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Hamiltonian and recursion operators for a discrete analogue of the Kaup-Kupershmidt equation

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Abstract

In this paper we study the algebraic properties of a discrete analogue of the Kaup-Kupershmidt equation. This equation can be seen as a deformation of the modified Narita-Itoh-Bogoyavlensky equation and has the Kaup-Kupershmidt equation in its continuous limit. Using its Lax representation we explicitly construct a recursion operator for this equation and prove that it is a Nijenhuis operator. Moreover, we present the bi-Hamiltonian structures for this equation.

1 Introduction

In a recent paper [1], Mikhailov, Novikov and Wang formulated necessary integrability conditions for evolutionary differential-difference equations (DΔEs) using the perturbative symmetry approach and obtained a family of new integrable DΔEs:

$$u_t = (1 + u^2) \left(u_p \prod_{k=1}^{p-1} \sqrt{1 + u_k^2} - u_{-p} \prod_{k=1}^{p-1} \sqrt{1 + u_{-k}^2} \right), \quad p \in \mathbb{N}, \quad (1)$$

where $u = u(n, t)$ is a complex function of one discrete variable $n \in \mathbb{Z}$ and one continuous independent variable t . Here we use the standard notations

$$u_t = \partial_t(u), \quad u_k = \mathcal{S}^k u(n, t) = u(n + k, t), \quad k \in \mathbb{Z}$$

and \mathcal{S} is the shift operator. Equation (1) can be viewed as an inhomogeneous deformation of the Narita-Itoh-Bogoyavlensky (NIB) equation [2]. Indeed, after the re-scaling

$$u_k \rightarrow \varepsilon^{-1} u_k, \quad t \rightarrow \varepsilon^{p+1} t \quad (2)$$

and in the limit $\varepsilon \rightarrow 0$ it turns into

$$u_t = u^2 \left(\prod_{k=1}^p u_k - \prod_{k=1}^p u_{-k} \right). \quad (3)$$

The authors of [1] have proved the integrability of (1) for any $p \in \mathbb{N}$ by constructing its Lax representation and provided the r matrix approach to produce the hierarchy of commuting symmetries. They showed that equation (1) possesses a Lax representation $L_t = [A, L]$ with a rational (pseudo-difference) operator $L = Q^{-1}P$, where

$$Q = u - u_1 w \mathcal{S}^{-1}, \quad P = (u w_1 \mathcal{S} - u_1) \mathcal{S}^{p-1}, \quad w = \sqrt{1 + u^2}, \quad (4)$$

and a skew-symmetric difference operator $A = L_+ - (L_+)^{\dagger}$. Here L_+ denotes the polynomial part of the Laurent series for the rational operator L and \dagger denotes the adjoint of an operator. The hierarchy of commuting symmetries for (1) can be generated by

$$L_{t_l} = [\pi_+(L^l), L], \quad \pi_+(L^l) = \left(L^l \right)_+ - \left(\left(L^l \right)_+ \right)^{\dagger}, \quad l \in \mathbb{N}, \quad (5)$$

where $t_1 = t$ corresponds to the equation itself.

For every member p of the family (1), if we use the substitution

$$\begin{aligned} u(n, t) &= \frac{2\sqrt{p}}{p+1} i - \frac{(p-1)^2}{12(p+1)} \sqrt{p} i \epsilon^2 U(x + A(p)\epsilon t, \tau + B(p)\epsilon^5 t), \quad x = \epsilon n, \\ A(p) &= -2p \left(\frac{p-1}{p+1} \right)^p, \quad B(p) = \frac{p^3(p^2-1)}{180} \left(\frac{p-1}{p+1} \right)^p, \end{aligned} \quad (6)$$

where $i^2 = -1$ is the imaginary unit, we always obtain the Kaup-Kupershmidt equation

$$U_{\tau} = 5U^2 U_x + \frac{25}{2} U_x U_{xx} + 5U U_{xxx} + U_{xxxx}$$

for a function $U(\tau, x)$ as $\epsilon \rightarrow 0$. This is analogous to what happens for the NIB family, whose members have a continuous limit in the Korteweg-de Vries equation.

This paper is devoted to the study of recursion operators and Hamiltonian structures of the equation (1) for the case when $p = 2$, that is,

$$u_t = (1 + u^2)(u_2 \sqrt{1 + u_1^2} - u_{-2} \sqrt{1 + u_{-1}^2}), \quad (7)$$

which was first obtained in the symmetry classification of a class of five-point differential-difference equations [3]. Its integrability was established in [4], where Garifullin and Yamilov constructed a Lax representation for (7), which is in a different form from (4) when $p = 2$ (see Remark 3 in [1] for the comparison). As shown also in [4], the continuous limit of (7) is the Kaup-Kupershmidt equation using the substitution

$$u(n, t) = \frac{2\sqrt{2}}{3} i - \frac{i}{18\sqrt{2}} \epsilon^2 U \left(x - \frac{4}{9} \epsilon t, \tau + \frac{2}{135} \epsilon^5 t \right), \quad x = \epsilon n,$$

which is the case of (6) when $p = 2$.

