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Degree growth of lattice equations defined on a 3×3 stencil

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Abstract

We study complexity in terms of degree growth of one-component lattice equations defined on a 3×3 stencil. The equations include two in Hirota bilinear form and the Boussinesq equations of regular, modified and Schwarzian type. Initial values are given on a staircase or on a corner configuration and depend linearly or rationally on a special variable, for example $f_{n,m} = \alpha_{n,m}z + \beta_{n,m}$, in which case we count the degree in z of the iterates. Known integrable cases have linear growth if only one initial values contains z, and quadratic growth if all initial values contain z. Even a small deformation of an integrable equation changes the degree growth from polynomial to exponential, because the deformation will change factorization properties and thereby prevent cancellations.

Dedicated to the memory of Decio Levi.

1 Introduction

The concept of integrability is associated with the dynamics being regular (as opposed to chaotic), without being simple. Since the equations are nonlinear, regularity means there must be some underlying “controlling” mathematical structures. For example, solutions to integrable equations are often associated with elliptic functions. Since many different types of (nonlinear) equations can show regular behavior there cannot be a strict all-encompassing definition of integrability.

It is more fruitful to consider “indicators” of integrability, each with their own range of applicability. For example, the three-soliton condition (3SC) is a good indicator of integrability for partial differential or partial difference equations in Hirota bilinear form, while the Painlevé property is applicable more widely for differential equations but not so easily for difference equations. These two are algorithmic ways to prove integrability, that is, following a given procedure one can prove or disprove integrability; usually it is easier to disprove. Certain other integrability indicators require the construction of additional structures, such as conserved quantities, or Lax pairs, or Bäcklund transformations etc. Such constructions may require a lot of ingenuity.

In this paper, we only consider integrability in the context of partial difference equations (for an overview, see [1, 2]). For them one algorithmic and powerful method is the study of the growth of complexity under iterations, characterized by "algebraic entropy" [3, 4, 5, 6, 7, 8, 9, 10]. We will briefly discuss this in Section 2.

Growth properties of 2D one-component equations defined on a single quadrilateral of the $\mathbb{Z}^2$ lattice have been studied by several authors, see e.g. [11, 12, 13, 14, 15, 16]. If the starting configuration is on the quadrilateral $(0,0)-(1,1)$ of the Cartesian lattice and the evolution is to the NE direction, then one can look for factorization and cancellation first at the point $(2,2)$ (and this was used as a condition in the search for possible integrable equations in [17]). As for lattice equations defined on a larger than 2×2 stencil, some results exist for the Toda lattice, defined on a star shaped 5-point stencil [18].

The main results of this paper concern the numerical analysis of degree growth for partial difference equations defined on a 3×3 stencil. The methodology is discussed in Section 4. In Section 5, we will first study two equations in Hirota bilinear form, one integrable and one non-integrable, which depend on seven point of the 3×3 stencil. These equations are relatively simple and we can study integrability from various points of view, in order to establish the validity of the computational method. Then in Section 6 ,we analyze one-component equations of Boussinesq type, which involve all nine points in the 3×3 stencil.

2 Degree growth and cancellations

When one studies discrete dynamics (maps), one way to quantify the idea of "regularity" is to study the complexity of their iterates. This association was made already in the 1990's by Veselov [3] and others. When a rational map is iterated, its complexity can be quantified as the degree of the computed numerator (or denominator) with respect to the initial values. In general, the degrees grow exponentially with the number of iterations n but the growth will be reduced if the numerator and denominator have a common factor which can be canceled. This was analyzed in detail in [4, 5, 6, 7, 8, 9] and it was found that for integrable maps the cancellations are strong enough to convert the exponential degree growth to polynomial growth with respect to n. In fact the conjecture relating degree growth (after cancellations) to integrability is as follows:

- If the degree growth is linear in n then the equation is linearizable.
- If the degree growth is polynomial in n then the equation is integrable.
- If the degree growth is exponential in n then the equation is chaotic.

In order to observe the cancellations, it is best to formulate a higher order map as a multi-component first order map in projective space. Then, instead of a rational expression, we have multi-component polynomial maps, and cancellations take place when, after some number of iterations, the components have a nontrivial common factor.

The existence of a common factor after n iterations means also that the initial values, for which the common factor vanishes, are singular points, because starting from those points the iterations eventually lead outside the projective space. Thus cancellations and singularities are two sides of the same phenomenon.

Associated to this is also the method of "singularity confinement" [19, 20, 21]. In this method one studies the singularity further by starting from a point infinitesimally close to the singular point and checking whether after passing the singularity the dynamics becomes regular eventually. This is a very effective method and been used extensively in order to de-autonomize discrete maps.

Still another approach is to study the singularity itself using methods from algebraic geometry, namely "resolution of singularities" [22, 23].

3 Lattice equations and their initial data

Among the integrable partial differential equations we have first order evolution equations like the Korteweg – de Vries (KdV) equation and also second order equations like the Boussinesq (BSQ) equation. The main difference is the amount of initial data needed to define evolution, say one function at $t = 0$ for KdV, or one function and its derivative w.r.t time for BSQ. This has its analogue for partial difference equations.

3.1 Initial data configuration

For equations defined on the $\mathbb{Z}^2$ lattice there are several possibilities. If the equation is defined on the elementary quadrilateral of the lattice, such as the discrete versions of the KdV equation, we need initial data on a line, for example on a corner or on a staircase as illustrated in Figure 1. (For more exotic initial data see [15].)

