1}^{2k-1}\sum_{j=1}^{i}\binom{i}{i-j+1}\left([a_{2k-1-i}, I_{2n-1,j}^{(i-j+1)}] + (-1)^{i-j}[a_{2k-1-i}^{(i-j+1)}, I_{2n-1,j}]\right). \tag{63}$$

In the non-commutative case the definition of densities of local conservation laws has to be modified, since

$$\partial_{t_{2k+1}}(\varrho_{2n}) \in \mathrm{Span}[\mathfrak{B}_0, \mathfrak{B}_0] \oplus D(\mathfrak{B}_0).$$

Here $\mathrm{Span}[\mathfrak{B}_0, \mathfrak{B}_0]$ is a linear subspace generated by all commutators of elements from $\mathfrak{B}_0$.

A $\mathbb{C}$-linear space of functionals is defined as $\mathfrak{B}_0^\sharp = \mathfrak{B}_0 / \big(\mathrm{Span}[\mathfrak{B}_0, \mathfrak{B}_0] \oplus D(\mathfrak{B}_0)\big)$, see [10], [23], [24]. The polynomials ϱ_{2n} as elements of $\mathfrak{B}_0^\sharp$ are constants of motion of the nonabelian KdV hierarchy (59).

3.2 Frobenius–Hochschild algebras over free associative algebra.

Denote by $\xi_k = (j_1, \ldots, j_k)$ sequences of non-negative integers of length $k \geqslant 1$. Let $u_{\xi_k} = u_{j_1} \cdots u_{j_k}$. We obtain $|u_{\xi_k}| = 2k + \sum\limits_{s=1}^{k} j_s$.

We will consider $\mathfrak{B}_0$ as a graded algebra $\mathfrak{B}_0 = \mathbb{C} \oplus \widetilde{\mathfrak{B}}_0$, where $\widetilde{\mathfrak{B}}_0 = \oplus_m \mathfrak{B}_{0,m}$, $m \geqslant 2$, and $\mathfrak{B}_{0,m}$ is a graded finite-dimensional $\mathbb{C}$-linear space with an additive lexicographically by indices ordered monomial basis $\{u_{\xi_k},\, |u_{\xi_k}| = m\}$.

For example, $\{u_3, u_1 u, u u_1\}$ is a monomial and lexicographically ordered basis $u_3 \succ u_1 u \succ u u_1$ in $\mathfrak{B}_0^5$.

Let $\xi_k = (j_1, \ldots, j_k)$ be a multindex of the monomial u_{ξ_k} and T_k be a generator of the cyclic permutation group of order k: $T_1(\xi_1) = \xi_1$ and $T_k(\xi_k) = (j_2, \ldots, j_k, j_1)$, $k \geqslant 2$. Let us denote by $T(\xi_k)$ the maximal index set in $T(\xi_k) = \max_\succ \{\xi_k, T_k(\xi_k), \ldots, T_k^{k-1}(\xi_k)\}$ with respect to the lexicographic ordering. We define the linear homomorphism $\mathcal{T}\colon \mathfrak{B}_{0,m} \to \mathfrak{B}_{0,m}$ by its action on the monomial basis elements $\mathcal{T}(u_{\xi_k}) = u_{T(\xi_k)}$.

For example, $\mathcal{T}(u_1 u) = \mathcal{T}(u u_1) = u_1 u$.

Proposition 22. Homomorphism $\mathcal{T}\colon \mathfrak{B}_0 \to \mathfrak{B}_0$ is a projector such that $\ker \mathcal{T} = \mathrm{Span}[\mathfrak{B}_0, \mathfrak{B}_0]$ and $\mathrm{Im}\mathcal{T} \simeq \mathfrak{B}_0^\natural = \mathfrak{B}_0/\mathrm{Span}[\mathfrak{B}_0, \mathfrak{B}_0]$.

Proof. It follows directly from the definition that $\mathcal{T} = \mathcal{T}^2$. The properties of $\mathcal{T}$ follow from the following facts:

1. $[u_{\xi'_k}, u_{\xi''_s}] = u_{\xi_{k.s}} - u_{T^k_{k+s}(\xi_{k.s})}$, where $\xi_{k.s} = (\xi'_k, \xi''_s)$ is the concatination of ξ'_k and ξ''_s;

2. $u_{\xi_k} - u_{T(\xi_k)} = [u_{j_1}, u_{j_2} \cdots u_{j_k}]$ for $\xi_k = (j_1, \ldots, j_k)$, $k > 1$.

■

It follows from Proposition 22 that the projector $\mathcal{T}$ gives the splitting of the exact sequence

$$0 \to \mathrm{Span}[\mathfrak{B}_0, \mathfrak{B}_0] \to \mathfrak{B}_0 \longrightarrow \mathfrak{B}_0^\natural \to 0. \tag{64}$$

It enables us to identify the element $\mathcal{T}(b)$, $b \in \mathfrak{B}_0$ with its canonical projection in $\mathfrak{B}_0^\natural$.

Theorem 23. The algebra $\mathfrak{B}_0^D$ is the $FH(\mathfrak{B}_0, \mathfrak{B}_0^\natural)$-algebra in which the bilinear form $\Phi = \overline{\sigma} \colon \mathfrak{B}_0^D \otimes_{\mathfrak{B}_0} \mathfrak{B}_0^D \to \mathfrak{B}_0^\natural$ is uniquely given by the formula

$$\overline{\sigma}(D^n, bD^m) = \begin{cases} \binom{n}{n+m+1}\mathcal{T}(b^{(n+m)}), & \text{if } n+m \geqslant 0,\ nm < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{65}$$

Proof. Let us first explain the expression $\mathcal{T}(b^{(n+m)})$. The space Span$[\mathfrak{B}_0, \mathfrak{B}_0]$ is closed under the differentiation D. Therefore, the operator D on $\mathfrak{B}_0$ uniquely determines the linear operator $\overline{D} : \mathfrak{B}_0 \mapsto \mathfrak{B}_0^\natural$, and for any $b \in \mathfrak{B}_0$ the formula $\overline{D}(b) = \mathcal{T}(D(b))$ holds.

Let $\xi_k = (j_1, \ldots, j_k)$, $k \geqslant 1$, and $T(\xi_k) = (j_{1,*}, \ldots, j_{k,*})$. If $\xi_k = (j_1, \ldots, j_1) = (j_1)^k$, $k \geqslant 1$, then $\overline{D}(u_{\xi_k}) = k u_{j_1+1} u_{j_1}^{k-1}$. If there are at least two distinct elements in the set ξ_k, then $\overline{D}(u_{\xi_k})$ is a sum of monomials with the leading monomial $u_{j_1+1,*} u_{j_2,*} \cdots u_{j_{k,*}}$. Thus, in a strictly ordered basis, the homomorphism $\overline{D} \colon \mathfrak{B}_{0,m} \to \mathfrak{B}_{0,m+1}^\natural$ is given by an upper triangular matrix with a non-zero diagonal. It induces the monomorphism $\overline{D} \colon \mathfrak{B}_{0,m}^\natural \to \mathfrak{B}_{0,m+1}^\natural$. Following the proof of Theorem 6 and using Lemma 20, it is easy to complete the proof of Theorem 23.