From the formulas (5), we are able to explicitly compute its second member of the hierarchy for $p = 2$, namely,

$$u_{t_2} = w^2(u_4w_1w_2^2w_3 + u_1u_2u_3w_1w_2 + uu_2^2w_1^2 + u_{-1}u_1u_2ww_1 - u_{-2}u_{-1}u_1w_{-1}w - u_{-2}^2uw_{-1}^2 - u_{-3}u_{-2}u_{-1}w_{-2}w_{-1} - u_{-4}w_{-3}w_{-2}^2w_{-1}), \quad (8)$$

which can also be obtained via $u_{t_2} = \mathfrak{R}(u_t)$. Here \mathfrak{R} is defined by

$$\begin{aligned} \mathfrak{R} &= \mathfrak{R}^{(1)} + \mathfrak{R}^{(3)}, \quad (9) \\ \mathfrak{R}^{(1)} &= w^2 \left((u_1 - \mathcal{S}^{-1}u) (\mathcal{S}w - w\mathcal{S}^{-1})^{-1} (u\mathcal{S} + u_1) + \mathcal{S}w\mathcal{S} + \mathcal{S}^{-1}w\mathcal{S}^{-1} \right) \frac{1}{w^2}; \\ \mathfrak{R}^{(3)} &= w^2 \left((u_1 - \mathcal{S}^{-1}u) (\mathcal{S}w - w\mathcal{S}^{-1})^{-1} (u\mathcal{S} - u_1) + \mathcal{S}w\mathcal{S} - \mathcal{S}^{-1}w\mathcal{S}^{-1} \right) \\ &\quad u (\mathcal{S}^2 + \mathcal{S} + 1) (\mathcal{S}^2 - 1)^{-1} \frac{u}{w^2} \end{aligned}$$

In this paper we are going to prove that the operator \mathfrak{R} is a Nijenhuis recursion operator for (7). Note that under the same rescaling (2) and the limit in ϵ , the operator (9) goes to

$$u^2\mathcal{S}^{-1}(\mathcal{S}^3 - 1)u(\mathcal{S}u - u\mathcal{S}^{-1})^{-1}u(1 - \mathcal{S}^{-3})\mathcal{S}u(\mathcal{S}^2 + \mathcal{S} + 1)(u\mathcal{S}^2 - u)^{-1},$$

which is the recursion operator of (3) for $p = 2$ in [5] reflecting the fact that equation (7) is its inhomogeneous deformation. The derivation of \mathfrak{R} is based on the structures of Lax representations for the symmetry hierarchy proposed in [6] for partial differential equations. This approach was soon adapted for D Δ E in [7] and further developed in [5, 8] taking into account the reduction groups of Lax representations [9]. The detailed derivation will be described in Section 3, where we also prove that the operator \mathfrak{R} is Nijenhuis since it can be written as a rational operator of a preHamiltonian pair [10, 11]. In the same section, we represent the recursion operator in a new form involving the action of the reflection operator \mathcal{T} , which is also an automorphism defined by $\mathcal{T}u_n = u_{-n}$, that is,

$$\begin{aligned} \mathfrak{R}' &= w^2(1 - \mathcal{T}) \left[u_1 (\mathcal{S}w - w\mathcal{S}^{-1})^{-1} \left((u\mathcal{S} + u_1) \frac{1}{u} (\mathcal{S}^2 - 1) + uu_1 (\mathcal{S}^3 - 1) \right) + \right. \\ &\quad \left. + w_1\mathcal{S}^2 \left(\frac{1}{u} (\mathcal{S}^2 - 1) + u (\mathcal{S}^2 + \mathcal{S} + 1) \right) \right] (\mathcal{S}^2 - 1)^{-1} \frac{u}{w^2}. \quad (10) \end{aligned}$$

The presence of the operator $(1 - \mathcal{T})$ in the expression above reveals that all the symmetries in the hierarchy are odd with respect to \mathcal{T} . Mathematically, operators \mathfrak{R} and \mathfrak{R}' are not equal while they give the same values on the image of $(1 - \mathcal{T})$. Thus they generate the same hierarchy of symmetries starting from the equation itself (7).

In section 4, we explicitly present a Hamiltonian operator

$$\mathcal{H} = w^2 (\mathcal{S}w\mathcal{S} - \mathcal{S}^{-1}w\mathcal{S}^{-1}) w^2 + w^2 (u_1 - \mathcal{S}^{-1}u) (\mathcal{S}w - w\mathcal{S}^{-1})^{-1} (u\mathcal{S} - u_1)w^2. \quad (11)$$

Equation (7) is Hamiltonian with respect to it. To prove that \mathcal{H} is Hamiltonian in theorem 4.1, we apply the formalism of rational and pre-Hamiltonian operators described in [10, 12]. Combining the expression of the Hamiltonian operator \mathcal{H} and the Nijenhuis recursion

operator \mathfrak{R} we show that $\mathcal{K} = \mathfrak{R}\mathcal{H}$ is also a Hamiltonian operator compatible with \mathcal{H} . Recursively we can prove the same result for all $\mathfrak{R}^k\mathcal{H}$, $k \in \mathbb{N}$. To increase the readability, we begin with recalling some basic definitions such as symmetries, recursion operators and Hamiltonian operators for integrable DΔEs, and end with a short discussion on the relevant open problems.

2 Basic definitions for differential-difference equations

In this section we will give the definitions for some basic concepts such as symmetries, recursion operators and Hamiltonian operators for integrable DΔEs. By no means this is a complete overview of it. We only mention the properties and theorems that we are going to use later. More details on these concepts and related theories for both differential and differential-difference equations can be found in [13, 14] and a recent book [15].