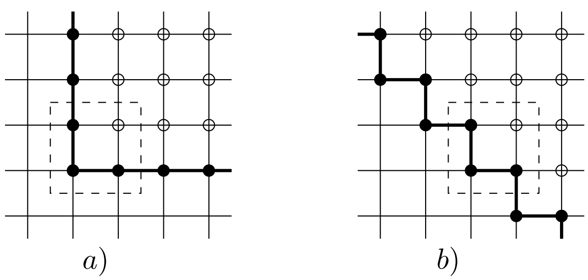

Figure 1. Two ways to give initial data for an equation defined on the elementary 2×2 quadrilateral. Here a) gives the corner configuration and b) the staircase configuration. The initial data is given on the vertices marked by solid black circles and the values on vertices marked by open circles are to be computed. The points involved in the first step of computations are bounded by the dashed line.

Some equations are defined on a 2×3 stencil, for example the discrete KdV equation in Hirota bilinear form. Two possible ways to give initial data are shown in Figure 2.

In this paper, we study one component equations defined on a 3×3 stencil on the $\mathbb{Z}^2$ lattice. For them two lines of initial data is necessary, as seen in Figure 3.

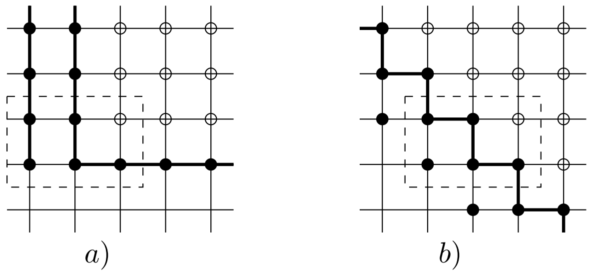

Figure 2. Initial data required for equations defined on a 2×3 stencil, a) for corner and b) for staircase. In both cases, one can say that now the initial data needs to be given on "1.5 lines".

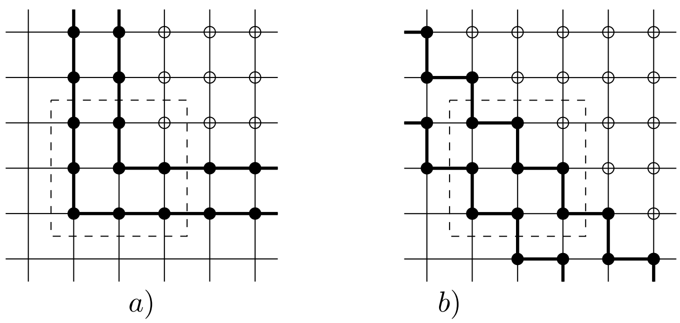

Figure 3. Initial data required for equations defined on a 3×3 stencil, a) for corner, b) for staircase. In this case, the initial data needs to be given on 2 infinite lines. For the staircase the two lines can be defined by $0 \le n+m \le 3$.

3.2 The equations under study

The first pair of equations are in the Hirota bilinear form. We use the short-hand notation of writing only the shift with respect to (n,m) thus $f_{n+1,m} \equiv f_{1,0}$ etc. The two equations considered are:

- Four-term seven point integrable equation in Hirota bilinear form (see Figure 4a))

$$2f_{2,2}f_{0,0} + 2f_{1,2}f_{1,0} - f_{2,1}f_{0,1} - 3f_{1,1}^2 = 0. \tag{1}$$

- Four-term seven point non-integrable equation in Hirota bilinear form (see Figure 4b))

$$12\, f_{2,2}f_{0,0} - 3\, f_{2,0}f_{0,2} + 16\, f_{1,2}f_{1,0} - 25\, f_{1,1}^2 = 0. \tag{2}$$

The numerical values of the parameters in (1) and (2) have no effect on integrability; they have been chosen for convenience of integrability analysis using the three-soliton-condition (3SC), done in the Appendix. There are also integrable three-term Hirota bilinear equations depending on 5 points of the 9 point stencil. They are related to the Toda lattice and we will not discuss them here.

The second set of equations consist of one-component equations of Boussinesq type. (For a comprehensive review see [24].) Their original three-component forms are defined

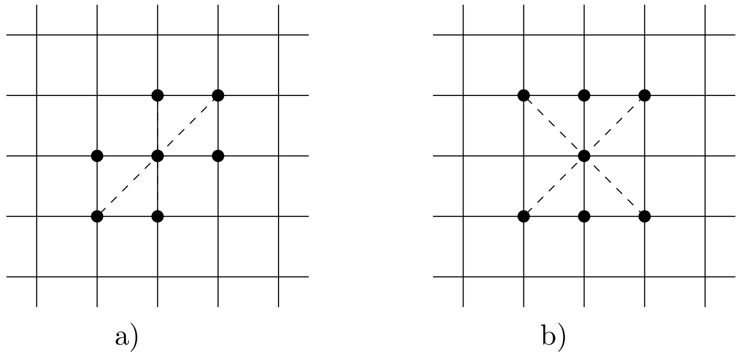

Figure 4. a): Graph of the integrable 4-term 7-point equation (1) b): Graph of the non-integrable 4-term 7-point equation (2). The (0,0) point at is the lower left black disk.

on the elementary quadrilateral and on its boundaries and are known to be integrable by Consistency-Around-the-Cube method [25]. By eliminating two variables one obtains these one-component forms on a larger stencil. For them no independent proof of integrability has been given so far. In order to get a non-integrable version of these equations we will add a multiplier ($\neq 1$) to the $x_{0,0}$ terms. In the following P, Q are the lattice parameters.