■

Corollary 24. The algebra $\mathfrak{B}_0^D$ is the $FH(\mathfrak{B}_0, \mathfrak{B}_0^\natural)$-algebra with the bilinear form $\widehat{\sigma}(A, B)$ (62).

Proof. It follows immediately from Theorem 23 and the fact that $\widehat{\sigma}(A, B) - \overline{\sigma}(A, B) \in \text{Span}[\mathfrak{B}_0, \mathfrak{B}_0]$ for any $A, B \in \mathfrak{B}_0^D$. ■

3.3 Non-commutative N-th Novikov equation and its hierarchy.

Let $\mathfrak{B} = (\mathcal{A}\langle u, u_1, \ldots\rangle, D)$ be the differential ring, where $\mathcal{A} = \mathbb{C}[\alpha_4, \ldots]$, $D(u_k) = u_{k+1}$, $D(\alpha_{2n}) = 0$, and α_{2n} are commuting parameters, i.e centre elements of $\mathfrak{B}$. Similar to the commutative case, we fix a positive integer N and define a homogenous polynomial $\mathfrak{F}_{2N+2} \in \mathfrak{B}$:

$$\mathfrak{F}_{2N+2} = \varrho_{2N+2} + \sum_{k=0}^{N-1} \alpha_{2(N-k+1)} \varrho_{2k}. \tag{66}$$

Let $\mathfrak{I}_N \subset \mathfrak{B}$ be a two-sided differential ideal generated by the polynomials $\mathfrak{F}_{2N+2}$ and $D^k(\mathfrak{F}_{2N+2})$, $k \in \mathbb{N}$. The quotient ring $\mathfrak{B}_N = \mathfrak{B}/\mathfrak{I}_N$ is a graded finitely generated free ring $\mathfrak{B}_N = \mathcal{A}\langle u_0, u_1, \ldots, u_{2N-1}\rangle$ over $\mathcal{A}$. Similar to Proposition 16 we can show that equations of the hierarchy (59) are all compatible, therefore $D_{2k-1}(\mathfrak{I}_N) \subset \mathfrak{I}_N$. The derivations D_{2k-1} induce derivations $\mathcal{D}_{2k-1}$ of the quotient algebra $\mathfrak{B}_N$.

The equation $\mathfrak{F}_{2N+2} = 0$ can be written in the form

$$u_{2N} = \mathcal{G}_{2N+2}(u_0, u_1, \ldots, u_{2N-2}) = \widehat{\varrho}_{2N+2} + 2^{2N+1} \sum_{k=0}^{N-1} \alpha_{2(N-k+1)} \varrho_{2k}, \tag{67}$$

which is called the non-commutative N-th Novikov equation.

In the non-commutative case the definition of first integrals has to be modified [20], [25]. Let $\mathrm{Span}_{\mathcal{A}}[\mathfrak{B}_N,\mathfrak{B}_N]$ be a $\mathcal{A}$-linear subspace generated by the commutators of all elements in $\mathfrak{B}_N$. We would like to emphasise that $\mathrm{Span}_{\mathcal{A}}[\mathfrak{B}_N,\mathfrak{B}_N]$ is not an ideal in $\mathfrak{B}_N$.

Let $\mathfrak{B}_N^\natural=\mathfrak{B}_N/\mathrm{Span}_{\mathcal{A}}[\mathfrak{B}_N,\mathfrak{B}_N]$ be the corresponding quotient linear space. It follows from the Leibniz rule that derivations of $\mathfrak{B}_N$ are well defined on $\mathfrak{B}_N^\natural$. There is a short split exact sequence (compare with (64):

$$0\ \to\ \mathrm{Span}_{\mathcal{A}}[\mathfrak{B}_N,\mathfrak{B}_N]\ \to\ \mathfrak{B}_N\ \to\mathfrak{B}_N^\natural\ \to\ 0. \tag{68}$$

All constructions and results associated with exact sequence (64) carry over to the case of exact sequence (68).

Definition 25. A non-constant element $\mathcal{H}\in\mathfrak{B}_N^\natural$ is called a first integral of the N-th Novikov equation (respectively of the non-commutative N-th Novikov hierarchy) if $\mathcal{T}(\mathcal{D}\mathcal{H})=0$ (resp. $\mathcal{T}(\mathcal{D}_{2k-1}\mathcal{H})=0$, $k=1,2,\ldots,N$).

Thus, an element $\mathcal{H}\in\mathfrak{B}_N$ is a representative of a first integral, if and only if $\mathcal{T}(\mathcal{H})\neq 0$ and $\mathcal{T}(\mathcal{D}\mathcal{H})=0$.

It follows from (60) and Corollary 21 that

$$\widehat{\mathcal{H}}_{2n+1,2N+1}=\widehat{\sigma}_{2n+1,2N+1}+\sum_{k=1}^{N-1}\alpha_{2N-2k+2}\widehat{\sigma}_{2n+1,2k-1},\quad n\in\mathbb{N} \tag{69}$$

are first integrals of the non-commutative N-th Novikov equation. In the case of free algebra $\mathfrak{B}_N$ we get infinitely many first integrals, since they are algebraically independent. Also the KdV hierarchy (59) reduced to $\mathfrak{B}_N$ is infinite, since the derivations $\mathcal{D}_{2k-1}$, $k>N$, cannot be represented as liner combinations of $\mathcal{D}_{2k-1},\ 1\leqslant k\leqslant N$, with coefficients from a ring of constants (as it takes place in the commutative case (49)).

Example. The $N=1$ Novikov equation (Newton equation on the free algebra $\mathfrak{B}_1=\langle u,u_1\rangle$ with the force $3u^2+8\alpha_4$)

$$\begin{array}{lll} \partial_{t_1}(u) &=& u_1; \\ \partial_{t_1}(u_1) &=& 3u^2+8\alpha_4; \end{array}\qquad \begin{array}{lll} \partial_{t_3}(u) &=& 0; \\ \partial_{t_3}(u_1) &=& 0; \end{array}$$

has an infinite hierarchy of commuting symmetries

$$\begin{aligned} 16\partial_{t_5}(u) &= (u_1u^2+u^2u_1)-2uu_1u-16\alpha_4u_1; \\ 16\partial_{t_5}(u_1) &= -(uu_1^2+u_1^2u)+2u_1uu_1-48\alpha_4u^2-128\alpha_4^2; \end{aligned} \tag{70}$$

$$\begin{aligned} 32\partial_{t_7}(u) &= 8\alpha_4uu_1+8\alpha_4u_1u+2u_1u^3-u_1^3+2u^3u_1-u^2u_1u-uu_1u^2; \\ 32\partial_{t_7}(u_1) &= 128\alpha_4^2u-8\alpha_4u_1^2+64\alpha_4u^3-2u^2u_1^2+uu_1uu_1-2uu_1^2u+u_1u^2u_1+ \\ &\qquad +u_1uu_1u-2u_1^2u^2+6u^5; \end{aligned} \tag{71}$$

$$\begin{aligned} 16\partial_{t_9}(u) &= 3\alpha_4(u^2u_1-2uu_1u+u_1u^2-8\alpha_4u_1); \\ 16\partial_{t_9}(u_1) &= 3\alpha_4(64\alpha_4^2+24\alpha_4u^2+uu_1^2-2u_1uu_1+u_1^2u). \end{aligned} \tag{72}$$

