Open Communications in Nonlinear Mathematical Physics]ocnmp[Special Issue 1, 2024 pp 1–11

This article is part of an OCNMP Special Issue in Memory of Professor Decio Levi

Soliton equations: admitted solutions and invariances via Bäcklund transformations

Sandra Carillo[1,2] *and Cornelia Schiebold*[3]

[1] *Dipartimento Scienze di Base e Applicate per l'Ingegneria*
"Sapienza" Università di Roma, 16, Via A. Scarpa, 00161 Rome, Italy

[2] *Gr. Roma1, IV - Mathematical Methods in NonLinear Physics*
National Institute for Nuclear Physics (I.N.F.N.), Rome, Italy

[2] *Department of Engineering, Mathematics and Science Education (IMD),*
Mid Sweden University, Sundsvall, Sweden

Received November 1, 2023; Accepted January 15, 2024

To Decio

This short note is dedicated to the memory of a friend who was an inspiration for me further to be an important scientist in soliton investigation. I will not mention the many important results he obtained in his long carreer but I prefer to mention a personal memory by S. Carillo:"Among the many occasions I was so lucky to share my time with Decio, I may recall that Decio introduced me to *who is doing what* in the field playing the *older brother* role with me. Thus, the first NEEDS conference I attended to in Balaruc, was really a key step in my career and, possibly, in my life. My deep gratitude and memory to Decio and his kind and measured style will never be forgotten."

Abstract

A couple of applications of Bäcklund transformations in the study of nonlinear evolution equations is here given. Specifically, we are concerned about third order nonlinear evolution equations. Our attention is focussed on one side, on proving a new invariance admitted by a third order nonlinear evolution equation and, on the other one, on the construction of solutions. Indeed, via Bäcklund transformations, a *Bäcklund chart*, connecting Abelian as well as non Abelian equations can be constructed. The importance of such a net of links is twofold since it indicates invariances as well as allows to construct solutions admitted by the nonlinear evolution equations it relates. The present study refers to third order nonlinear evolution equations of KdV type. On the basis of the Abelian wide Bäcklund chart which connects various different third order nonlinear evolution equations an invariance admitted by the *Korteweg-deVries interacting soliton* (int.sol.KdV) equation is obtained and a related new explicit solution is constructed. Then, the corresponding non-Abelian *Bäcklund chart*, shows how to construct matrix solutions of the mKdV equations: some recently obtained solutions are reconsidered.

1 Introduction

The relevance of Bäcklund transformations [34, 35] in the study of *soliton equations* is well known according to some of the most well known books concerned about them [1, 3, 40, 37, 39, 27]. In particular, the focus is on the Korteweg-de Vries (KdV) equation, one of the most studied third order nonlinear evolution equations, which, both in the scalar case [6, 5] as well as in the non-Abelian one, [14, 9, 8, 11], turns out to be connected to many other third order nonlinear evolution equations. Specifically, a novel invariance admitted by the KdV interacting soliton equation, introduced by Fuchssteiner [24] is proved. Furthermore, recent results concerned about operator equations which represent the non-Abelian counterpart of the *scalar* equations are reconsidered. A wide net of Bäcklund transformations, we termed Bäcklund chart [14, 9, 8, 11], relates the KdV equation, the potential Korteweg-de Vries (pKdV), the modified Korteweg-de Vries (mKdV), the KdV eigenfunction (KdV eig.) to the KdV singularity manifold equation (KdV sing.) [46]. The constructed Bäcklund chart represents a key tool to show invariances enjoyed by the equations it connects as well as to construct solutions they admit. Third order nonlinear evolution equations of KdV type are considered. Specifically, they are all connected via Bäcklund transformations. A *Bäcklund chart*, depicts the many links which connect the different nonlinear evolution equations under investigation. The latest Bäcklund chart is illustrated in [11]; its construction is directly related to results in [25, 5] further developped. Generalisations, in the case of noncommutative nonlinear evolution equations are comprised [14, 15, 9, 8] while a comparison between the two different cases Abelian and non-Abelian, respectively, is studied in [11].

The opening Section 2 is focussed on the KdV interacting soliton equation, as it was termed by Fuchssteiner [24]: it is, now, proved to admit a non trivial invariance which seems to be new. As a consequence, via such an invariance, we construct a family of stationary solutions, again new, admited by the KdV interacting soliton equation.