Let u be a function of a discrete variable $n \in \mathbb{Z}$ and of a continuous variable t . An evolutionary differential-difference equation of dependent variable u is of the form

$$u_t = f(u_p, \dots, u_q), \quad p \leq q, \quad p, q \in \mathbb{Z}. \quad (12)$$

It is an abbreviated form to encode the infinite sequence of ordinary differential systems of equations

$$\partial_t u(n, t) = f(u(n+p, t), \dots, u(n+q, t)), \quad n \in \mathbb{Z}.$$

For convenience we drop the subscript when it is zero as $u_0 = u$. We say function f is of order (p, q) if $\partial_{u_p} f \neq 0$ and $\partial_{u_q} f \neq 0$. The difference between p and q , $q - p$, is referred to as the total order of f . For example, equation (7) is of order $(-2, 2)$ with total order 4.

All such functions, depending on a finite number of shifts of u , form a difference ring denoted \mathcal{A} with the shift operator \mathcal{S} as its automorphism defined by

$$\mathcal{S} : a(u_p, \dots, u_q) \mapsto a(u_{p+1}, \dots, u_{q+1}), \quad \mathcal{S} : \alpha \mapsto \alpha; \quad a \in \mathcal{A}, \alpha \in \mathbb{C}.$$

The reflection \mathcal{T} of the lattice \mathbb{Z} defined by

$$\mathcal{T} : a(u_p, \dots, u_q) \mapsto a(u_{-p}, \dots, u_{-q}), \quad \mathcal{T} : \alpha \mapsto \alpha; \quad a \in \mathcal{A}, \alpha \in \mathbb{C} \quad (13)$$

is another automorphism of \mathcal{A} . The composition $\mathcal{S}\mathcal{T}\mathcal{S}\mathcal{T} = \text{Id}$ is the identity map.

On a difference ring \mathcal{A} we can define derivations. A derivation (vector field) is said to be evolutionary if it commutes with the shift operator \mathcal{S} . Such vector field is completely determined by a function $g \in \mathcal{A}$. We call it the characteristic of the vector field. Evolutionary equation (12) corresponds to an evolutionary vector field of characteristic f .

For any two evolutionary vector fields with characteristics f and g , we define a Lie bracket as follows

$$[f, g] = g_*(f) - f_*(g),$$

where $f_* = \sum_{j \in \mathbb{Z}} \frac{\partial f}{\partial u_j} \mathcal{S}^j$ is the Fréchet derivative of a function f . We say that an evolutionary vector field with characteristic $g \in \mathcal{A}$ is a symmetry of equation (12) if and only

if $[f, g] = 0$. If an equation possesses infinitely many commuting higher order symmetries, it is said to be integrable.

Often the symmetries of integrable equations can be generated by recursion operators [16]. Roughly speaking, a recursion operator is a linear operator \mathfrak{R} mapping a symmetry to a new symmetry. For an evolutionary equation (12), it satisfies

$$\mathfrak{R}_*[f] = [f_*, \mathfrak{R}], \quad (14)$$

where $\mathfrak{R}_*[f]$ is the Fréchet derivative of \mathfrak{R} along the evolutionary vector field of characteristic f .

Recursion operators for nonlinear integrable equations are often Nijenhuis operators, that is,

$$[\mathfrak{R}a, \mathfrak{R}b] - \mathfrak{R}[\mathfrak{R}a, b] - \mathfrak{R}[a, \mathfrak{R}b] + \mathfrak{R}^2[a, b] = 0, \quad \text{for all } a, b \in \mathcal{A}. \quad (15)$$

The Nijenhuis property ensures that recursion operators generate commuting symmetries.

We now introduce briefly the concept of Hamiltonian operators for DΔE. For any element $h \in \mathcal{A}$, we define an equivalent class (or a functional) $\int h$ by saying that two elements $h, g \in \mathcal{A}$ are equivalent if $h - g \in \text{Im}(\mathcal{S} - 1)$. The space of functionals is denoted by \mathcal{A}' , which does not inherit a ring structure of \mathcal{A} . For any functional $\int h \in \mathcal{A}'$ (simply written $h \in \mathcal{A}'$ without confusion), we define its difference variational derivative (Euler operator) denoted by $\delta_u f$ as

$$\delta_u f = \sum_{i \in \mathbb{Z}} \mathcal{S}^{-i} \frac{\partial}{\partial u_i} f.$$

We say an evolutionary equation (12) is a Hamiltonian equation if there exists a Hamiltonian operator \mathcal{H} and a Hamiltonian $h \in \mathcal{A}'$ such that

$$u_t = \mathcal{H} \delta_u h.$$

A difference operator \mathcal{H} is called a Hamiltonian if the bracket defined on \mathcal{A}' by $\{\int f, \int g\} = \int \delta_u f \mathcal{H} \delta_u g$ is a Lie bracket, that is, the bracket is skew-symmetric and satisfies the Jacobi identity. Often Hamiltonian operators are rational instead of local. We will follow the pre-Hamiltonian formalism developed in [10, 12] to give the definition.

A difference operator A is said to be pre-Hamiltonian if there exists a two form on \mathcal{A} denoted by ω_A such that

$$A_*[Aa](b) - A_*[Ab](a) = A(\omega_A(a, b)), \quad \text{for all } a, b \in \mathcal{A}. \quad (16)$$

The Fréchet derivative of a difference operator $A(\cdot)$ is a bi-difference operator $A_*\cdot$, by convention we indicate with round parenthesis the argument of the operator itself and with square bracket the direction of the Fréchet derivative determined by the characteristic of the evolutionary vector field.