There are infinitely many algebraically independent first integrals (in the sense of Definition 25) given by (69). For example, it follows from (69) that in $\mathfrak{B}_1^\natural$

$$\widehat{\mathcal{H}}_{3,3} = -\frac{3}{16}\left(\frac{1}{2}u_1^2 - u^3 - 8\alpha_4 u\right); \tag{73}$$

$$\mathcal{T}(\widehat{\mathcal{H}}_{5,3}) = \frac{5}{128}\mathcal{T}\left(64\alpha_4^2 + 5uu_1^2 - 10u_1uu_1 + 5u_1^2u\right) = \frac{5}{2}\alpha_4^2; \tag{74}$$

$$\mathcal{T}(\widehat{\mathcal{H}}_{7,3}) = \frac{7}{265}(u_1uu_1u - u_1^2u^2) - \frac{7}{3}\alpha_4\widehat{\mathcal{H}}_{3,3}; \tag{75}$$

$$\mathcal{T}(\widehat{\mathcal{H}}_{9,3}) = \frac{9}{512}\left(u_1uu_1u^2 - u_1^2u^3\right) - \frac{9}{2}\alpha_4^3 + \frac{1}{2}\mathcal{T}(\widehat{\mathcal{H}}_{3,3}^2). \tag{76}$$

We have

$$\mathcal{D}(\widehat{\mathcal{H}}_{3,3}) = -\frac{3}{64}(u^2u_1 - 2uu_1u + u_1u^2); \qquad \mathcal{D}(\mathcal{T}(\widehat{\mathcal{H}}_{5,3})) = 0;$$

$$\mathcal{D}(\mathcal{T}(\widehat{\mathcal{H}}_{7,3})) = \frac{7}{256}\left(8\alpha_4uu_1u - 8\alpha_4u_1u^2 + u_1uu_1^2 - u_1^2uu_1 + 3u^3u_1u - 3u^2u_1u^2\right) - \frac{7}{3}\alpha_4\mathcal{D}(\widehat{\mathcal{H}}_{3,3}),$$

and $\mathcal{T}(\mathcal{D}(\widehat{\mathcal{H}}_{3,3})) = \mathcal{T}(\mathcal{D}(\widehat{\mathcal{H}}_{5,3})) = \mathcal{T}(\mathcal{D}(\widehat{\mathcal{H}}_{7,3})) = \mathcal{T}(\mathcal{D}(\widehat{\mathcal{H}}_{9,3})) = 0$.

3.4 Self-adjointness of the KdV and N–Novikov hierarchies.

Let $\mathfrak{B}_\mathcal{A} = \mathcal{A}\langle u, u_1, \ldots\rangle$. The set of differential operators

$$\mathfrak{B}_\mathcal{A}[D] = \left\{A_+ = \sum_{i=0}^m a_iD^i \,|\, a_i \in \mathfrak{B}_\mathcal{A},\ a_m \neq 0,\ m \in \mathbb{Z}_{\geqslant 0}\right\}$$

and the set of differential formal series

$$\mathfrak{B}_\mathcal{A}^D = \mathfrak{B}_\mathcal{A}[D][[D^{-1}]] = \left\{A = \sum_{i\leqslant m} a_iD^i \,|\, a_i \in \mathfrak{B}_\mathcal{A},\ a_m \neq 0,\ m \in \mathbb{Z}\right\}$$

are non-commutative associative algebras in which multiplication is defined by formula (13). According to formula (13), a conjugation anti-automorphism

$$\dagger\colon \mathfrak{B}_\mathcal{A}^D \to \mathfrak{B}_\mathcal{A}^D \,:\, (AB)^\dagger = B^\dagger A^\dagger$$

is defined on the ring $\mathfrak{B}_\mathcal{A}^D$.

Lemma 26. 1. The operator † is uniquely defined by the conditions

$$u^\dagger = u, \quad D^\dagger = -D, \quad \alpha_{2k}^\dagger = \alpha_{2k}, \quad z^\dagger = \bar{z},\ z \in \mathbb{C}.$$

2. On the ring $\mathfrak{B}_\mathcal{A}$ the operators D and † commute, i.e.

$$(D(a))^\dagger = D(a^\dagger) \ \text{ for any } a \in \mathfrak{B}_\mathcal{A}. \tag{77}$$

Proof. We have $u_1 = Du - uD$. Therefore, $u_1^\dagger = -uD + Du = u_1$. Using the formula $u_{k+1} = Du_k - u_kD$, by induction on k, $k \geqslant 1$, we obtain that $u_{k+1}^\dagger = u_{k+1}$. Using now the formula $DD^k = D^{k+1}$, by induction on $k \geqslant 1$ and $k \leqslant -1$ we obtain that $(D^k)^\dagger = (-1)^k D^k$, where $k \in \mathbb{Z}$ and $D^0 = 1$ is the identity operator.

Statement 1 now follows from the fact that elements u_k, $k \geqslant 0$, D and D^{-1}, where $DD^{-1} = 1$, multiplicatively generate the ring $\mathfrak{B}_{\mathcal{A}}^D$ as a module over the ring $\mathcal{A}$.

Assertion 2 is verified directly. Let $a \in \mathfrak{B}_{\mathcal{A}}$. Then

$$(D(a))^\dagger = (Da - aD)^\dagger = -a^\dagger D + Da^\dagger = D(a^\dagger).$$

■

A differential series $A \in \mathfrak{B}_{\mathcal{A}}^D$ is called self-adjoint if $A^\dagger = A$, and anti-self-adjoint if $A^\dagger = -A$.

Examples.

1. The operator $L = D^2 - u$ is self-adjoint.

2. The series $\mathcal{L} = D - \frac{1}{2}uD^{-1} + \ldots$, $\mathcal{L}^2 = L$, is anti-self-adjoint.

Let $A, B \in \mathfrak{B}_{\mathcal{A}}^D$. Then

$$[A, B]^\dagger = (AB)^\dagger - (BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = -[A^\dagger, B^\dagger].$$

3. Let A and B be self-adjoint (or anti-self-adjoint) series. Then series $[A, B]$ is anti-self-adjoint.

4. Let one of the series A or B be self-adjoint and the other be anti-self-adjoint. Then series $[A, B]$ is self-adjoint.

Consider the series $A = \sum\limits_{i \leqslant m} a_i D^i$. From the formula (13), we obtain

$$A^\dagger = \sum_{i \leqslant m} (-1)^i D^i a_i^\dagger = \sum_{i \leqslant m} (-1)^i \sum_{j \geqslant 0} \binom{i}{j} D^j(a_i^\dagger) D^{i-j}. \tag{78}$$

According to (78), from conditions $j \geqslant 0$ and $i = j - 1$ we obtain that $j = 0$ and $i = -1$, i.e.

$$\operatorname{res}(A^\dagger) = -(\operatorname{res} A)^\dagger. \tag{79}$$

The representation $A = A_+ + A_-$ is uniquely defined. So according to (78), we have

$$(A_+)^\dagger = (A^\dagger)_+ \quad \text{and} \quad (A_-)^\dagger = (A^\dagger)_- \tag{80}$$

Corollary 27. $\varrho_{2k}^\dagger = \varrho_{2k}$, $k \geqslant 1$.