The subsequent Section 3, is devoted to the non-Abelian Bäcklund chart, constructed in [14, 9, 8], and, in particular, the matrix mKdV equation. Some solutions it admits are shown. Specifically, a general theorem obtained in [15] is applied to derive solutions in the case of the 2×2 matrix mKdV equation according to the results in [12, 17, 18].

In the closing Section 4 some perspectives and open problems are mentioned.

2 The KdV interacting soliton equation

This Section is devoted to the *KdV interacting soliton* equation, denoted, for short, as int. sol. KdV, is introduced by Fuchssteiner in [24]. This equation was then, in [25], connected to the KdV, mKdV and KdV sing. and, more recently, [5, 11] also to the KdV eigenfunction equation [32, 7]. Bäcklund transformations as well known can be applied to reveal new symmetry properties, as well as to construct solutions of non-linear evolution equations they connect. Both these viewpoints are adopted in the present short note. Hence, first of all the definition of Bäcklund transformation is recalled. Then, the connections among third order non-linear evolution equations are retrieved and, finally, an invariance, enjoyed by the int. sol. KdV equation, is readily constructed. Notably, such an invariance as well as the corresponding solution seem to be new. Indeed, the enjoyed invariance allows to construct a non trivial solution of the int. sol. KdV equation.

According to [25], the int. sol. KdV equation [1] equation is connected via Bäcklund transformations to the Korteweg deVries (KdV), the modified Korteweg deVries (mKdV), and the *Korteweg deVries singuarity manifold* (KdV sing.), introduced by Weiss in [46] via the *Painlevè test* of integrability.

[1]The int. sol. KdV equation appears also in [2] where third order nonlinear evolution equations and their linearizability are studied.

2.1 Invariance

In this subsection an invariance property enjoyed by the int.sol.KdV equation is proved. It can be trivially checked to be *scaling invariant* since on substitution of $\alpha s, \forall \alpha \in \mathbb{C}$, to s it remains unchanged. Remarkably, on application of results in [5, 11], the following further nontrivial invariances can be proved.

Proposition 2.1

The int.sol.KdV equation $s_t = s_{xxx} - 3\frac{s_x s_{xx}}{s} + \frac{3}{2}\frac{{s_x}^3}{s^2}$ *is invariant under the transformation*

$$\mathrm{I}: \quad \widehat{s} = \frac{as(cD^{-1}(s)+d)d - cs(aD^{-1}(s)+b)}{(cD^{-1}(s)+d)^2}, \quad a,b,c,d \in \mathbb{C} \ s.t.\ ad-bc \neq 0, \tag{1}$$

where D^{-1} *is chosen such that* $D \circ D^{-1}$ *is the identity.*[2]

The proof, according to [5], is based on the invariance under the Möbius group of transformations

$$\mathrm{M}: \quad \widehat{\varphi} = \frac{a\varphi + b}{c\varphi + d}, \qquad a,b,c,d \in \mathbb{C} \qquad \text{such that} \quad ad-bc \neq 0. \tag{2}$$

enjoyed by the KdV singularity manifold equation

$$\varphi_t = \varphi_x\{\varphi; x\}, \quad \text{where} \quad \{\varphi; x\} := \left(\frac{\varphi_{xx}}{\varphi_x}\right)_x - \frac{1}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2. \tag{3}$$

Combination of such an invariance with the connection between the KdV eigenfunction and the KdV singularity manifold equation allows to prove the proposition. Indeed, let

$$\mathrm{M}: \widehat{\varphi} = \frac{a\varphi + b}{c\varphi + d} \quad , \quad \forall a,b,c,d \in \mathbb{C}|\ ad-bc \neq 0 \tag{4}$$

then, the Bäcklund chart in Fig. 1, where the Bäcklund transformations B and $\widehat{\mathrm{B}}$ are, respectively:

$$\begin{tikzcd}
\boxed{\varphi_t \ = \ \varphi_x\{\varphi; x\}} \arrow[r, dash, "\mathrm{B}"] \arrow[d, leftrightarrow, "\mathrm{M}"] & \boxed{s^2 s_t = s^2 s_{xxx} - 3 s s_x s_{xx} + \frac{3}{2}{s_x}^3} \arrow[d, leftrightarrow, "\mathrm{I}"] \\
\boxed{\hat{\varphi}_t \ = \ \hat{\varphi}_x\{\hat{\varphi}; x\}} \arrow[r, dash, "\widehat{\mathrm{B}}"] & \boxed{\hat{s}^2 \hat{s}_t = \hat{s}^2 \hat{s}_{xxx} - 3 \hat{s}\hat{s}_x \hat{s}_{xx} + \frac{3}{2}\hat{s}_x^3}
\end{tikzcd}$$

