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Conjectures analogous to the Collatz conjecture

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Abstract

In this paper we introduce some conjectures analogous to the well-known Collatz conjecture.

1 Introduction

The Collatz conjecture (also called "$3n + 1$ conjecture", or with various other names) [1] [2] is, so far, neither *proven*, nor shown to be *unprovable*. It is considered "one of the most famous unsolved problems in mathematics" [1]. In this paper we briefly describe the Collatz conjecture, some of its known variants, and some related, presumably new, analogous conjectures.

This paper is dedicated to the memory of *Mario Bruschi* and *Decio Levi*.

Mario Bruschi (8th August 1948-1st July 2021) was a pupil (almost 13 years younger), and then friend, colleague and, over time, co-author of several scientific papers with the second author of this paper (FC). *Mario* was a very remarkable man, with many interests in life besides science; he authored 2 papers related to the Collatz conjecture [3, 4], and it was him who first brought the attention of FC to this topic, and who made the numerical calculations reported in a joint paper never published (the last typed copy of this paper is dated February 4th, 2019, and we recently put it on arXiv [5]).

Also *Decio Levi* got his Laurea at the Department of Physics of the University of Rome "La Sapienza" by defending a dissertation prepared under the supervision of FC. Both *Mario* and *Decio* were only half a generation younger than FC, so we subsequently became quite good personal friends among ourselves; they both co-authored some papers with FC and many more were co-authored by the 2 of them, and several papers also with other members (of their generation or somewhat younger) belonging to the small group of Italian theoretical physicist working in Rome on analogous topics (say, on "integrable systems"); including in particular the 4 co-editors of this book dedicated to *Decio* (but it should also be mentioned that the main mentor of *Decio*, as well as his close friend and co-author of very many papers, was *Pavel Winternitz*). It is sad that all 3 of them—*Mario*, *Decio*, *Pavel* (all of them younger than FC)—are no more with us.

2 The Collatz conjecture

Consider the sequence of *positive integers* n_j $(j = 0, 1, 2, ...)$ generated by the following 2 simple rules (hereafter we assume all these rules to be applied *sequentially*, so that they generate a *uniquely* defined sequence of *positive integers*): $n_{j+1} = n_j/2$ if n_j is *even*, $n_{j+1} = 3n_j + 1$ if n_j is *odd*. The Collatz conjecture states that, for *every* positive integer *initial* value n_0, this sequence ends up into the cycle $\{1, 4, 2, 1\}$. This conjecture is attributed to Lothar Collatz (1910-1990), who perhaps first thought of it in 1937 [2]; it has been checked (in 2020) for *all* positive integers numbers n_0 up to $2^{68} \approx 2.95 \cdot 10^{20}$, and some computer experts continue to labor in order to increase this upper bound [1]; it is *obviously true* for certain *arbitrarily large* initial values (for instance for $n_0 = 2^N$ with N *any positive integer*); but it has not yet been *proven* nor *disproved*, nor shown to be *unprovable*; recently it has been shown that it is *extremely probable* that this conjecture is *true*, in an appropriate mathematical framework where this statement has a *precise* meaning [6]. The reason why it might *not* be true is, of course, because there might exist an *initial* positive integer n_0 such that the sequence generated by the simple rules detailed above either *diverges* or leads to a *different* cycle than the *single* one mentioned above.

Many analogous conjectures have been proposed over time (see for instance the book [7]). In this paper we mention some, possibly new, analogous *conjectures*.

3 The "$5n + 1$ Conjecture"

Consider the sequence of *positive integers* n_j $(j = 0, 1, 2, ...)$ generated by the following 3 simple rules: $n_{j+1} = n_j/2$ if n_j is *even*, $n_{j+1} = n_j/3$ if n_j is *odd and divisible by* 3, $n_{j+1} = 5n_j + 1$ if n_j is *odd and not divisible by* 3. The "$5n + 1$ Conjecture" states that, for *every* positive integer *initial* value n_0, this sequence ends up into the cycle $\{1, 6, 3, 1\}$. This conjecture is *not new*: it is in fact sometimes called the *Kakutani* $5n + 1$ *conjecture*, and it has been checked for all positive integers up to 10^{16}, see for instance the Chapter entitled *Empirical verification of the* $3x + 1$ *and related conjectures* by Tomás Oliveira e Silva (kindly made available to us by its author) in the book [7].

3.1 A modified version of the "$5n + 1$ Conjecture"

Note that, if we eliminate the *second* rule namely we do *not* divide n_j by 3 if n_j is divisible by 3, we find several cycles: (e.g. $\{3, 16, 8, 4, 2, 1, 6, 3\}$; $\{26, 13, 66, 33, 166, 83, 416, 208, 104, 52, 26\}$; $\{13, 66, 33, 166, 83, 416, 208, 104, 52, 26, 13\}$); while starting from other numbers (e.g. $7, 11$,...) the sequence *seems* to diverge.

4 The "$7n + 1$ Conjecture"

Consider the sequence of *positive integers* n_j $(j = 0, 1, 2, ...)$ generated by the following 4 simple rules: $n_{j+1} = n_j/2$ if n_j is *even*, $n_{j+1} = n_j/3$ if n_j is *odd and divisible by* 3, $n_{j+1} = n_j/5$ if n_j is *odd and divisible by* 5, $n_{j+1} = 7n_j + 1$ if n_j is *odd and not divisible by* 3 and 5. The "$7n + 1$ Conjecture" states that, for *every* positive integer *initial* value n_0, this sequence ends up into the cycle $\{1, 8, 4, 2, 1\}$. But also this conjecture is *not new*: it is called the *Kakutani* $7n + 1$ *conjecture*, and it has been checked for all positive integers up to 10^{14}, see again the Chapter entitled *Empirical verification of the* $3x + 1$ *and related conjectures* by Tomás Oliveira e Silva in [7].