Proof. $\varrho_{2k}^\dagger = \left(\operatorname{res}\mathcal{L}^{2k-1}\right)^\dagger = -\operatorname{res}\left((\mathcal{L}^{2k-1})^\dagger\right) = \operatorname{res}\mathcal{L}^{2k-1} = \varrho_{2k}$. ■

Explicit formulas $\varrho_{2k}^\dagger = \varrho_{2k}$ in the case $k = 1, 2, 3, 4$ see (57) and (58).

Lemma 28. The operators $\partial_{t_{2k-1}}$, $k = 1, 2, \ldots$, participating in the noncommutative hierarchy KdV, commute with the conjugation operator †, i.e.

$$\left(\partial_{t_{2k-1}}(a)\right)^\dagger = \partial_{t_{2k-1}}(a^\dagger) \tag{81}$$

for any $a \in \mathfrak{B}_{\mathcal{A}}$.

Proof. Applying the Leibniz rule for the operator $\partial_{t_{2k-1}}$, we find that it suffices to prove formula (81) only for multiplicative generators u_i, $i \geqslant 0$, of the ring $\mathfrak{B}_{\mathcal{A}}$. Since $u_i = D^i(u)$ and the operators $\partial_{t_{2k-1}}$ commute with the operator D, it suffices to verify formula (81) only for the case $a = u$. By the definition of the hierarchy KdV, on a free associative algebra we have $\partial_{t_{2k-1}}(u) = -2D(\varrho_{2k})$ where by definition $\varrho_{2k} = \mathrm{res}\mathcal{L}^{2k-1}$. Using the formula (77) and Corollary 20, we obtain

$$\big(\partial_{t_{2k-1}}(u)\big)^\dagger = -2\big(D(\varrho_{2k})\big)^\dagger = -2D(\varrho_{2k}^\dagger) = -2D(\varrho_{2k}) = \partial_{t_{2k-1}}(u).$$

■

Let us now turn to the case of the N–Novikov hierarchy described in Section 3.3.

Lemma 29.

1. For any $N \geqslant 1$, the conjugation operator $\dagger\colon \mathfrak{B}_N \to \mathfrak{B}_N$ is defined.
2. The operators D_{2k-1}, of the N–Novikov hierarchy, commute with the operator $\dagger$.

Proof. We have $\mathfrak{B}_N = \mathfrak{B}/\mathfrak{I}_N$ where $\mathfrak{I}_N$ is a two-sided ideal generated by polynomials $\mathfrak{F}_{2N+2}$ and $D^k(\mathfrak{F}_{2N+2})$, $k \in \mathbb{N}$. Since $\varrho_{2k}^\dagger = \varrho_{2k}$ and $\big(D(a)\big)^\dagger = D(a^\dagger)$, then all generators of the ideal $\mathfrak{I}_N$ are self-adjoint polynomials. This proves assertion 1.

Assertion 2 follows directly from Lemma 28. ■

Let us consider the short exact sequence (68). The operator $\dagger\colon \mathfrak{B}_N \to \mathfrak{B}_N$ moves the linear space $\mathrm{Span}_{\mathcal{A}}[\mathfrak{B}_N, \mathfrak{B}_N]$ into itself. Therefore, the conjugation operator $\dagger\colon \mathfrak{B}_N^\natural \to \mathfrak{B}_N^\natural$ is defined. The first integrals of the N–Novikov hierarchy are given by formula (69), where by definition $\widehat{\sigma}_{2n+1,2k-1} = \widehat{\sigma}(\mathcal{L}_+^{2n+1}, \mathcal{L}_-^{2k-1})$.

Lemma 30. The self-adjoint polynomials

$$\widehat{\mathcal{H}}_{2n+1,2N+1} = \sum_{k=1}^{N+1} \alpha_{2N-2k+2}\,\widehat{\sigma}_{2n+1,2k-1}, \quad \text{where } \alpha_0 = 1, \tag{82}$$

are first integrals of the N–Novikov hierarchy.

The proof follows from Corollary 21 and Lemma 29.

4 Quantisation of Novikov's equations.

4.1 Quantum Novikov equations.

In this section we define a quantisation ideal $\mathfrak{Q}_N$ and the quantum N-th Novikov equation. Let $\mathcal{Q}_N$ be a commutative graded algebra of parameters

$$\mathcal{Q}_N = \mathbb{C}[\alpha_{2j+2}, q_{i,j}, q_{i,j}^\omega \mid 0 \leqslant i < j \leqslant 2N-1,\ 0 \leqslant |\omega| < i+j+4]$$

where $|q_{i,j}| = 0$, $\omega = (i_{2N-1}, \dots, i_1, i_0) \in \mathbb{Z}_{\geqslant}^{2N}$, $|\omega| = (2N+1)i_{2N-1} + \cdots + 3i_1 + 2i_0$, $|q_{i,j}^\omega| = i+j+4-|\omega|$. The parameters are constant in a sense that for any $a \in \mathcal{Q}_N$ we have $\mathcal{D}_{2k-1}(a) = 0$. Let $u^\omega = u_{2N-1}^{i_{2N-1}} \cdots u_1^{i_1} u_0^{i_0}$, and thus $|u^\omega| = |\omega|$. Let $\mathfrak{B}_N(q)$ denote the graded ring

$$\mathfrak{B}_N(q) = \mathcal{Q}_N\langle u_0, \dots, u_{2N-1}\rangle.$$

with parameters, which are in the centre of the ring.

Let $\mathfrak{Q}_N = \langle p_{i,j} \,|\, 0 \leqslant i < j \leqslant 2N-1\rangle \subset \mathfrak{B}_N(q)$ be a two-sided $\mathcal{D} = \mathcal{D}_1$ differential homogeneous ideal generated by the polynomials

$$p_{i,j} = u_i u_j - q_{i,j} u_j u_i + \sum_{0 \leqslant |\omega| < i+j+4} q_{i,j}^\omega u^\omega, \quad 0 \leqslant i < j \leqslant 2N-1, \quad q_{i,j} \neq 0. \tag{83}$$

Let us consider the graded associative algebra $\mathfrak{C}_N = \mathfrak{B}_N(q)/\mathfrak{Q}_N$ and the ring epimorphism $\mathfrak{B}_N(q) \to \mathfrak{C}_N$ preserving the grading.

Definition 31. The ideal $\mathfrak{Q}_N$ is called the Poincaré–Birkhoff–Witt ideal (briefly, the PBW-ideal) if the image of the set of monomials u^ω, $\omega \in \mathbb{Z}_{\geqslant}^{2N}$, forms a non-degenerate additive basis in the $\mathcal{Q}$-module $\mathfrak{C}_N$.

Definition 32. The ideal $\mathfrak{Q}_N$ is a *quantisation ideal* of the N-th Novikov equation if:

1. it is the PBW-ideal;

2. it is invariant with respect to the derivation $\mathcal{D}$.

Condition 2 of Definition 32 reduces to a system of polynomial algebraic equations in $\mathcal{Q}_N$.