Figure 1. Induced invariance Bäcklund chart.

$$\mathrm{B}: \quad s - \varphi_x = 0 \quad \text{and} \qquad \widehat{\mathrm{B}}: \quad \hat{s} - \hat{\varphi}_x = 0 \ ,$$

shows how the invariance I is constructed. Indeed, such an invariance follows via combination of the Möbius transformation M with the two Bäcklund transformations B and $\widehat{\mathrm{B}}$. An application of the invariance I indicates how to construct solutions of the KdV interacting soliton equation.

[2]Often one assumes that $s(x,t)$ belongs to the Schwartz space S of *rapidly decreasing functions* for each fixed t. Here $S(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} < \infty, \forall \alpha, \beta \in \mathbb{N}_0^n\}$, where $\|f\|_{\alpha,\beta} := sup_{x\in\mathbb{R}^n}\left|x^\alpha D^\beta f(x)\right|$, and $D^\beta := \partial^\beta/\partial x^\beta$; throughout this article $n = 1$. Then one may define D^{-1} by $D^{-1} := \int_{-\infty}^x d\xi$. In calculations other choices may be useful.

2.2 An admitted explicit solution

An example of a new solution admitted be the KdV interacting soliton equation is readily obtained starting from its invariance proved in the previous subsection. To construct it, it is easy to check that $s(x,t) = k, \forall k \in \mathbb{R}\backslash\{0\}$, represents a solution of the KdV interacting soliton equation

$$s_t = s_{xxx} - 3\frac{s_x s_{xx}}{s} + \frac{3}{2}\frac{{s_x}^3}{s^2} \ . \tag{5}$$

When, in the Möbius group we let the parameters be

$$a = d = 0, \ b = c = 1, \tag{6}$$

it follows, from the invariance I, that a further solution of the KdV interacting soliton equation is represented by

$$\widehat{s}(x,t) = -\frac{1}{kx^2} \ , \quad \forall k \in \mathbb{R}\backslash\{0\} \ . \tag{7}$$

Further solutions can be obtained in the same way. Notably, the same way of reasoning allows to constrcut solutions also in the non-Abelian corresponding case.

3 Non-Abelian case: solutions of mKdV equation

This section is devoted to non-Abelian equations. In [14, 15], operator equations, which can be considered as non-Abelian counterparts of third order nonlinear evolution equations of KdV type are studied and an extended Bäcklund chart is constructed [8]. The comparison between the Abelian and the non-Abelian Bäcklund chart [11] shows a richer structure in the non-commutative case. In particular, we consider the special case when the operator is finite dimensional so that it admits a matrix representation. Thus, the aim is to emphasise the importance of Bäcklund transformations also when solutions admitted by non-Abelian soliton equations are looked for. Solutions admitted by the matrix equations are a subject of interest in the literature. The study presented, based on previous results [14, 15] further developed in [12, 17], is consistent with multisoliton solutions of the matrix KdV equation obtained by Goncharenko [26], via a generalisation of the Inverse Scattering Method. Accordingly, Theorem 3 in [15] represents a generalisation of Goncharenko's multisoliton solutions. Solutions of the matrix mKdV equation obtained in [17] (motivated by [12]), where the solution formula in the case of a $d \times d$-matrix equation is presented, are reconsidered. Note that this is a particular case of the operator formula obtained in [15]. For further matrix solutions we refer to [21, 26, 36, 43, 44, 29, 42].

As an example, some 2×2-matrix solutions of the mKdV equation are recalled from [17]. They are contructed on application of the following theorem.

Theorem 3 ([13], see also [17])
For $N \in \mathbb{N}$, let $k_1, \ldots, k_N$ be complex numbers such that $k_i + k_j \neq 0$ for all i, j, and let $B_1, \ldots, B_N$ be arbitrary $\mathsf{d} \times \mathsf{d}$-matrices.