Let $\mathcal{H} = AB^{-1}$ be a skew-symmetric rational operator, where both A and B are difference operators. We say that \mathcal{H} is Hamiltonian if A is pre-Hamiltonian with ω_A determined by (16) and B satisfies

$$B_*[Aa](b) - B_*[Ab](a) + B_*(a)^\dagger[Ab] + A_*(a)^\dagger[Bb] = B(\omega_A(a, b)), \quad (17)$$

where A^\dagger is the adjoint of the difference operator A . Given two pre-Hamiltonian operators A, B they form a pre-Hamiltonian pair (A, B) if any linear combination $A + \zeta B$ of them is a pre-Hamiltonian operator for any number $\zeta \in \mathbb{C}$.

Pre-Hamiltonian operators are also closely linked with Nijenhuis operators. It has been proven in [10, 11] that any Nijenhuis operator that can be written in a minimal rational decomposition $\mathfrak{R} = AB^{-1}$ with B pre-hamiltonian generates a pre-Hamiltonian pair (A, B) and, vice-versa, any pre-hamiltonian pair (A, B) generates a Nijenhuis operator as $\mathfrak{R} = AB^{-1}$. This provides an efficient way to prove that a certain rational operator is Nijenhuis by checking if its factors form a pre-Hamiltonian pair, instead of vanishing of the Nijenhuis torsion (15).

3 Construction of recursion operators

For an integrable evolutionary equation, if it is known its Lax representation, there is a powerful approach to construct its recursion operator [6]. This idea has been extended for Lax pairs that are invariant under the reduction groups [5, 8]. In this section, we use the same idea to construct a recursion operator of an equation (7) from its Lax representation.

We rewrite the scalar Lax representation (4) for $p = 2$ in the matrix form with the auxiliary system

$$\mathcal{S}\Phi = U(\lambda)\Phi \quad \Phi_t = V(\lambda)\Phi,$$

where $\lambda \in \mathbb{C}$ is the spectral parameter and

$$U(\lambda) = \begin{pmatrix} \frac{u_1}{uw_1} & \frac{\lambda}{w_1} & -\lambda\frac{u_1w}{uw_1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (18a)$$

$$V(\lambda) = \begin{pmatrix} \lambda + \frac{u_2w_1}{u} & -uu_1 & -\left(w + \lambda\frac{u_2ww_1}{u}\right) \\ \frac{u_1w^2}{u} + \frac{1}{\lambda}\frac{u_{-1}w}{u} & \lambda - \frac{1}{\lambda} & -\left(\frac{u_{-1}w^2}{u} + \lambda\frac{u_1w}{u}\right) \\ w + \frac{1}{\lambda}\frac{u_{-2}ww_{-1}}{u} & uu_{-1} & -\left(\frac{1}{\lambda} + \frac{u_{-2}w_{-1}}{u}\right) \end{pmatrix}. \quad (18b)$$

The compatibility condition of $U(\lambda)$ and $V(\lambda)$, simply writing as U and V , leads to

$$U_t = \mathcal{S}(V)U - UV.$$

Notice that matrices $U(\lambda)$ and $V(\lambda)$ are invariant with respect to the transformation:

$$V(\lambda) = -\mathcal{J}TV(\lambda^{-1})\mathcal{T}\mathcal{J}, \quad \mathcal{S}^{-1}U^{-1}(\lambda)\mathcal{S} = \mathcal{J}TU(\lambda^{-1})\mathcal{T}\mathcal{J}, \quad (19)$$

where the reflection operator \mathcal{T} is defined by (13). The symbol \mathcal{J} stands for a ‘‘reflected’’ identity matrix

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

which is sometimes called the backward identity or exchange matrix. The transformation (19) reflects the symmetry $\mathcal{T}(u_t) = -u_t$ of the equation (7). The primitive automorphic function of the group generated by the transformation $\lambda \mapsto \lambda^{-1}$ is

$$\mu(\lambda) = \lambda + \lambda^{-1} \quad (20)$$

For a given matrix U , we can build up a hierarchy of nonlinear systems by choosing a matrix V' with the degree of λ from $-l$ to l . The way to construct a recursion operator directly from a Lax representation is to relate the different operators \bar{V}' using the ansatz

$$\bar{V}' = \mu(\lambda)V' + B$$

and then to find the relation between the two flows corresponding to \bar{V}' and V' . Here B is the remainder and we assume that it has the same symmetry as V , and thus is of the form:

$$B = \begin{pmatrix} b_{11}^{(1)} & b_{12}^{(1)} & b_{13}^{(1)} \\ b_{21}^{(1)} & b_{22}^{(1)} & b_{23}^{(1)} \\ b_{31}^{(1)} & b_{32}^{(1)} & b_{33}^{(1)} \end{pmatrix} \lambda + \begin{pmatrix} b_{11}^{(0)} & b_{12}^{(0)} & b_{13}^{(0)} \\ b_{21}^{(0)} & b_{22}^{(0)} & -\mathcal{T}b_{21}^{(0)} \\ -\mathcal{T}b_{13}^{(0)} & -\mathcal{T}b_{12}^{(0)} & -\mathcal{T}b_{11}^{(0)} \end{pmatrix} - \begin{pmatrix} \mathcal{T}b_{33}^{(1)} & \mathcal{T}b_{32}^{(1)} & \mathcal{T}b_{31}^{(1)} \\ \mathcal{T}b_{23}^{(1)} & \mathcal{T}b_{22}^{(1)} & \mathcal{T}b_{21}^{(1)} \\ \mathcal{T}b_{13}^{(1)} & \mathcal{T}b_{12}^{(1)} & \mathcal{T}b_{11}^{(1)} \end{pmatrix} \lambda^{-1} \quad (21)$$