4.1 A modified version of the "$7n + 1$ Conjecture"

Consider the sequence of *positive integers* n_j $(j = 0, 1, 2, ...)$ generated by the following 3 simple rules: $n_{j+1} = n_j/2$ if n_j is *even*, $n_{j+1} = n_j/3$ if n_j is *odd and divisible by* 3, $n_{j+1} = 7n_j + 1$ if n_j is *odd and not divisible by* 3 and 5. Then, for *every* positive integer *initial* value n_0, this sequence ends up into one of the *two* cycles $\{1, 8, 4, 2, 1\}$ and $\{19, 134, 67, 470, 235, 1646, 823, 5762, 2881, 20168, 10084,$
$5042, 2521, 17648, 8824, 4412, 2206, 1103, 7722, 3861, 1287, 429, 143,$
$1002, 501, 167, 1170, 585, 195, 65, 456, 228, 114, 57, 19\}$. We checked this statement (only) for all positive *integers* n_0 up to 10^9. This is the first case we found, of a conjecture, somewhat analogous to the Collatz conjectures, which seems to yield (*only*) 2 cycles.

5 The "$11n + 1$ Conjecture"

Following the same idea that had led us to write the "$5n+1$ Conjecture" and the "$7n+1$ Conjecture", one might be tempted to propose many quite analogous conjectures (based on *primes*); even stating that, starting from *any* positive integer, one always reaches a *unique* cycle containing 1. However, the following observation disproves this naive idea (although some variations of this idea might merit further exploration; see below).

Indeed, consider the sequence of *positive integer numbers* n_j $(j = 0, 1, 2, ...)$ generated by the following 5 simple rules: $n_{j+1} = n_j/2$ if n_j is *even*, $n_{j+1} = n_j/3$ if n_j is *odd and divisible by* 3, $n_{j+1} = n_j/5$ if n_j is *odd and divisible by* 5, $n_{j+1} = n_j/7$ if n_j is *odd and divisible by* 7, $n_{j+1} = 11n_j + 1$ if n_j is *odd and not divisible by* 3, 5 and 7. The "$11n + 1$ *Conjecture*" states that, for *every* positive integer *initial* value n_0, this sequence ends up into one of the (*only*) 2 cycles $\{1, 12, 6, 3, 1\}$ and $\{17, 188, 94, 47, 518, 259, 37, 408, 204, 102, 51, 17\}$. We checked its validity (only) for *all* positive integers n_0 up to 10^9.

5.1 The "Modified $11n + 1$ Conjecture"

Surprisingly, if we modify the "$11n + 1$ Conjecture" by eliminating the *fourth* rule, we seem to only get 1 cycle. Indeed, consider the sequence of *positive integer numbers* n_j ($j = 0, 1, 2, ...$) generated by the following 4 simple rules: $n_{j+1} = n_j/2$ if n_j is *even*, $n_{j+1} = n_j/3$ if n_j is *odd and divisible by* 3, $n_{j+1} = n_j/5$ if n_j is *odd and divisible by* 5, $n_{j+1} = 11n_j + 1$ if n_j is *odd and not divisible by* 3 and 5. The "Modified $11n + 1$ Conjecture" states that, for *every* positive integer *initial* value n_0, this sequence ends up into the cycle $\{1, 12, 6, 3, 1\}$. We checked this statement (only) for *all positive integers* n_0 up to 10^9.

6 Final remarks

Note that, for the "Modified $7n + 1$ conjecture" and the "$11n + 1$ conjecture", one seems to have *only* 2 cycles. So—assuming these conjectures are indeed *true*—this entails the possibility to divide, in 2 different ways, *all positive integers* in 2 classes, identified by the property to generate—when used as initial value—a sequence that eventually ends up into one of these 2 cycles.

The results mentioned above clearly suggest the possibility to introduce *many*—indeed, presumably an *infinity*—of *somewhat analogous* conjectures to those described above. A very interesting *open* issue is whether the *hypothesis* that one of these conjecture be *true* (or, alternatively, *false*) might be shown to imply *that another* one of these conjectures is *as well true* (or, alternatively, *false*); and whether progress in this direction might eventually provide an avenue to finally *prove* or *disprove* the validity of *some*, or even of *all*, these conjectures.

Let us end this paper by recalling the simple idea [5], that if the Collatz conjecture is *true*, then the 3-*times iterated Collatz procedure* (as described in **Section 2**) implies that *all* positive integers can be divided into 3 *distinct categories*: those such that, when taken as *initial values* of the sequence generated by that procedure, entail that it converge eventually to 1, or to 2 or to 4; hence the related *conjecture* —relevant of course only if the Collatz conjecture is true—that these 3 categories of integers are *equally* populated, a statement meant to be true in the same sense as the assertion that there are *as many even positive integers as odd positive integers* [5]. Clearly analogous ideas and conjectures can be associated to all the conjectures formulated above.

References

[1] See the item "Collatz conjecture" in Wikipedia, the References quoted there, and, if need be, the References quoted in those References (the full list would add up to *several hundred* items...).

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