Define the $N\mathsf{d} \times N\mathsf{d}$-matrix function $L = L(x,t)$ as block matrix $L = (L_{ij})_{i,j=1}^N$ with the $\mathsf{d} \times \mathsf{d}$-blocks

$$L_{ij} = \frac{\ell_i}{k_i + k_j} B_j,$$

where $\ell_i = \ell_i(x,t) = \exp(k_i x + k_i^3 t)$.

Then

$$V = \begin{pmatrix} B_1 & B_2 & \ldots & B_N \end{pmatrix} \Big(I_{N\mathsf{d}} + L^2 \Big)^{-1} \begin{pmatrix} \ell_1 I_{\mathsf{d}} \\ \vdots \\ \ell_N I_{\mathsf{d}} \end{pmatrix}$$

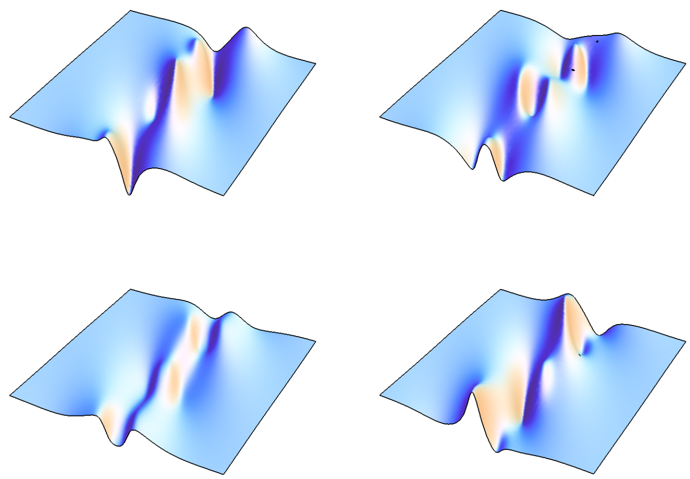

Figure 2. The solution depicted in the case $\mathsf{d} = 2$, $k_1 = 1+i$, $k_2 = \overline{k_1} = 1-i$, $B_1 = \begin{pmatrix} i & -2 \\ 1+i & 2-i \end{pmatrix}$, $B_2 = \overline{B_1}$, $-5 \leq x \leq 5$ and $0 \leq t \leq 2$, plot range $(-3.5, 3.5)$.

is a solution of the matrix modified KdV equation

$$V_t = V_{xxx} + 3\{V^2, V_x\} \quad , \quad \textit{where} \quad \{V^2, V_x\} := V^2 V_x + V_x V^2 \quad (\textit{anticommutator})$$

with values in the $\mathsf{d} \times \mathsf{d}$*-matrices on every domain* Ω *on which* $\det(I_{N\mathsf{d}} + L^2) \neq 0$*.*

Various kinds of solutions are obtained when complex parameters are considered. According to what happens in the case of the scalar mKdV equation, where the input data k, $\overline{k}$, b, $\overline{b}$ produce a breather[3], a bound state of a soliton and an antisoliton [47], also the matrix mKdV equation admits breather solutions.

Example 1 First examples are obtained when we set $k = 1 + \mathrm{i}$. For the corresponding scalar breather this implies velocity $= 2$, and hence the plots are drawn for $(x + 2t, t)$ giving a stationary picture.

To give an idea of some significant solutions admitted by matrix mKdV equation in the case $\mathsf{d} = 2$, in Fig. 2 and Fig. 3 the solution is depicted for the matrix parameters

a) Figure 2 $\quad B_1 = B = \begin{pmatrix} \mathrm{i} & -2 \\ 1+\mathrm{i} & 2-\mathrm{i} \end{pmatrix}, B_2 = \overline{B},$

b) Figure 3 $\quad B_1 = B = \begin{pmatrix} \mathrm{i} & -2\mathrm{i} \\ 3\mathrm{i}-1 & -1 \end{pmatrix}, B_2 = \overline{B}.$

Next we turn to the 2-soliton case with real input data k_1, k_2, b_1, b_2.

Example 2 Here we focus on the input data $N = 2$, $k_1 = 1$, $k_2 = \sqrt{2}$ in Theorem 3.

[3]Here $\overline{k}$ denotes the complex conjugate of k.