Lemma 33. Let $\mathfrak{Q}_N \subset \mathfrak{B}_N$ be a quantisation ideal of the N-th Novikov equation. Then $q_{i,j} = 1$.

Proof. The Lemma can be proven by induction. Let us show that $q_{2N-2,2N-1} = 1$. Applying $\mathcal{D}$ to the polynomial $p_{2N-2,2N-1}$ we get

$$\mathcal{D}(p_{2N-2,2N-1}) = (1 - q_{2N-2,2N-1})u_{2N-1}^2 + f_{2N-2,2N-1}.$$

where $f_{2N-2,2N-1}$ is a polynomial whose leading monomial $\mathrm{Lm}(f_{2N-2,2N-1}) < u_{2N-1}^2$. Thus $q_{2N-2,2N-1} = 1$ is the necessary condition for $\mathcal{D}(\mathfrak{Q}_N) \subset \mathfrak{Q}_N$. Let us assume that $q_{i,j} = 1$ for all $i < j$, such that $i + j \geqslant k$. For a polynomial $p_{i,j}$ with $i < j$ and $i + j < k$ we get

$$\mathcal{D}(p_{i,j}) = (u_{i+1}u_j + u_i u_{j+1}) - q_{i,j}(u_{j+1}u_i + u_j u_{i+1}) + \sum q_{i,j}^\omega \mathcal{D}(u^\omega).$$

By the induction assumption $q_{i,j+1} = 1$. Therefore, $\mathcal{D}(p_{i,j}) = (1 - q_{i,j})u_{j+1}u_i + f_{i,j}$, where $f_{i,j}$ is a polynomial such that $\mathrm{Lm}(f_{i,j}) \prec u_{j+1}u_i$ in the additive basis of $\mathfrak{C}_N$. It follows from $\mathcal{D}(p_{i,j}) \in \mathfrak{Q}_N$ that $q_{i,j} = 1$. ∎

Corollary 34.

1. The relation

$$[u_i, u_j] = -h_{ij}, \quad 0 \leqslant i < j \leqslant 2N-1,$$

holds in the ring $\mathfrak{C}_N$, where

$$h_{ij} = \sum_{0\leqslant|\omega|<i+j+4} q_{i,j}^{\omega} u^{\omega}, \quad q_{i,j}^{\omega} \in \mathcal{Q}_N.$$

2. For any $P \in \mathfrak{C}_N$, polynomial $[u_k, P] \in \mathfrak{C}_N$ is a linear combination of polynomials h_{ij} with coefficients from $\mathfrak{C}_N$.

Since the ideal $\mathfrak{Q}_N \subset \mathfrak{B}_N(q)$ is $\mathcal{D}$-invariant, i.e. $\mathcal{D}(\mathfrak{Q}_N) \subset \mathfrak{Q}_N$, then the derivation $\mathcal{D}$ induces a well defined derivation ∂_{t_1} on the quotient algebra $\mathfrak{B}_N(q)/\mathfrak{Q}_N$ and a quantum dynamical system defined by the quantum N-th Novikov equation

$$\partial_{t_1}(u_k) = u_{k+1},\ k = 0, \dots, 2N-2, \qquad \partial_{t_1}(u_{2N-1}) = \mathfrak{G}_{2N+2}(u_0, \dots, u_{2N-2}),$$

where $\mathfrak{G}_{2N+2}(u_0, \dots, u_{2N-2}) \in \mathfrak{B}_N(q)/\mathfrak{Q}_N$ is a canonical projection of $\mathcal{G}_{2N+2}(u_0, \dots, u_{2N-2}) \in \mathfrak{B}_N$.

Theorem 35. The ideal $\mathfrak{Q}_N$ is the quantisation ideal of the N-th Novikov equation if and only if the set of polynomials $h_{ij} \in \mathfrak{C}_N$ is a solution of the following systems in the ring $\mathfrak{C}_N$:

I. the system of algebraic equations linear in h_{ij}

$$[h_{ij}, u_k] + [h_{jk}, u_i] = [h_{ik}, u_j] \tag{84}$$

for all triples $(i, j, k),\ 0 \leqslant i < j < k \leqslant 2N-1$;

II. the system of differential equations linear in h_{ij}

$$h'_{ij} = \widetilde{h}_{i+1,j} + \widetilde{h}_{i,j+1} \tag{85}$$

where $h'_{ij} = \partial_{t_1}(h_{ij})$ and

$$\widetilde{h}_{i+1,j} = \begin{cases} h_{i+1,j}, & \text{if } i+1<j; \\ 0, & \text{if } i+1=j; \end{cases} \qquad \widetilde{h}_{i,j+1} = \begin{cases} h_{i,j+1}, & \text{if } j+1<2N; \\ [u_i, \mathfrak{G}_{2N+2}], & \text{if } j=2N-1. \end{cases}$$

Proof. In [18], V. Levandovskyy obtained necessary and sufficient conditions on polynomials $p_{i,j}$ of the form (83) under which the ideal $\mathfrak{Q}_N$ is the BPW-ideal. Under the additional condition $q_{i,j} = 1$ (see Lemma 33), Lemma 2.1 from [18] provides a proof of Statement I. The proof of Statement II follows from Statements 1–2 of Corollary 4. ■

Let the ideal $\mathfrak{Q}_N \subset \mathfrak{B}_N$ be a quantization ideal that is invariant under derivations $\mathcal{D}_{2k-1}$, $k = 2, \dots, N$, on the ring $\mathfrak{B}_N$. Then derivations $\partial_{t_{2k-1}}$, $k = 2, \dots, N$, are defined on the ring $\mathfrak{C}_N$ such that $\partial_{t_{2k-1}}(u) = \partial_{t_1}(\widehat{\varrho}_{2k})$, where $\widehat{\varrho}_{2k} \in \mathfrak{C}_N$ is the image of the polynomial $\varrho_{2k} \in \mathfrak{B}_N$.

Corollary 36. The polynomials $h_{ij} \in \mathfrak{C}_N$ satisfy the following system of differential equations linear in h_{ij}

$$\partial_{t_{2k-1}}(h_{ij}) = [u_i, \partial_{t_1}^{j+1}(\widehat{\varrho}_{2k})] - [u_j, \partial_{t_1}^{i+1}(\widehat{\varrho}_{2k})], \quad k = 1, \dots, N,\ 0 \leqslant i < j \leqslant 2N-1.$$

4.2 Quantum $N = 1$ Novikov equation.

Let

$$u_2 = \mathcal{G}_4(u_0) = 3u^2 + 8\alpha_4 \tag{86}$$

and $\mathfrak{I}_1 \subset \mathfrak{B}$ is the two-sided differential ideal generated by Novikov equation relatively the derivation D, such that $D(u_k) = u_{k+1}$. Then in $\mathfrak{B}_1 = \mathfrak{B}/\mathfrak{I}_1 = \mathcal{A}\langle u_0, u_1\rangle$ the induced derivation $\mathcal{D}$ can be defined by its action on the generators

$$\mathcal{D}(u) = u_1, \quad \mathcal{D}(u_1) = 3u^2 + 8\alpha_4.$$

Let us consider a homogeneous two sided ideal $\mathfrak{Q}_1 \subset \mathfrak{B}_1(q)$ generated by one polynomial

$$p_{0,1} = uu_1 - u_1 u - q^{(0,2)}u^2 - q^{(1,0)}u_1 - q^{(0,1)}u - q^{(0,0)}$$

with arbitrary constants q^{ω}. Let us find conditions on these constants under which the ideal $\mathfrak{Q}_1$ becomes the quantization ideal. According to Theorem 35 we obtain

$$h'_{01} = [u, 3u^2 + 8\alpha_4] = 0.$$

Thus, $q^{(0,2)} = q^{(1,0)} = q^{(0,1)} = 0$ and $q^{(0,0)}$ is a free parameter. Let us denote $q^{(0,0)} = 8i\hbar$, then the quotient algebra $\mathfrak{C}_1 = \mathfrak{B}_1(q)/\mathfrak{Q}_1$ coincides with the Heisenberg (Weyl) algebra $\mathbb{C}[\alpha_4, \hbar]\langle u, u_1\rangle/\langle uu_1 - u_1 u - 8i\hbar\rangle$ in quantum mechanics. Thus we have proved the following statement.