We need to stress that, due to the action of the reduction group, there is one more relation we should consider

$$b_{22}^{(0)} = -\mathcal{T}b_{22}^{(0)}. \quad (22)$$

Assume that V' associates to the symmetry flow u_τ and \bar{V}' to the flow u_t . The compatibility condition leads to

$$U_t = \mu(\lambda)U_\tau + S(B)U - UB. \quad (23)$$

When we apply these assumptions to (23), we get a over-determined system counting 34 equations of 14 parameters. We proceed considering some of these equations to find a representation of \mathfrak{R} and at the same time checking the others as the compatibility conditions to ensure the existence of a solution, which leads to the following statement:

Theorem 3.1. *A recursion operator of the equation (7) is given by (9).*

Proof. We consider the ansatz in (21) and apply it to (23). The result is a matrix system involving the spectral parameter λ , that allow us to decompose it even further considering that every coefficient of λ needs to be equal to zero independently. By inspection, some of these relations are trivial. We notice that two parameters are directly zero

$$b_{21}^{(1)} = 0 \quad b_{31}^{(1)} = 0 \quad (24)$$

and some are connected with one to one relations, such as

$$b_{11}^{(1)} = \mathcal{S}b_{22}^{(1)} \quad b_{21}^{(1)} = \mathcal{S}b_{32}^{(1)} = 0. \quad (25)$$

We consider (23) as the matrix equation $\Delta_{ij} = 0$. Notice that, because of the structure of U , all the equations coming from $i = 2, 3$ and $j = 1, 2, 3$ do not involve t -derivative and they can be used to reduce the number of parameters.

First we express all parameters in B in terms of 4 parameters, namely, $b_{11}^{(1)}, b_{12}^{(1)}, b_{33}^{(1)}$ and $b_{22}^{(0)}$, and we have

$$b_{11}^{(1)} = (\mathcal{S}^{-2} - 1)^{-1} \frac{u}{w^2} u_\tau. \quad (26)$$

We then find $b_{33}^{(1)}$ and $b_{12}^{(1)}$ in terms of $b_{11}^{(1)}$ as follows:

$$b_{33}^{(1)} = - \left(\frac{1}{u^2} (\mathcal{S}^2 - 1) + \mathcal{S}^2 + \mathcal{S} \right) \mathcal{S}^{-2} b_{11}^{(1)} \quad (27)$$

$$(\mathcal{S}w - w\mathcal{S}^{-1}) b_{12}^{(1)} = uu_1 \left((\mathcal{S} + 1) \frac{1}{u^2} (\mathcal{S}^2 - 1) + (\mathcal{S}^3 - 1) \right) \mathcal{S}^{-2} b_{11}^{(1)} \quad (28)$$

Finally, we analyse the equations involving u_t and get

$$\begin{aligned} \frac{1}{w^2} u_t = & -u (\mathcal{S}^2 + \mathcal{S} + 1) \mathcal{S}^{-1} b_{22}^{(0)} - \left((\mathcal{S}w\mathcal{S} + \mathcal{S}^{-1}w\mathcal{S}^{-1}) u^{-1} (\mathcal{S}^2 - 1) + \right. \\ & \left. + (\mathcal{S}w\mathcal{S} - \mathcal{S}^{-1}w\mathcal{S}^{-1}) u (\mathcal{S}^2 + \mathcal{S} + 1) \right) \mathcal{S}^{-2} b_{11}^{(1)} - (u_1 - \mathcal{S}^{-1}u) b_{12}^{(1)}, \end{aligned} \quad (29)$$

where the only unknown parameter left is $b_{22}^{(0)}$ satisfies

$$\begin{aligned} ((\mathcal{S}^2 + \mathcal{S} + 1) u^2 + \mathcal{S}^2 - 1) b_{22}^{(0)} = & \frac{u_1 w}{u} \mathcal{S}^{-1} (\mathcal{S}w - w\mathcal{S}^{-1}) b_{12}^{(1)} \\ & - \frac{u_1 w}{u} \mathcal{S}^{-1} \left(u_1 u \left(\mathcal{S} \frac{w^2}{u^2} - \frac{w^2}{u^2} \mathcal{S}^{-2} \right) + \left(\frac{u_1}{u} - \frac{u}{u_1} \mathcal{S}^{-1} \right) \right) b_{11}^{(1)}. \end{aligned} \quad (30)$$

Substituting (28) into (30), we obtain its right-handed side equals zero, which leads to

$$b_{22}^{(0)} = 0.$$

We now substitute it together with (26) and (28) into (29), which leads to the recursion operator (9) as stated. \blacksquare