- The regular lattice Boussinesq equation in one-component form

$$(P-Q)\left(\frac{1}{x_{2,0}-x_{1,1}}-\frac{1}{x_{1,1}-x_{0,2}}\right)+b_0(x_{0,1}-x_{1,0}+x_{2,1}-x_{1,2}) \\ -(x_{2,2}-x_{0,1})(x_{2,1}-x_{1,2})-(x_{0,0}-x_{2,1})(x_{1,0}-x_{0,1})=0. \quad (3)$$

  This was first given in [26], except for the parameter b_0 found in [25]. [1]

- The modified lattice Boussinesq equation in one-component form was first given in [26], we use its reversed form with $x_{2,2}$ in the numerator

$$\left(\frac{P\,x_{1,1}-Q\,x_{2,0}}{x_{2,0}-x_{1,1}}\right)\frac{x_{1,0}}{x_{2,1}}-\left(\frac{P\,x_{0,2}-Q\,x_{1,1}}{x_{1,1}-x_{0,2}}\right)\frac{x_{0,1}}{x_{1,2}}=\frac{x_{2,2}}{x_{1,2}}-\frac{x_{2,2}}{x_{2,1}}-\frac{x_{1,0}}{x_{0,0}}+\frac{x_{0,1}}{x_{0,0}}. \quad (4)$$

- The Schwarzian Boussinesq equation in one-component form

$$\frac{(x_{2,2}-x_{1,2})(x_{0,2}-x_{1,1})(x_{0,1}-x_{0,0})}{(x_{2,2}-x_{2,1})(x_{1,1}-x_{2,0})(x_{1,0}-x_{0,0})}= \\ \frac{(x_{1,1}-x_{0,2})(b_0\,x_{0,1}+b_1)+(x_{1,2}-x_{0,2})(x_{0,1}-x_{1,1})\,P-(x_{1,2}-x_{1,1})(x_{0,1}-x_{0,2})\,Q}{(x_{2,0}-x_{1,1})(b_0\,x_{1,0}+b_1)+(x_{2,1}-x_{1,1})(x_{1,0}-x_{2,0})\,P-(x_{2,1}-x_{2,0})(x_{1,0}-x_{1,1})\,Q}. \quad (5)$$

  This was first given in [28] except for the parameters b_0, b_1 found in [25].

[1] It turns out that the b_0 term in (3) can be eliminated by the transformation $x_{n,m} \to x_{n,m} + \frac{1}{3}b_0(n+m)$. However, the parameter b_0 cannot be eliminated from the three-component form and in fact has effects on the solutions[27].

4 Degree growth computations

4.1 Limits of analytical computations

For precise degree growth analysis one could in principle use analytical computations. In order to get an idea of its feasibility, let us consider the integrable equation (4). For computations we write equation (4) for $\mathbb{C}P^2$ with a pair of polynomial maps. Defining polynomials y, a by

$$x_{n,m} = \frac{y_{n,m}}{a_{n,m}},$$

we get the maps

$$\begin{aligned} y_{n,m} &= [(a_{n-2,m}\, y_{n-1,m-1}\, Q - a_{n-1,m-1}\, y_{n-2,m}\, P) \times \\ &\quad (a_{n-1,m-1}\, y_{n,m-2} - a_{n,m-2}\, y_{n-1,m-1})\, a_{n-1,m-2}\, a_{n-1,m}\, y_{n-2,m-1}\, y_{n,m-1} \\ &\quad -(a_{n-1,m-1}\, y_{n,m-2}\, Q - a_{n,m-2}\, y_{n-1,m-1}\, P) \times \\ &\quad (a_{n-2,m}\, y_{n-1,m-1} - a_{n-1,m-1}\, y_{n-2,m}) a_{n-2,m-1}\, a_{n,m-1}\, y_{n-1,m-2}\, y_{n-1,m}] y_{n-2,m-2} \\ &\quad +(a_{n-1,m-1}\, y_{n,m-2} - a_{n,m-2}\, y_{n-1,m-1})\, (a_{n-2,m}\, y_{n-1,m-1} - a_{n-1,m-1}\, y_{n-2,m}) \times \\ &\quad (a_{n-2,m-1}\, y_{n-1,m-2} - a_{n-1,m-2}\, y_{n-2,m-1})\, a_{n-2,m-2}\, y_{n-1,m}\, y_{n,m-1}, \\ a_{n,m} &= (a_{n-2,m}\, y_{n-1,m-1} - a_{n-1,m-1}\, y_{n-2,m})\, (a_{n-1,m-1}\, y_{n,m-2} - a_{n,m-2}\, y_{n-1,m-1}) \times \\ &\quad (a_{n-1,m}\, y_{n,m-1} - a_{n,m-1}\, y_{n-1,m})\, a_{n-2,m-1}\, a_{n-1,m-2}\, y_{n-2,m-2}. \end{aligned}$$

We take the staircase configuration and as initial values all $x_{n,m}$ for $n + m = 0, 1, 2, 3$ are free parameters. In practice, we take $y_{n,m} = f_{n,m}$, $a_{n,m} = 1$, when $n + m = 0, 1, 2, 3$. The first computed values at $n + m = 4$, for example $y_{2,2}$, $a_{2,2}$, will respectively be of degrees 5 and 4 in f, and will have 16 resp. 8 terms,

Continuing with $n + m = 5$, we get $y_{3,2}$ and $a_{3,2}$ of degrees 13 and 12 (with 1184 and 528 terms), respectively. But there are cancellations. We find

$$\mathrm{GCD}(y_{3,2}, a_{3,2}) = (f_{3,0} - f_{2,1})(f_{2,1} - f_{1,2})(f_{2,0} - f_{1,1})f_{2,1},$$

and consequently after cancellation the final degrees are 9 and 8, respectively (with 220 and 112 terms). The GCD also clearly indicates that initial values with $f_{n,m} - f_{n-1,m+1} = 0$ form singular lines in the initial value space coordinatized by $f_{n,m}$, $n + m = 0, 1, 2, 3$.