Proposition 37. The ideal $\mathfrak{Q}_1 \subset \mathfrak{B}_1(q)$ is the quantization ideal if and only if $uu_1 - u_1 u = 8i\hbar$, where $\hbar$ is an arbitrary parameter.

In the case $N = 1$ the N-th Novikov equation $u_2 = 3u^2 + 8\alpha_4$ has the form of the classical Newton equation. According to Proposition 37, the quantum $N = 1$ Novikov equation is unique and can be written in the Heisenberg form

$$i\hbar\partial_{t_1}(u) = [u, \mathfrak{H}_{3,3}] = i\hbar u_1, \qquad i\hbar\partial_{t_1}(u_1) = [u, \mathfrak{H}_{3,3}] = i\hbar(3u^2 + 8\alpha_4), \tag{87}$$

where the Hamiltonian operator $\mathfrak{H}_{3,3} = -\frac{2}{3}\widehat{\mathcal{H}}_{3,3} = \frac{1}{16}(u_1^2 - 2u^3 - 16\alpha_4 u)$ is self-adjoint and $[u, u_1] = 8i\hbar$. Here $\widehat{\mathcal{H}}_{3,3}$ is given by (82). It follows from (87) that the t_1 derivative of any element of $a \in \mathfrak{C}_1$ can be written in the form $\partial_{t_1}(u) = \frac{i}{\hbar}[\mathfrak{H}_{3,3}, a]$. In particular the Hamiltonian $\mathfrak{H}_{3,3}$ is a quantum constant of motion $\partial_{t_1}(\mathfrak{H}_{3,3}) = \frac{i}{\hbar}[\mathfrak{H}_{3,3}, \mathfrak{H}_{3,3}] = 0$.

We have $\partial_{t_3}(u) = 0$. Higher symmetries (70), (71),(72) on the free associative algebra $\mathfrak{B}_1$ after the reduction to $\mathfrak{C}_1$ take the self-adjoint form

$$\begin{aligned}
\partial_{t_5}(u) &= -\alpha_4 u_1; \\
\partial_{t_5}(u_1) &= -\alpha_4(3u^2 + 8\alpha_4);
\end{aligned}$$

$$\begin{aligned}
32\partial_{t_7}(u) &= 64i\alpha_4\hbar + 16\alpha_4 u_1 u + 24iu^2\hbar + 2u_1 u^3 - u_1^3; \\
32\partial_{t_7}(u_1) &= 64\alpha_4 u^3 + 128\alpha_4^2 u - 8\alpha_4 u_1^2 - 48iu_1 u\hbar + 6u^5 - 3u_1^2 u^2 + 192\hbar^2;
\end{aligned}$$

$$\begin{aligned}
2\partial_{t_9}(u) &= 3\alpha_4^2 u_1; \\
2\partial_{t_9}(u_1) &= 3\alpha_4^2(3u^2 + 8\alpha_4).
\end{aligned}$$

4.3 Quantum $N = 2$ Novikov hierarchy.

In the case $N = 2$ the N-th Novikov equation on the free associative algebra $\mathfrak{B} = \mathbb{C}[\alpha_4, \alpha_6]\langle u, u_1, \ldots\rangle$ can be written in the form

$$u_4 = \mathcal{G}_6(u_0, u_1, u_2), \tag{88}$$

where

$$\mathcal{G}_6 = 5(u_2 u + u u_2) + 5u_1^2 - 10u^3 - 16\alpha_4 u + 32\alpha_6. \tag{89}$$

It defines two commuting derivations $\mathcal{D}, \mathcal{D}_3$ on the quotient ring $\mathfrak{B}_2 = \mathbb{C}[\alpha_4, \alpha_6]\langle u, u_1, u_2, u_3\rangle = \mathfrak{B}/\mathfrak{I}_2$ which results in the 2-KdV hierarchy consisting of two compatible nonabelian systems. The first system of the hierarchy $\big(\partial_{t_1}(u_k) = \mathcal{D}(u_k)\big)$

$$\partial_{t_1}(u) = u_1; \qquad \partial_{t_1}(u_1) = u_2; \qquad \partial_{t_1}(u_2) = u_3; \qquad \partial_{t_1}(u_3) = \mathcal{G}_6 \tag{90}$$

is equation (88) written in the form of a first order system. The second system of the hierarchy is

$$4\partial_{t_3}(u_k) = \partial_{t_1}^{k+1}(u_2 - 3u^2), \quad k = 0, 1, 2, 3. \tag{91}$$

Let us introduce a two-sided ideal

$$\widehat{\mathfrak{Q}}_2 = \langle [u_i, u_j] - h_{ij};\ [h_{ij}, u_k];\ 0 \leqslant i < j \leqslant 3,\ 0 \leqslant k \leqslant 3\rangle \subset \mathfrak{B}_2(q)$$

and set

$$\widehat{\mathfrak{C}}_2 = \mathfrak{B}_2(q)/\widehat{\mathfrak{Q}}_2.$$

Lemma 38. The ideal $\widehat{\mathfrak{Q}}_2$ is a quantisation ideal of the 2-nd Novikov equation if and only if polynomials $h_{ij} \in \widehat{\mathfrak{C}}_2$ are the solution of the following linear in h_{ij} system of differential equations:

$$\begin{array}{lll} h'_{01} = h_{02}; & h'_{02} = h_{12} + h_{03}; & h'_{12} = h_{13}; \\ h'_{03} = h_{13} + P_8; & h'_{13} = h_{23} + P_9; & h'_{23} = P_{10}, \end{array}$$

where $h'_{ij} = \partial_{t_1}(h_{ij}),\ P_8 = (h_{01}u)' = h_{02}u + h_{01}u_1,\ P_9 = 10(h_{12}u - h_{01}u_2) + 16\alpha_4 h_{01},$ $P_{10} = 10h_{02}(-u_2 + 3u^2) - 10h_{12}u_1 + 16\alpha_4 h_{02}$.