The recursion operator (9) can be written in a form involving the reflection operator \mathcal{T} : we start with an equation for u_t containing the \mathcal{T} -reflection operator, different from (29) and we then solve all the other equations in the same way as before. In this way, we obtain the following form which is equivalent to (9):

$$\begin{cases} u_t = w^2(1 - \mathcal{T})u_1 \left(b_{12}^{(1)} + \frac{u_2 w_1}{u_1} \left(\frac{1}{u^2} (\mathcal{S}^2 - 1) + \mathcal{S}^2 + \mathcal{S} + 1 \right) b_{11}^{(1)} \right) \\ b_{12}^{(1)} = (\mathcal{S}w - w\mathcal{S}^{-1})^{-1} uu_1 \left((\mathcal{S} + 1) \frac{1}{u^2} (\mathcal{S}^2 - 1) + \mathcal{S}^3 - 1 \right) \mathcal{S}^{-2} b_{11}^{(1)} \\ b_{11}^{(1)} = (\mathcal{S}^{-2} - 1)^{-1} \frac{u}{w^2} u_\tau \end{cases}$$

which leads to (10). It is obvious that the whole hierarchy is anti-symmetric (odd) under \mathcal{T} operator as $\mathfrak{R} = w^2(1 - \mathcal{T})\hat{\mathfrak{R}}$, where $\hat{\mathfrak{R}}$ is a rational operator of \mathcal{S} . The presence of \mathcal{T} prevents us from directly exploiting a series of theorems proved for rational operators of \mathcal{S} in [10]. On the other hand, this alternative and more compact form for the recursion operator shows an important property of the symmetries that they are all odd with respect to \mathcal{T} .

One of important properties of recursion operators is to generate an infinite hierarchy of commuting symmetries, which is guaranteed by their Nijenhuis property (15). To show that \mathfrak{R} defined by (9) is a Nijenhuis operator, instead of directly checking the vanishing of the Nijenhuis torsion, we use the pre-Hamiltonian formalism, in particular Theorems 3 in [10]: if two difference operators A and B form a preHamiltonian pair, then their ratio AB^{-1} is Nijenhuis.

Theorem 3.2. *The recursion operator \mathfrak{R} in (9) is a Nijenhuis operator*

Proof. The recursion operator in (9) can be rewritten as follows:

$$\mathfrak{R} = w^2 (A + BC^{-1}D) E^{-1} \frac{1}{w^2}, \quad (31)$$

where each symbol above corresponds to a difference operator

$$\begin{aligned} A &= \left((\mathcal{S}w\mathcal{S} + \mathcal{S}^{-1}w\mathcal{S}^{-1}) \frac{1}{u} (\mathcal{S}^2 - 1) + (\mathcal{S}w\mathcal{S} - \mathcal{S}^{-1}w\mathcal{S}^{-1}) u (\mathcal{S}^2 + \mathcal{S} + 1) \right) \\ B &= (u_1 - \mathcal{S}^{-1}u) \\ C &= (\mathcal{S}w - w\mathcal{S}^{-1}) \\ D &= uu_1 \left((\mathcal{S} + 1) \frac{1}{u^2} (\mathcal{S}^2 - 1) + \mathcal{S}^3 - 1 \right) \\ E &= \frac{1}{u} (\mathcal{S}^2 - 1) \end{aligned} \quad (32)$$

From this setting, it is possible to rearrange the terms such that the recursion operator appears as a rational operator $\mathfrak{R} = MN^{-1}$. In doing so we need first to simplify $C^{-1}D$ and we convert it into a right fraction. We know from [10] that the ring of difference operators satisfies the Ore condition, by consequence we have

$$\begin{aligned} C^{-1}D &= (uu_1)^{-1} (u_{-1}u_1w\mathcal{S}^2 + uu_2w_1\mathcal{S}) \\ &\quad + C^{-1} ((u_1u_2 - uu_3w_1w_2)\mathcal{S} - (u_{-1}u - u_{-2}u_1w_{-1}w)) \\ &= (uu_1)^{-1} (u_{-1}u_1w\mathcal{S}^2 + uu_2w_1\mathcal{S}) + GI^{-1}(u_1u_{-1})^{-1}\mathcal{S} \end{aligned}$$

where operators G and I are given by

$$\begin{aligned} G &= p_1 (p_2 (p_1w_1 - p_3w_2) \mathcal{S} - p (p_1w - p_{-1}w_{-1})) & p &= uu_{-1} - u_1u_{-2}ww_{-1} \\ I &= p_2w (p_3w_2 - p_1w_1) \mathcal{S} + (p_{-1}p_2w_{-1}w_1 - pp_1w^2) + p_{-1}w (p_{-2}w_{-2} - pw_{-1}) \mathcal{S}^{-1} \end{aligned} \quad (33)$$

This leads to

$$M = w^2 \left(AI + B (uu_1)^{-1} (u_{-1}u_1w\mathcal{S}^2 + uu_2w_1\mathcal{S}) I + BG \right), \quad N = w^2 E \mathcal{S}^{-1} u_1 u_{-1} I.$$

We now prove that M and N form a preHamiltonian pair by checking that the operator $M + \lambda N$ is preHamiltonian for any constant λ using computer algebra Mathematica, similar to what we did for Theorem 7 in [10]. Here we only present how we prove that the operator N is a preHamiltonian operator. We notice that the first part of N given by

$$\frac{w^2}{u} (\mathcal{S}^2 - 1) \mathcal{S}^{-1} u_1 u_{-1} \quad (34)$$

is a very simple pre-Hamiltonian operator, which we recognise via (16) defining an associate bi-difference operator

$$\omega(a, b) = \frac{u_{-3}u_1w_{-1}^2ab_{-2} - u_{-3}u_1w_{-1}^2a_{-2}b + u_{-1}u_3w_1^2a_2b - u_{-1}u_3w_1^2ab_2}{u_{-1}u_1}.$$