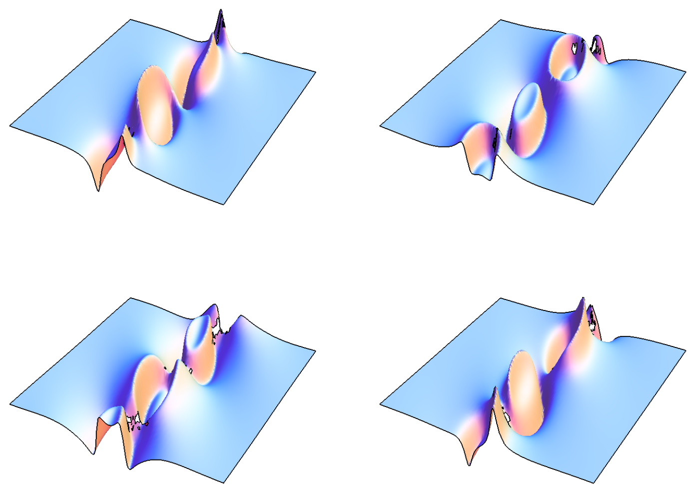

Figure 3. The solution depicted in the case $\mathsf{d} = 2$, $k_1 = 1+i$, $k_2 = \overline{k_1} = 1-i$, $B_1 = \begin{pmatrix} i & -2i \\ 3i-1 & -1 \end{pmatrix}$, $B_2 = \overline{B_1}$, $-5 \le x \le 5$ and $-1 \le t \le 1$, plot range $(-5.5, 5.5)$.

For comparison, first of all we depict, in Fig. 4, the *scalar* 2-soliton (i. e. the case $\mathsf{d} = 1$ in Theorem 3) generated with $b_1 = b_2 = 1$.

Then we turn to the case $\mathsf{d} = 2$, where in Fig. 5 and Fig. 6. the solutions are depicted generated with the matrix parameters

a) Figure 5 $\quad B_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$,

b) Figure 6 $\quad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Comments and observations:

- Obviously, solutions which correspond to the choice $k_1, \ldots, k_N \in \mathbb{R}$ and $B_1 = \ldots = B_N =: B$ (up to a common real multiple) where B is real, are real-valued.
- For $N = 1$, Theorem 3 gives

$$V = \Big(I_d + \big(\frac{1}{2k}\ell B\big)^2 \Big)^{-1} \ell B,$$

where I_d denotes the d-dimensional identity matrix.

An example of a solution with a real spectral matrix B which has complex Jordan form is given in [17]. Specifically, B is the rotation by the angle $-\frac{\pi}{4}$,

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

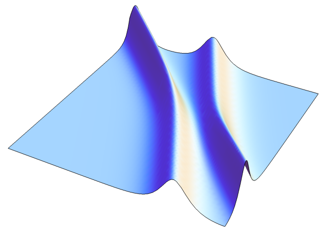

Figure 4. $\mathsf{d} = 1$ with the input data $N = 2$, $k_1 = 1$, $k_2 = \sqrt{2}$, and $b_1 = b_2 = 1$ in Theorem 3

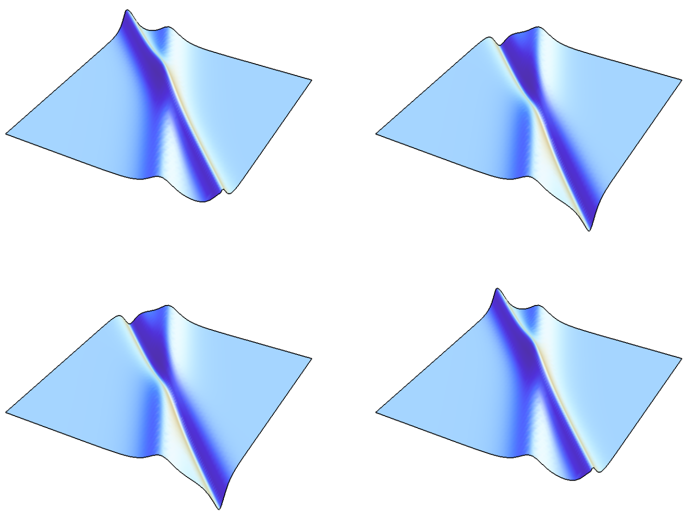

Figure 5. The solution depicted represents the case $\mathsf{d} = 2$, $k_1 = 1$, $k_2 = \sqrt{2}$, $B_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, when $-10 \leq x \leq 10$ and $-5 \leq t \leq 5$, plot range $(-\sqrt{2}, \sqrt{2})$.