Proof. Using the formula

$$\mathcal{G}_6 = 10(u_2 u - u^3) + 5(h_{02} + u_1^2) - 16\alpha_4 u + 32\alpha_6 \in \widehat{\mathfrak{C}}_2,$$

we obtain that the assertion of this lemma is an immediate consequence of Theorem 35. ■

Proposition 39. The ideal $\mathfrak{Q}_2 \subset \mathfrak{B}_2(q)$ is the quantization ideal if and only if

$$[u_i, u_j] = 0 \ \text{ for } i + j < 3 \ \text{ or } i + j = 4; \quad [u, u_3] = [u_2, u_1] = 32i\hbar; \quad [u_2, u_3] = 320i\hbar u,$$

where $\hbar$ is an arbitrary parameter.

Proof. Set $h_{01} = \xi \in \widehat{\mathfrak{C}}_2$. Then, accordingly to Lemma 38, in the ring $\widehat{\mathfrak{C}}_2$ we have:

$$\begin{aligned}
&h_{02} = \xi'; \quad 2h_{03} = \xi'' + 10\xi u + 2\alpha, \ \text{ where } \ \alpha = const, \ |\alpha| = 7;\\
&2h_{12} = \xi'' - 10\xi u - 2\alpha; \quad 2h_{13} = \xi''' - 10(\xi u)';\\
&2h_{23} = \xi^{(4)} - 10(\xi u)'' - 20(h_{12}u - \xi u_2) - 16\alpha_4\xi;\\
&\xi^{(5)} - 10(\xi u)''' - 20(h_{12}u - \xi u_2)' - 16\alpha_4\xi' = 20\xi'(-u_2 + 3u^2) - 20h_{12}u_1 + 32\alpha_4\xi'.
\end{aligned}$$

Therefore, the polynomial ξ must satisfy the equation

$$\xi^{(5)} - 20\xi'''u - 30\xi''u_1 + \xi'(10u_2 + 40u^2 - 48\alpha_4) + 10\xi(u_3 + 10u_1u) = 0. \tag{92}$$

Accordingly to formula (83), the solution to this equation should be sought in the form:

$$\xi = \beta_1u^2 + \beta_2u_1 + \beta_3u + \beta_5, \quad |\beta_k| = k. \tag{93}$$

Substituting expression (93) into equation (92) and using the PBW-basis in $\widehat{\mathfrak{C}}_2$, we obtain that $\xi = 0$ in $\widehat{\mathfrak{C}}_2$. Then the polynomial $\xi \in \mathfrak{B}_2(q)$ must belong to the ideal $\widehat{\mathfrak{Q}}_2$, but this is possible only when $[u, u_1] = 0$.

Under condition $[u, u_1] = 0$, the system of differential equations (85) in the case $N = 2$ takes the form

$$\begin{aligned}
&h_{01} = 0; \quad h_{02} = 0; \quad h_{12} + h_{03} = 0;\\
&h_{13} = h'_{12} = h'_{03}; \quad h_{23} = h'_{13} - 5(h_{12}u + uh_{12});\\
&h'_{23} = -5(h_{12}u_1 + u_1h_{12}).
\end{aligned}$$

If $\alpha = -32i\hbar$, then we obtain:

$$[u, u_3] = [u_2, u_1] = 32i\hbar; \quad [u_2, u_3] = 320i\hbar u.$$

■

Proposition 40. The following statements are equivalent

1. The ideal $\mathfrak{Q}_2$ is $\mathcal{D}$-invariant: $\mathcal{D}(\mathfrak{Q}_2) \subseteq \mathfrak{Q}_2$.
2. The ideal $\mathfrak{Q}_2$ is $\mathcal{D}_3$-invariant: $\mathcal{D}_3(\mathfrak{Q}_2) \subseteq \mathfrak{Q}_2$.
3. The ideal $\mathfrak{Q}_2$ is generated by the commutation relations

$$\begin{aligned}
&[u_i, u_j] = 0 \text{ for } \ i + j < 3 \text{ or } \ i + j = 4;\\
&[u, u_3] = [u_2, u_1] = 32i\hbar, \quad [u_2, u_3] = 320i\hbar u_0,
\end{aligned}$$

where $\hbar$ is an arbitrary parameter.

Proposition 41. Quantum $N = 2$ KdV hierarchy has the quantum Hamiltonians

$$\mathfrak{H}_{3,5} = -\frac{2}{3}\widehat{\mathcal{H}}_{3,5}, \qquad \mathfrak{H}_{5,5} = -\frac{2}{5}\widehat{\mathcal{H}}_{5,5}$$

such that $[\mathfrak{H}_{3,5}, \mathfrak{H}_{5,5}] = 0$. Here $\widehat{\mathcal{H}}_{3,5}$, $\widehat{\mathcal{H}}_{5,5}$ are given by (82).

Theorem 42. For $N = 2$ the quantum N-th Novikov equation, corresponding to the derivations ∂_{t_1}, can be written in the Heisenberg form

$$\partial_{t_1}(u_k) = \frac{i}{\hbar}[\mathfrak{H}_{3,5}, u_k] = \begin{cases} u_{k+1}, & 0 \leqslant k \leqslant 2, \\ 32\alpha_6 - 16\alpha_4 u + 5u_1^2 + 10u_2 u - 10u^3, & k = 3. \end{cases}$$

The quantum dynamical system $\mathfrak{C}_2$, corresponding to the derivations ∂_{t_3}, can be written in the Heisenberg form

$$4\partial_{t_3}(u_k) = \frac{i}{\hbar}[\mathfrak{H}_{5,5}, u_k] = \partial_{t_1}^k(u_3 - 6u_1 u), \qquad k = 0, 1, 2, 3.$$

4.4 Quantum $N = 3$ and $N = 4$ Novikov hierarchy.

For $N = 3$ non-commutative N-th Novikov's equation has a form

$$u_6 = \mathcal{G}_8(u_0, u_1, u_2, u_3, u_4), \tag{94}$$

where

$$\mathcal{G}_8 = 7(u_4 u + uu_4) + 14(u_3 u_1 + u_1 u_3) + 21u_2^2 - 21(u_2 u^2 + u^2 u_2) - 28(u_1^2 u + uu_1^2) - \\ - 28uu_2 u - 14u_1 uu_1 + 35u^4 - 16\alpha_4(u_2 - 3u^2) - 64\alpha_6 u + 128\alpha_8 = \mathcal{G}_8^\dagger. \tag{95}$$

In $\mathfrak{B}_3 = \mathcal{A}\langle u_0, u_1, \dots, u_5\rangle$ the non-commutative N-th Novikov equation is a generator of the two-sided ideal $\mathfrak{I}_3$.

Proposition 43. The following statements are equivalent

1. The ideal $\mathfrak{Q}_3$ is $\mathcal{D}$-invariant: $\mathcal{D}(\mathfrak{Q}_3) \subseteq \mathfrak{Q}_3$.
2. The ideal $\mathfrak{Q}_3$ is $\mathcal{D}_3$-invariant: $\mathcal{D}_3(\mathfrak{Q}_3) \subseteq \mathfrak{Q}_3$.
3. The ideal $\mathfrak{Q}_3$ is $\mathcal{D}_5$-invariant: $\mathcal{D}_5(\mathfrak{Q}_3) \subseteq \mathfrak{Q}_3$.
4. The ideal $\mathfrak{Q}_3$ is generated by the commutation relations

$$[u_i, u_j] = 0 \text{ if } i + j < 5 \text{ or } i + j = 6,$$
$$[u, u_5] = [u_4, u_1] = [u_2, u_3] = \eta, \quad [u_2, u_5] = [u_4, u_3] = 7 \cdot 2\eta u,$$
$$[u_5, u_3] = 14\eta u_1, \quad [u_4, u_5] = 2\eta\hbar(63u^2 + 14u_2 - 8\alpha_4)$$

where $\eta \in \mathbb{C}$ is an arbitrary parameter.