Getting data for points with $n + m = 6$ is already very demanding, but with judicious choices in the order of computations we eventually find that $\mathrm{GCD}(y_{3,3}, a_{3,3})$ is a product of various $f_{n,m} - f_{n-1,m+1}$ terms and when they are divided out $y_{3,3}, a_{3,3}$ will have 4672 and 2592 terms of degrees 14 and 13, respectively. Thus with some effort we have found the beginning part of the degree growth sequence to be (c.f. Section 6.2)

$$1,\ 5,\ 9,\ 14, \ldots$$

It is now clear that with full analytical computations we cannot hope to get a sufficient number of data points for growth analysis. And for the more complicated (5) the situation is worse still. Thus, we must resort to other techniques.

4.2 Computational method

In the following, we will compute numerically the degree growth for the equations mentioned in Section 3.2. We take the equations in Hirota bilinear form as test cases, because for them we have an independent integrability test. Thus, from these equations we get an indication on how the computed degrees can be used to distinguish between integrable and non-integrable models. We then go on to compute degrees for the Boussinesq equations.

As we saw a full analytic computation not feasible, but for longer growth data even the numerical computations must be streamlined:

- We must reduce the number of variables that are tracked for determining the degree. It is not necessary to keep all initial values as independent variables, instead one can introduce one dedicated variable, say z, for this purpose and give the initial data in terms of that variable, for example:

  1. Use the dedicated variable at one point only, say $f_{0,0} = z$, all other initial values being random numbers.
  2. Set all initial values as linear functions of the dedicated variable z:

     $$f_{n,m} = \alpha_{n,m} z + \beta_{n,m}, \tag{6}$$

     with random integer coefficients α, β. Then we take $y_{n,m} = f_{n,m}$, $a_{n,m} = 1$.
  3. Set all initial values as rational functions of z [12]

     $$f_{n,m} = \frac{\alpha_{n,m} z + \beta_{n,m}}{\gamma_{n,m} z + \delta_{n,m}}. \tag{7}$$

     For polynomial computations we set $y_{n,m} = \alpha_{n,m} z + \beta_{n,m}, a_{n,m} = \gamma_{n,m} z + \delta_{n,m}$.

- During iterations the expressions start to contain huge numbers as coefficients and this can be made manageable using modular arithmetic with respect to a large prime p. It is best to use random numbers for all constants (such as those appearing in the equation P, Q, b_j, as well as those appearing in initial conditions $\alpha_{n,m}, \beta_{n,m}, \dots$). The necessary operations of polynomial algebra, including computation of GCD, have been implemented for modular arithmetic. The programming language C++ provides tools for this purpose through its NTL package. In principle, there could be spurious zeroes if by accident some number is congruent to 0 mod p. In a suspicious situation one could redo computations with a different prime.

- Initial value geometry. The choices we will use here, corner and staircase, were already illustrated in Figures 2 and 3. We do not consider exotic geometries such as those used in [15].

5 Degree growth for Hirota bilinear equations

Integrability analysis of equations (1) and (2) based on the three-soliton-condition has been done in the Appendix. In this section we study the degree growth. We will consider different geometries and distributions of z.

0	0	0	0	0	0	2	5	8	11	14	18	22	26	30	34	38	42
0	0	0	0	0	0	2	5	8	11	14	18	22	26	30	34	38	38
0	0	0	0	0	0	2	5	8	11	14	18	22	26	30	34	34	34
0	0	0	0	0	0	2	5	8	11	14	18	22	26	30	30	30	30
0	0	0	0	0	0	2	5	8	11	14	18	22	26	26	26	26	26
0	0	0	0	0	0	2	5	8	11	14	18	22	22	22	22	22	22
○	0	0	0	0	0	2	5	8	11	14	18	18	18	18	18	18	18
○	○	0	0	0	0	2	5	8	11	14	14	14	14	14	14	14	14
○	○	○	0	0	0	2	5	8	10	11	11	11	11	11	11	11	11
○	○	○	○	0	0	2	5	6	8	8	8	8	8	8	8	8	8
·	○	○	○	○	0	2	4	5	5	5	5	5	5	5	5	5	5
·	·	○	○	○	◎	②	2	2	2	2	2	2	2	2	2	2	2
·	·	·	○	◎	⦿	◎	0	0	0	0	0	0	0	0	0	0	0
·	·	·	·	◎	◎	○	○	0	0	0	0	0	0	0	0	0	0
·	·	·	·	·	○	○	○	○	0	0	0	0	0	0	0	0	0
·	·	·	·	·	·	○	○	○	○	0	0	0	0	0	0	0	0

Figure 5. Degrees for equation (1). The variable z is only at $(1,1)$. The corner configuration is within the dashed lines. For $n,m>5$ the degrees are $\deg_{n,m}=4\min(n,m)-10$.

5.1 Only one tracking variable among the initial value

The integrable case (1). First we will consider the integrable equation (1) and put the tracking variable z in only one initial value.[2]

In Figure 5, we have the staircase configuration with the initial value z at $(1,1)$, it is marked by a black disk. The other initial values are random numbers and marked by open circles. We have also displayed the stencil relevant to this equation using larger circles. The displayed numbers give the degree of the numerator of $x_{n,m}$, i.e., the degree of $y_{n,m}$ in z. For $n,m>5$ the degrees are given by

$$\deg_{n,m}=4\min(n,m)-10, \tag{8}$$

and therefore grow linearly.

In the same figure, we have indicated the corner configuration with dashed lines. The degrees are the same, because from staircase of initial values we can compute numerical values for all points for which $(n<2,\ m>2)$ or $(n>2, m<2)$, and this way create the initial values (random numbers) for the corner situation.