According to Lemma 3 in [12], given a generic pre-Hamiltonian difference operator P with associated bi-difference operator ω_P , then the operator PQ , where Q is a difference operator, is itself pre-Hamiltonian if and only if the following expression is in the image of the operator Q , i.e.,

$$Q_*[P(Q(a))](b) - Q_*[P(Q(b))](a) + \omega_P(Q(a), Q(b)) \in \text{Im}(Q)$$

Using computer algebra Mathematica, we verify this property on operators (34) and I given by (33). Thus prove that the operator N is a preHamiltonian operator. ■

4 Hamiltonian operator

In this section we will show that (7) is a Hamiltonian equation via the pre-Hamiltonian theory [10, 12]. Let

$$\mathcal{H} = w^2 (\mathcal{S}w\mathcal{S} - \mathcal{S}^{-1}w\mathcal{S}^{-1}) w^2 + w^2 (u_1 - \mathcal{S}^{-1}u) (\mathcal{S}w - w\mathcal{S}^{-1})^{-1} (u\mathcal{S} - u_1)w^2. \quad (35)$$

Note that (7) can be written as $u_t = \mathcal{H}\delta_u \ln(w)$. We are going to prove that (35) is a Hamiltonian operator.

Theorem 4.1. *The operator \mathcal{H} given by (35) is a Hamiltonian operator.*

Proof. We prove the statement by a direct computation. For simplicity, we write \mathcal{H} as follows:

$$\mathcal{H} = w^2 \left(F - BC^{-1}B^\dagger \right) w^2, \quad (36)$$

where operators B, C are defined by (32) and F is defined as

$$F = \mathcal{S}w\mathcal{S} - \mathcal{S}^{-1}w\mathcal{S}^{-1}. \quad (37)$$

To show that \mathcal{H} is a Hamiltonian operator, we will use the definition given in Section 2: in particular, we are going to verify identities (16) and (17). We first note that \mathcal{H} is skew-symmetric since $\mathcal{H} + \mathcal{H}^\dagger = 0$. We then need to express \mathcal{H} in (35) as a rational operator. Note that

$$\begin{aligned} C^{-1}B^\dagger &= GI^{-1} & p &= u_{-1}uw_1 - u_1u_2w \\ G &= u_{-1}ww_1^2w_2p_1\mathcal{S} - u_2w_{-1}w^2w_1p_{-1} \\ I &= w_1^2w_2p_1\mathcal{S} + w_{-1}w_1 (u^2w_{-1}w_1 - u_{-2}u_2w^2) - w_{-2}w_{-1}^2p_{-2}\mathcal{S}^{-1} \end{aligned}$$

As we did in theorem 3.2, due to the Ore property of the ring of the difference operators [10], we obtain

$$\mathcal{H} = MN^{-1}, \quad M = w^2 (FI + BG); \quad N = w^{-2}I.$$

We now prove that the operator M is pre-Hamiltonian using (16). The calculations are straightforward though not so neat. The two form ω_M is skew-symmetric and of order

$(-5, 5)$ for any $a, b \in \mathcal{A}$. Here we explicitly write down the terms in $\omega_M(a, b)$ involving fifth order shifts for both a and b :

$$\omega_M^{(5)}(a, b) = -\frac{u_1 w w_2^2 w_3 w_4^2 w_5^2 w_6 (u_6 u_7 w_5 - u_4 u_5 w_6) (a_5 b - a b_5)}{u_1 u_2 w - u_{-1} u w_1}$$

The last step is straightforward to verify (17) by substituting the expressions of M , N and ω_M into it, which is verified by using computer algebra Mathematica. \blacksquare

By now we have obtained a recursion operator \mathfrak{R} (9), which is Nijenhuis and a Hamiltonian operator \mathcal{H} (35) for equation (7). We will go further looking for Hamiltonian operators compatible with \mathcal{H} . Similar to pre-Hamiltonian operators, two Hamiltonian operators \mathcal{H}, \mathcal{K} are said to be compatible and form a Hamiltonian pair $(\mathcal{H}, \mathcal{K})$ if any their linear combination $\mathcal{H} + \lambda \mathcal{K}$ is a Hamiltonian operator for any constant λ . Theorem 4 in [12] states that the operator $\mathcal{K} = \mathfrak{R}\mathcal{H}$ obtained from Nijenhuis recursion operator \mathfrak{R} in (9) and the Hamiltonian operator \mathcal{H} in (35), is also Hamiltonian and compatible with \mathcal{H} if \mathcal{K} is skew-symmetric.

Corollary 4.1. *The operator $\mathcal{K} = \mathfrak{R}\mathcal{H}$ obtained from the recursion operator in (9) and the Hamiltonian operator in (35) is a Hamiltonian operator and it is compatible with \mathcal{H} .*