Setting $\eta = 2^7 i\hbar$, where $\hbar$ is an arbitrary real parameter, we get.

Proposition 44. Quantum $N = 3$ KdV hierarchy has three self-adjoint commuting quantum Hamiltonians

$$\mathfrak{H}_{3,7} = -\frac{2}{3}\widehat{\mathcal{H}}_{3,7}, \quad \mathfrak{H}_{5,7} = -\frac{2}{5}\widehat{\mathcal{H}}_{5,7}, \quad \mathfrak{H}_{7,7} = -\frac{2}{7}\widehat{\mathcal{H}}_{7,7}$$

Theorem 45. For $N = 3$ the quantum N-th Novikov equation, corresponding to the derivation ∂_{t_1}, can be written in the Heisenberg form

$$\partial_{t_1}(u_k) = \frac{i}{\hbar}[\mathfrak{H}_{3,7}, u_k] = \begin{cases} u_{k+1}, & 0 \leqslant k \leqslant 4, \\ \mathfrak{G}_8, & k = 5. \end{cases}$$

where

$$\mathfrak{G}_8 = 28u_3u_1 + 21u_2^2 + 35u^4 - 14(u_4 - 5u_2u + 5u_1^2)u - 16\alpha_4(u_2 - 3u^2) - 64\alpha_6 u + 128\alpha_8.$$

The quantum dynamical systems in $\mathfrak{C}_3$, corresponding to the derivations ∂_{t_3} and ∂_{t_5} can be written in the Heisenberg form

$$4\partial_{t_3}(u_k) = \frac{i}{\hbar}[\mathfrak{H}_{5,7}, u_k] = \partial_{t_1}^k(u_3 - 6u_1u),$$
$$16\partial_{t_5}(u_k) = \frac{i}{\hbar}[\mathfrak{H}_{7,7}, u_k] = \partial_{t_1}^k(u_5 - 20u_2u_1 - 10u_3u + 30u_1u^2).$$

In the case $\mathbf{N = 4}$ the invariant ideal of quantisation $\mathfrak{Q}_4$ is generated by the commutation relations ($\eta = 2^9 i\hbar$):

$$[u, u_7] = [u_6, u_1] = [u_2, u_5] = [u_4, u_3] = \eta, \quad [u_2, u_7] = [u_6, u_3] = [u_4, u_5] = 18\eta u,$$
$$[u_7, u_3] = 2[u_4, u_6] = 36\eta u_1, \quad [u_4, u_7] - 18\eta u_2 = [u_6, u_5] = \eta(198u^2 + 60u_2 - 16\alpha_4),$$
$$[u_7, u_6] = \eta(858u_1^2 - 1980u_2u - 1716u^3 - 54u_4 + 64\alpha_6 + 416\alpha_4 u),$$
$$[u_7, u_5] = \eta(396u_1u + 60u_3), \quad [u_i, u_j] = 0 \text{ if } i + j < 7 \text{ or } i + j = 8.$$

The corresponding quantum hierarchy takes the form

$$2^{2n-2}\partial_{t_{2n-1}}(u_k) = \frac{i}{\hbar}[\mathfrak{H}_{2n+1,9}, u_k], \qquad n = 1, 2, 3, 4, \quad k = 0, \dots, 7,$$

where

$$\mathfrak{H}_{2n+1,9} = -\frac{2}{2n+1}\widehat{\mathcal{H}}_{2n+1,9}, \quad n = 1, 2, 3, 4.$$

The first equation of this fine hierarchy is the $N = 4$ Novikov equation written in the Heiseberg form

$$\partial_{t_1}(u_k) = \frac{i}{\hbar}[\mathfrak{H}_{3,9}, u_k] = \begin{cases} u_{k+1}, & 0 \leqslant k \leqslant 6, \\ \mathfrak{G}_{10}, & k = 7. \end{cases}$$

where

$$\mathfrak{G}_{10} = 512\alpha_{10} - 160\alpha_4 u^3 + 192\alpha_6 u^2 + 160\alpha_4 u_2 u - 256\alpha_8 u + 80\alpha_4 u_1^2$$
$$-16\alpha_4 u_4 - 64\alpha_6 u_2 - 126u^5 + 420u_2u^3 + 630u_1^2u^2 - 126u_4u^2$$
$$-378u_2^2u - 504u_1u_3u + 18u_6u + 69u_3^2 - 462u_1^2u_2 + 114u_2u_4 + 54u_1u_5,$$

and the other three equations are ($k = 0, \dots, 7$):

$$\begin{aligned}
4\partial_{t_3}(u_k) &= \frac{i}{\hbar}[\mathfrak{H}_{5,9}, u_k] = \partial_{t_1}^k (u_3 - 6u_1 u),\\
16\partial_{t_5}(u_k) &= \frac{i}{\hbar}[\mathfrak{H}_{7,9}, u_k] = \partial_{t_1}^k (u_5 - 20u_2u_1 - 10u_3u + 30u_1u^2),\\
64\partial_{t_7}(u_k) &= \frac{i}{\hbar}[\mathfrak{H}_{9,9}, u_k] = \partial_{t_1}^k (u_7 - 140u_1u^3 + 70u_3u^2 + 280u_2u_1u - 14u_5u\\
&\quad + 70u_1^3 - 70u_3u_2 - 42u_4u_1).
\end{aligned}$$

There are two interesing observations. In all cases considered in this section, namely $N = 1, 2, 3, 4$:

- The form of the normally ordered quantum N-Novikov equations coincide with the corresponding classical equations in the commutative case.
- For the first N equations of the quantum N-th Novikov hierarchy the polynomials $\widehat{\mathcal{H}}_{2n+1,2N+1},\ n = 1, 2, \dots, N$ (82) are also quantum commuting Hamiltonians. We have shown that these polynomials are integrals for the non-commutative hierarchy on the free associative algebra $\mathfrak{B}_N$, i.e. $\partial_{t_{2k-1}}(\widehat{\mathcal{H}}_{2n+1,2N+1}) \in \mathrm{Span}[\mathfrak{B}_\mathrm{N}, \mathfrak{B}_\mathrm{N}]$, where $\partial_{t_{2k-1}},\ k = 1, \dots, N$. Apparently these derivations map these polynomials (naturally embedded in $\mathfrak{C}_N$) into the corresponding quantisation ideal $\mathfrak{Q}_N$. In other words
$$\partial_{t_{2k-1}} : \widehat{\mathcal{H}}_{2n+1,2N+1} \mapsto \mathrm{Span}[\mathfrak{B}_\mathrm{N}, \mathfrak{B}_\mathrm{N}] \bigcap \mathfrak{Q}_\mathrm{N}, \qquad 1 \leqslant \mathrm{n}, \mathrm{k} \leqslant \mathrm{N}.$$

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