[2]This method was first used in [15] where it was found that for the Liouville equation the degrees are then bounded, for the integrable KdV equation the degrees grow linearly, while in a non-integrable version of KdV the degrees were found to grow asymptotically as 4^n.

0	0	0	0	1	2	13	40	127	338	851	1914	4002	7682	13778	23034
0	0	0	0	1	2	12	36	109	276	659	1400	2755	4956	8324	13030
0	0	0	0	1	2	11	32	92	220	494	982	1802	3010	4702	6864
0	0	0	0	1	2	10	28	76	170	355	652	1103	1694	2451	3338
0	0	0	0	1	2	9	24	61	126	241	402	619	868	1167	1492
0	0	0	0	1	2	8	20	47	88	151	224	312	400	507	610
○	0	0	0	1	2	7	16	34	56	84	110	140	168	200	232
○	○	0	0	1	2	6	12	22	30	40	46	57	62	75	78
○	○	○	0	1	2	5	8	11	14	15	18	19	22	23	26
○	○	○	○	1	2	4	4	5	4	6	4	7	4	8	4
·	○	○	◎	◎	②	2	2	2	2	2	2	2	2	2	2
·	·	○	○	⦿	○	1	0	1	0	1	0	1	0	1	0
·	·	·	◎	◎	◎	○	0	0	0	0	0	0	0	0	0
·	·	·	·	○	○	○	○	0	0	0	0	0	0	0	0

Figure 6. Non-integrable equation (2) with initial single z at point $(1,1)$.

If we set the initial value z at $(0,0)$, the degree growth is slower. For example, the degree at the first calculated point $(2,2)$ is zero, because the variable z only appears in the denominator. For $n,m>2$ the degrees are in that case given by

$$\deg_{n,m} = 2\min(n,m) - 4. \tag{9}$$

In all cases, when there is only one z-dependent initial value the growth is linear for the integrable case.

The non-integrable case (2)**.** For the non-integrable case (2) the degrees in Figure 6 are for the initial value z only at $(1,1)$, in a staircase configuration. A longer list of values on the diagonal (dashed line in Figure 6) is given by

$$0,\ 1|\,2,\ 4,\ 8,\ 22,\ 56,\ 151,\ 402,\ 1103,\ 3010,\ 8324,\ 23034,\ 64171,\ 179096, 501810,\ 1408760,\ \ldots$$

This is approximately 0.726×2.813^n. The first few columns in the figure show regular growth but is still unlikely that the numbers can be given by some formula, especially since the equation itself is asymmetric.

5.2 All initial values contain z

Next we consider cases in which all initial values depend on z, either linearly, i.e., $\alpha_{n,m}z+\beta_{n,m}$ or rationally $(\alpha_{n,m}z+\beta_{n,m})/(\gamma_{n,m}z+\delta_{n,m})$, where $\alpha_{n,m}$, $\beta_{n,m}$, $\gamma_{n,m}$, $\delta_{n,m}$ are some random numbers. We will mostly use the staircase configuration extending from upper left to lower right. As indicated in Figure 3, the initial values will be given on points for which $0 \leq n+m \leq 4$. Due to translational invariance along the staircase the degrees only depend on $n+m$ and therefore the results can be given by one sequence of values.

Integrable/linear. In the staircase configuration the degrees for (1) are given by

$$\deg_{k=n+m>3} = k^2 - 7k + 14. \tag{10}$$

We also computed the degrees without canceling common factors, and got (for $k > 3$)

$$2, 4, 10, 28, 81, 237, 697, 2053, 6050, 17832, 52562, 154936, 456705, 1346233, 3968305, \dots,$$

leading to growth rate 2.94771^k, showing once more the essential influence of cancellations.

For the corner configuration and with linear initial data the degrees of the numerator follow the rule

$$\deg_{n,m} = 2nm - 3(n+m) + 6. \tag{11}$$

On the diagonal where $m = n$, we have $k = 2n$ and we can compare staircase and corner degrees: $\deg_{str} = 4n^2 - 14n + 14$ versus $\deg_{cor} = 2n^2 - 6n + 6$, i.e., in the staircase the growth is about twice as fast. This is because in the corner case the degrees on the corner boundaries (cf. Figure 3 a)) are those of the initial values (i.e. $= 1$) while for the staircase the degrees at the corresponding points are computed.

Integrable/rational. In Figure 7, we have rational initial data for the corner configuration. In the region above and to the right of the dashed line the degrees are given by

$$\deg_{n,m} = 6mn - 13(n+m) + 4\max(n,m) + 2\delta_{n,m} + 32. \tag{12}$$

Asymptotically these are three times bigger than in the linear case (11).

For the staircase we have the degrees starting at $k = 4$ as $7, 13, 26, 43, 62, 91, 122$ and then for $k > 10$,

$$\deg_{k=n+m} = 3k^2 - 26k + 84, \tag{13}$$

again about three times bigger than in (11).

Non-integrable/linear For the staircase configuration we have the degrees for $k := n+m$, starting with $k = 0$, as

$$1, 1, 1, 1, |2, 3, 7, 12, 22, 36, 61, 101, 174, 295, 508, 864, 1478, 2513, 4289, 7303, 12463,$$
$$21241, 36237, 61771, 105346, 179593, 306252, \dots.$$

This is approximately $\propto 1.705^k$ or on the $n = m$ diagonal as $\propto 2.908^n = 1.705^{2n}$.