Proof. From (31) and (36) we have

$$\mathcal{K} = \mathfrak{R}\mathcal{H} = w^2 \left(A + B^\dagger C^{-1} D \right) E^{-1} \left(F - B C^{-1} B^\dagger \right) w^2, \quad (39)$$

where we use the same definitions for operators involved as in (32) and in (37). It follows from Theorem 4 in [12] that we only need to check that the operator \mathcal{K} is skew-symmetric to show that it is a Hamiltonian operator compatible with \mathcal{H} , that is, to show that $\mathcal{K} + \mathcal{K}^\dagger = 0$. The expression (39) contains C^{-1} , we split the skew-symmetry condition, according to the positions of C^{-1} , into four parts denoted by $J^{(i)}, i = 1, \dots, 4$ as follows:

$$\mathcal{K} + \mathcal{K}^\dagger = J^{(1)} + J^{(2)} + J^{(3)} + J^{(4)} \quad (40a)$$

$$J^{(1)} = 2w^2 B C^{-1} (w^2 - w_1^2) C^{-1} B^\dagger w^2 \quad (40b)$$

$$J^{(2)} = 2w^2 B C^{-1} (u_1 w_1 \mathcal{S}^2 - u w \mathcal{S}^{-1}) w^2 \quad (40c)$$

$$J^{(3)} = 2w^2 (u_1 w_1 \mathcal{S} - u_{-1} w_{-1} \mathcal{S}^{-2}) C^{-1} B^\dagger \quad (40d)$$

$$J^{(4)} = 2w^4 (w_{-1}^2 - w_1^2) \quad (40e)$$

Let us define the operators \bar{B} and \bar{C} similarly to B and C in (39)

$$\bar{B} = u_1 + \mathcal{S}^{-1} u; \quad \bar{C} = \mathcal{S} w + w \mathcal{S}^{-1}.$$

Then we obtain the following identities

$$u_1 w_1 \mathcal{S}^2 - u w \mathcal{S}^{-1} = \frac{1}{2} \left(C \bar{B}^\dagger - \bar{C} B^\dagger \right);$$

$$u_1 w_1 \mathcal{S} - u_{-1} w_{-1} \mathcal{S}^{-2} = \frac{1}{2} (B \bar{C} + \bar{B} C),$$

Substituting them into $J^{(2)}$ (40c) and $J^{(3)}$ (40d), respectively, (40a) becomes

$$\mathcal{K} + \mathcal{K}^\dagger = w^2 BC^{-1} \left(w^2 - w_1^2 + \frac{1}{2} (C\bar{C} - \bar{C}C) \right) C^{-1} B^\dagger w^2$$

which, by direct computation is zero, proving that \mathcal{K} is skew-symmetric. Thus operator \mathcal{K} is a Hamiltonian operator compatible with \mathcal{H} defining a Hamiltonian pair $(\mathcal{H}, \mathcal{K})$. ■

An immediate consequence of this result is that the operator $\mathcal{R}^n \mathcal{H}$ is a Hamiltonian operator for all $n \in \mathbb{Z}$. Thus every symmetry in the hierarchy for equation (7) is also Hamiltonian. Indeed, we can write its next symmetry (8) as

$$u_{t_2} = \mathcal{H} \delta_u u_1 u_{-1} w = \mathcal{R} \mathcal{H} \delta_u \ln(w).$$

5 Discussion and Further Work

The main focus of this paper is to study the algebraic properties of the integrable equation (7), that is,

$$u_t = w^2 (u_2 w_1 - u_{-2} w_{-1}), \quad w = \sqrt{1 + u^2}.$$

We construct its recursion operator (9) based on its Lax representation, and show that it is a Hamiltonian system with Hamiltonian operator \mathcal{H} defined by (35). Moreover, we obtain a hierarchy of Hamiltonian operators compatible to \mathcal{H} following from the Nijenhuis property of the recursion operator.

There are the verifiable conditions to guarantee that a weakly nonlocal operator generates a local hierarchy starting from a proper seed [17]. The recursion operator (9) is not weakly nonlocal. We have not proven that it produces local symmetries even though the straightforward computation shows that starting from the equation itself, that is true for finite steps. A similar study as in [10] is required in this case.

As mentioned in Introduction, the operator (10) involving the reflection operator \mathcal{T} can be used to produce the hierarchy of infinite symmetries starting from equation itself just as the recursion operator (9). This is true for other scalar integrable differential-difference equations. For instance, for the Volterra chain

$$u_t = u(u_1 - u_{-1}),$$

its recursion operator is

$$\mathcal{R} = u\mathcal{S} + u_1 + u + u\mathcal{S}^{-1} + u(u_1 - u_{-1})(\mathcal{S} - 1)^{-1} \frac{1}{u}.$$

The corresponding \mathcal{R}' is

$$\mathcal{R}' = u(1 - \mathcal{T})(\mathcal{S} + 1)u\mathcal{S}(\mathcal{S} - 1)^{-1} \frac{1}{u}.$$

We'll explore more to uncover whether there are any advantages to present in this form.

The equation (1) can be written in a rational form

$$v_t = (1 + v^2) \left(\prod_{k=1}^{p-1} \frac{1 + v_k^2}{1 - v_k^2} \frac{v_p}{1 - v_p^2} - \prod_{k=1}^{p-1} \frac{1 + v_{-k}^2}{1 - v_{-k}^2} \frac{v_{-p}}{1 - v_{-p}^2} \right), \quad p \in \mathbb{N}$$

via a point transformation [18]¹

$$u = \frac{2v}{1 - v^2}.$$

All results obtained in terms of the variable u including the Lax representation, recursion operators and Hamiltonian structures can be naturally passed to the variable v .

In principle, the methods developed in this paper can be used to construct a recursion operator for the entire family of integrable equations (1). For any fixed p , we can write down the recursion operator although it is formed by more parts comparing to (9). We have also done it for $p = 3, 4$. Until now we are not able to present it in a neat way as it has been done for the Narita-Itoh-Bogoyavlensky lattice in [5].

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