- The solutions depicted in Fig. 2 - Fig. 3 and Fig. 5 - Fig. 6 provide only some examples of the wide variety of solutions which are covered by Theorem 3.
- In [18] first steps towards an asymptotic study of solutions from Theorem 3 are given in the case $N = 2$, see also [45].

4 Conclusions and perspectives

To complete our work we mention some of the themes which deserve to be further investigated. In particular, as we already pointed out, the interest on Bäcklund transformations is twofold since they allow to reveal new connections among equations, as well as they indicate a way to construct new solutions. A natural extension of the connections via Bäcklund transformations is represented

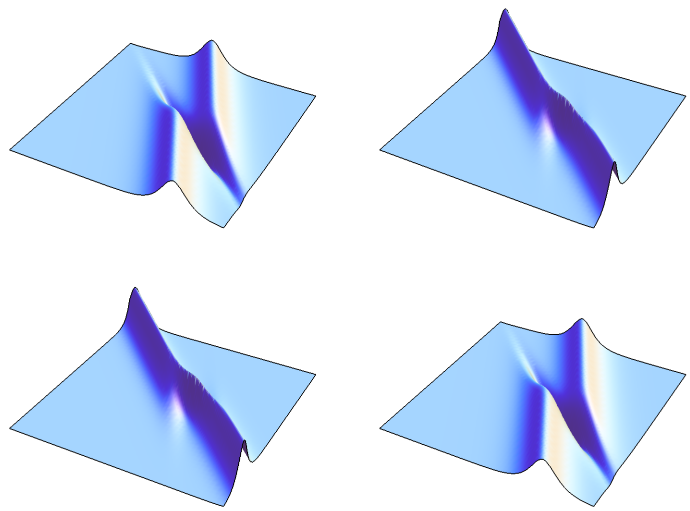

Figure 6. The solution depicted in the case $\mathsf{d}=2$, $k_1=1$, $k_2=\sqrt{2}$, $B_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $B_2=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, when $-10\le x\le 10$ and $-5\le t\le 5$, plot range $(-\sqrt{2},\sqrt{2})$.

by the extension to hierarchies. This aspect relies on the knowledge of the recursion operator admitted by at least one of the equations which appear in the Bäcklund chart. Then, according to [25] also in the non-Abelian case [14, 9, 8, 11], it follows that all the equations in the Bäcklund chart admit a recursion operator: it can be obtained from the known recursion operator via Bäcklund transformations. Hence, the same Bäcklund chart follows to link the hierarchies generated by the recursion operator applied to the considered nonlinear evolution equations. Indeed, the algebraic properties which characterise a hereditary recursion operator are preserved under Bäcklund transformations as well known [23, 22]. Notably, the involved algebraic properties are preserved via Bäcklund transformations also when non-Abelian nonlinear evolution equations are studied [14].

Indeed, promising perspectives as well as open problems can arise in the study of higher order nonlinear evolution equations.

Notably, the Bäcklund chart connecting 3rd oder (scalar) nonlinear evolution equations as well as the corresponding hierarchies [25], admit a non-Abelian counterpart. Such a non-Abelian Bäcklund chart is in [14, 15] with extensions in [9, 8]. As testified by the study on non-Abelian Burgers equation [33, 28, 30, 16, 10], no matter which is the order of the nonlinear evolution equations, the links established for the base members naturally extend to the corresponding whole hierarchies. In particular, the non-Abelian Burgers Bäcklund chart exhibits a structure which is richer than the corresponding Abelian one.

A Bäcklund chart [38, 6], connects the Caudrey-Dodd-Gibbon-Sawata-Kotera and Kaup-Kupershmidt hierarchies [20, 41, 31]. All the involved equations are 5th order nonlinear evolution equations; notably, the Bäcklund chart linking them all shows an impressive resemblance to the one connecting KdV-type equations. Again, such Bäcklund chart can be extended to the corresponding whole hierarchies [38, 6]. Some preliminary results are given in [19].

Acknowledgements

The financial support of Gr. Roma1, IV - Mathematical Methods in NonLinear Physics National Institute for Nuclear Physics (I.N.F.N.), Rome, Italy, of Sapienza University of Rome, Italy and of the National Mathematical Physics Group (G.N.F.M.) - I.N.d.A.M., is gratefully acknowledged.

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