Non-integrable/rational For the staircase the degree sequence is

$$1, 1, 1, 1, |7, 11, 24, 38, 64, 102, 176, 294, 514, 870, 1498, 2539, 4341, 7376, 12600,$$
$$21456, 36631, 62419, 106488, 181496, 309541, \dots.$$

The is about the same as for linear initial data

•	•	46	128	207	283	359	436	513	590	667	744	821	898	975	1054	1133	1214	1295	1376
•	•	43	119	192	262	332	403	474	545	616	687	758	829	902	975	1050	1125	1200	1275
•	•	40	110	177	241	305	370	435	500	565	630	695	762	829	898	967	1036	1105	1174
•	•	37	101	162	220	278	337	396	455	514	573	634	695	758	821	884	947	1010	1073
•	•	34	92	147	199	251	304	357	410	463	518	573	630	687	744	801	858	915	972
•	•	31	83	132	178	224	271	318	365	414	463	514	565	616	667	718	769	820	871
•	•	28	74	117	157	197	238	279	322	365	410	455	500	545	590	635	680	725	770
•	•	25	65	102	136	170	205	242	279	318	357	396	435	474	513	552	591	630	669
•	•	22	56	87	115	143	174	205	238	271	304	337	370	403	436	469	502	535	568
•	•	19	47	72	94	118	143	170	197	224	251	278	305	332	359	386	413	440	467
•	•	16	38	57	74	94	115	136	157	178	199	220	241	262	283	304	325	346	367
•	•	13	29	42	57	72	87	102	117	132	147	162	177	192	207	222	237	252	267
•	•	10	20	29	38	47	56	65	74	83	92	101	110	119	128	137	146	155	164
•	•	7	10	13	16	19	22	25	28	31	34	37	40	43	46	49	52	55	58
•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•
•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•	•

Figure 7. Degrees for equation (1) with rational initial data.

Summary. Depending on the initial data from which the degrees are computed, we have slightly different growth rates. In the integrable case of (1) a single z dependent initial data point gives linear growth, while linear and rational initial data give quadratic growth. For the chosen example of non-integrable equation in Hirota bilinear form (2) the growth is always exponential, about 2.8^n or 2.9^n. Thus, we see that the computationally simplest case of only one initial z-dependent point is enough to differentiate between integrable and non-integrable equations.

6 Degree growth of lattice Boussinesq equations

We now turn to the lattice Boussinesq equations (3), (4), and (5). These equations involve all points of the 3×3 stencil. The non-integrable versions are obtained by changing the coefficient of the $x_{0,0}$ term.

From the results for Hirota equations (1) and (2) we have observed first of all that a single non-numeric initial value is enough to differentiate between integrable and non-integrable equations. Furthermore, we found that even for equations defined on a larger stencil, the integrable case with linear and rational z dependence have quadratic degree growth, while for the non-integrable case the growth is typically 2.8^n. One may expect similar overall results for the Boussinesq equations since they are integrable.

However we may expect differences in the details. One difference is due to the observations that Boussinesq equations are in many ways associated with threefold symmetry, for example the solutions often involve cubic roots of unity. In the following, this manifests itself in the degree sequences where we need an indicator function for divisibility by 3, which we define as

$$\mathcal{D}_n(m) = \begin{cases} 1 & \text{when } n|m, \\ 0 & \text{otherwise.} \end{cases}$$

6.1 Regular lattice Boussinesq equation (3)

For numerical computations we use $P - Q = 3$, $b_0 = 1$. The value of b_0 seems to have no effect. Unless mentioned otherwise, we only consider the staircase configuration.

• **Single z in the initial values.** If initial value has a single z at $(0,0)$ we get for the integrable case degrees for the corner configuration as given in Figure 8,[3]

$$\deg_{n,m} = \min(n-1, m-1, \lfloor (n+m-1)/3 \rfloor) \tag{14}$$

The same degrees are obtained for the staircase configuration. In Figure 9, we have degrees for a case which is non-integrable due to a different coefficient for $x_{0,0}$.

• **Linear initial values:**
In the integrable case the degrees in the staircase configuration are

$$1,1,1,1\,|\,4,7,11,16,22,29,37,46,56,67,79,92,106,121,137,154,172,191,211,$$
$$232,254,277,301,...$$

Here and in the following, the first 4 degrees are initial values at $(0,0)$, $(0,1)$, $(1,1)$, $(1,2)$ The above sequence is given by

$$\deg_{k=n+m>3} = \tfrac{1}{2}(k^2 - 3k + 4). \tag{15}$$

In the non-integrable case with "$2\,x_{0,0}$" instead of "$x_{0,0}$" we have instead

$$1,1,1,1\,|\,4,7,13,24,47,93,180,353,695,1358,2655,5206,10192,19942,$$
$$39048,76447,149634,292944,573525,\ldots$$

with growth rate $0.112 \cdot 1.958^k$. First difference with respect to the integrable case is at $(3,3)$: 13 vs. 11.

• **Rational initial values**
Integrable case we have degrees

$$1,1,1,1\,|\,9,16,26,41,55,73,97,118,144,177,205,239,281,316,358,409,$$
$$451,501,561,610,668,737,793,\ldots$$

This is fitted with

$$\deg_{k=n+m>3} = \tfrac{1}{3}[4k^2 - 13(k-1) + (k-2)\mathcal{D}_3(k-1) - \mathcal{D}_3(k)]. \tag{16}$$

[3] $\lfloor x \rfloor$ = "floor" of x = ignore decimals of x.

○	○	1	2	3	4	5	6	7	8	8	8	9	9	9	10	10
○	○	1	2	3	4	5	6	7	7	8	8	8	9	9	9	10
○	○	1	2	3	4	5	6	7	7	7	8	8	8	9	9	9
○	○	1	2	3	4	5	6	6	7	7	7	8	8	8	9	9
○	○	1	2	3	4	5	6	6	6	7	7	7	8	8	8	9
○	○	1	2	3	4	5	5	6	6	6	7	7	7	8	8	8
○	○	1	2	3	4	5	5	5	6	6	6	7	7	7	8	8
○	○	1	2	3	4	4	5	5	5	6	6	6	7	7	7	8
○	○	1	2	3	4	4	4	5	5	5	6	6	6	7	7	7
○	○	1	2	3	3	4	4	4	5	5	5	6	6	6	6	6
○	○	1	2	3	3	3	4	4	4	5	5	5	5	5	5	5
○	○	1	2	2	3	3	3	4	4	4	4	4	4	4	4	4
○	○	1	2	2	2	3	3	3	3	3	3	3	3	3	3	3
○	○	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2
◎	◎	①	1	1	1	1	1	1	1	1	1	1	1	1	1	1
◎	◎	◎	○	○	○	○	○	○	○	○	○	○	○	○	○	○
⊙	◎	◎	○	○	○	○	○	○	○	○	○	○	○	○	○	○

Figure 8. Degrees for Boussinesq equation (3) with one z at (0,0). The degrees are given by $\min(n-1, m-1, \lfloor (n+m-1)/3 \rfloor)$. As an example, the dashed lines border the area with degree 5.

○	○	1	16	117	554	2019	6206	16873	41733	95644
○	○	1	14	89	371	1215	3406	8526	19547	41733
○	○	1	12	65	236	691	1758	4032	8526	16873
○	○	1	10	45	141	367	842	1758	3406	6206
○	○	1	8	29	78	179	367	691	1215	2019
○	○	1	6	17	39	78	141	236	371	554
○	○	1	4	9	17	29	45	65	89	117
○	○	1	2	4	6	8	10	12	14	16
○	○	1	1	1	1	1	1	1	1	1
○	○	○	○	○	○	○	○	○	○	○
•	○	○	○	○	○	○	○	○	○	○

Figure 9. Degrees for a non-integrable case. First difference with respect to the integrable case displayed in Figure 8 is at point $(3,3)$.

Note the period-3 components.

For the non-integrable case with "$2\,x_{0,0}$" instead of "$x_{0,0}$" we get

$$1,1,1,1\,|\,9,17,33,63,123,243,473,927,1823,3567,6977,13675,26777,\\ 52403,102599,200863,393179,769723,1506935,\dots$$

The growth is about $0.29\cdot 1.958^k$. With the $x_{0,0}$ term replaced by "$0\,x_{0,0}$" we get slightly smaller degrees

$$1,1,1,1\,|\,8,15,29,55,108,213,414,813,1598,3125,6115,11985,23464,\\ 45923,89915,176023,344559,674551,1320600,\dots$$

with about the same growth: $0.26\cdot 1.958^k$. Although the degrees start slower in the last case they exceed the integrable case already at $k=6$.

6.2 Modified lattice Boussinesq equation (4)

In general the degrees are close to those of the regular Boussinesq equation, sometimes even the same. In computations we use $p=1, q=3$.

• **Single z in the initial values.** The degrees are the same as for the regular Boussinesq equation, given in (14) and Figure 8.

• **Linear initial values:**
For the integrable case we get

$$1,1,1,1,\ |\,5,9,14,21,29,38,49,61,74,89,\ 105,122,141,161,182,205,\\ 229,254,281,309,338,369,401,...$$

This is fitted with

$$\deg_{k=n+m>3}=\tfrac{1}{3}[2k^2-6k+7-\mathcal{D}_3(n)]. \tag{17}$$

For the non-integrable case

$$1,1,1,1,\ |\,5,10,21,49,112,255,582,1329,3035,6930,15824,36134,\\ 82511,188411,430231,982420,2243327,...$$

The asymptotic degree growth is approximately $0.151\cdot 2.283^n$.

• **Rational initial values**
For the integrable case

$$1,1,1,1,\ |\,9,17,27,41,57,75,97,121,147,177,209,243,281,321,363,\\ 409,457,507,561,617,675,737,801,...$$

This is fitted with

$$\deg_{k=n+m>3}=\tfrac{1}{3}[4k^2-12k+11-2\mathcal{D}_3(k)]. \tag{18}$$

This bears some similarity to (16). Indeed we have

$$\deg^{(16)} - \deg^{(18)} = (\mathcal{D}_3(n-1)-1)\lfloor (n-1)/3 \rfloor \tag{19}$$

which means that every third degree value is the same.

The non-integrable case gives

$$1, 1, 1, 1, \mid 9, 18, 38, 90, 206, 469, 1071, \; 2446, 5586, 12755,$$
$$29125, 66507, 151867, 346783, 791869, \ldots$$

Now the growth is approximately $0.279 \cdot 2.283^n$.

6.3 Schwarzian lattice Boussinesq equation (5)

It turns out that in the integrable case the degrees are the same as for the modified Boussinesq equation. Degrees for the non-integrable cases are different, however.

6.3.1 A generalization

In [29] a generalization for the Schwarzian lattice Boussinesq equation was given in the form

$$\frac{\mathcal{Q}_{p,q}(x_{0,1}, x_{1,1}, x_{0,2}, x_{1,2})}{\mathcal{Q}_{p,q}(x_{1,0}, x_{2,0}, x_{1,1}, x_{2,1})} = \frac{(Q_a x_{0,0} - Q_b x_{0,1})\,(P_a x_{1,2} - P_b x_{2,2})\,(Q_a P_b x_{1,1} - P_a Q_b x_{0,2})}{(P_a x_{0,0} - P_b x_{1,0})\,(Q_a x_{2,1} - Q_b x_{2,2})\,(Q_a P_b x_{2,0} - P_a Q_b x_{1,1})}, \tag{20}$$

in which $\mathcal{Q}_{p,q}(x_{0,1}, x_{1,1}, x_{0,2}, x_{1,2})$ and $\mathcal{Q}_{p,q}(x_{1,0}, x_{2,0}, x_{1,1}, x_{2,1})$ can be obtained from

$$\begin{aligned} \mathcal{Q}_{p,q}(x_{0,0}, x_{1,0}, x_{0,1}, x_{1,1}) \quad &:= \quad P_a P_b (x_{0,0} x_{0,1} + x_{1,0} x_{1,1}) \\ &\quad -Q_a Q_b (x_{0,0} x_{1,0} + x_{0,1} x_{1,1}) - G(p,q)(x_{0,1} x_{1,0} + x_{0,0} x_{1,1}) \,, \end{aligned} \tag{21}$$

by m and n shifts, respectively. Furthermore, the parameters P, Q, G are given by

$$P_a^2 = g(p) - g(a),\; P_b^2 = g(p) - g(b),\; Q_a^2 = g(q) - g(a),\; Q_b^2 = g(q) - g(b),\; G(p,q) = g(p) - g(q), \tag{22}$$

where $g(x) = x^3 - \alpha_2 x^2 + \alpha_1 x$. Actually the form of the function g is irrelevant, because only parameters P, Q, G enter in the equation, and due to their additive definition they can be considered free, except for the following constraints

$$P_a^2 - P_b^2 = Q_a^2 - Q_b^2, \quad G(p,q) = P_a^2 - Q_a^2. \tag{23}$$

In numerical computations we took $P_a = 40$, $P_b = 32$, $Q_a = 24$, $Q_b = 7$. We found that this equation has the same degrees as the standard Schwarzian lattice Boussinesq equation.

If $P_a^2 = P_b^2$, $Q_a^2 = Q_b^2$ (20) reduces to (5) with $b_0 = b_1 = 0$.

It was surmised in [29] that from (20) one can obtain the other one-component lattice Boussinesq equations by suitable transformations and limits, but no rigorous proofs have been presented. If such connections exist they are not simple. For example (20) is homogeneous and scale invariant while the b_1 term in (5) breaks that. Furthermore the b_i terms in (3) and (5) arise from the α_i terms in $g(x)$ but in (20) the α_i terms are entirely hidden in P, Q, G.

7 Discussion

The lattice Boussinesq equations are usually given as three component equations residing on a single lattice plaquette and on its boundaries. By eliminating certain variables in favor of others one can obtain [24] several one component representation of the three kinds of Boussinesq equations. It turns out [24] that a particular one-component equation can represent different variables in different equations, but in any case only three different one component equations remain.

The degree growth computations in the present 3×3 stencil case confirm the many results that have been obtained for equations defined in the 2×2 and 2×3 stencil and stated in Section 2. One additional observation is that it is enough to have just one non-numeric initial value to differentiate between integrable and non-integrable equations: for integrable equations the growth is linear in this case.

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Appendix

Integrability by the three-soliton condition

For the two Hirota bilinear cases we can study integrability by computing multi-soliton solutions. It is well known that all one-component Hirota bilinear equations have one- and two-soliton solutions, but the existence of three-soliton solutions, without additional constraints, is possible only for integrable equations.

The parameters in equations (1) have been chosen so that we have simple one-soliton solutions: Defining the plane wave factor (corresponding to $e^{kx+\omega t}$) by

$$\rho_{n,m}(k) := c_k \left(\frac{k-1}{k+1}\right)^n \left(\frac{k}{k-1}\right)^m, \tag{24}$$

we have the one-soliton solution

$$\tau_{n,m} = 1 + \rho_{n,m}(k_1),$$

and the two-soliton solution

$$\tau_{n,m} = 1 + \rho_{n,m}(k_1) + \rho_{n,m}(k_2) + A_{1,2}\, \rho_{n,m}(k_1)\rho_{n,m}(k_2),$$

where the phase factor is

$$A_{k_i,k_j} := \frac{(k_i - k_j)^2}{k_i^2 + k_i k_j + k_j^2 - 1}. \tag{25}$$

The denominator is typical to Boussinesq type equations. One can then verify that (1) has the three-soliton solution[4]

$$\begin{aligned}\tau_{n,m} =&1+\rho_{n,m}(k_1)+\rho_{n,m}(k_2)+\rho_{n,m}(k_3)\\ &+A_{1,2}\,\rho_{n,m}(k_1)\rho_{n,m}(k_2)+A_{2,3}\,\rho_{n,m}(k_2)\rho_{n,m}(k_3)+A_{3,1}\,\rho_{n,m}(k_3)\rho_{n,m}(k_1)\\ &+A_{1,2}A_{2,3}A_{3,1}\,\rho_{n,m}(k_1)\rho_{n,m}(k_2)\rho_{n,m}(k_3),\end{aligned}$$

without any additional restrictions on k_j.

Turning now to the non-integrable case (2) with the given coefficients, we have the plane-wave factors

$$\rho_{n,m}(k) := c_k\left(\frac{-k+3}{9k+3}\right)^n\left(\frac{-k-1}{k-1}\right)^m,$$

and then the one-and two-soliton are solutions as above, but with the phase factor

$$A_{i,j}=\frac{-3(k_i-k_j)^2}{[k_ik_j+1][3k_i^2k_j^2-8k_ik_j(k_i+k_j)-3(k_i^2+k_ik_j+k_j^2)]}.$$

After these an attempt for a three-soliton solution yields the condition

$$k_1k_2k_3+k_1+k_2+k_3=0,$$

and thus the three line-solitons cannot be in general position. This is typical for the non-integrable case.

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