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Geometrical description of equations related to the $\mathrm{E}_8^{(1)}$ affine Weyl group

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Abstract

We present a method for the construction of the trajectory of a discrete Painlevé equation associated with the affine Weyl group $\mathrm{E}_8^{(1)}$ on the weight lattice of said group. The method is based on the geometrical description of the lattice and the construction of the fundamental Miura relation. To this end we introduce the relation between the nonlinear variables and the corresponding τ functions. Our approach is heuristic and makes use of some simple rules of thumb in order to derive the result. Once the latter is obtained, verifying that it does indeed correspond to the equation at hand is elementary. We apply our approach to the explicit construction of the trajectory of well-known, $\mathrm{E}_8^{(1)}$ associated, discrete Painlevé equations derived in previous works of ours. For each of them we investigate the possibility of defining an evolution by periodically skipping up to four intermediate points in the trajectory and identifying the resulting equation to one previously obtained, whenever the latter exists.

Keywords: discrete Painlevé equations, affine Weyl groups, singularity confinement

1 Introduction

Discrete Painlevé equations have, by now, been around for three decades [1]. To tell the truth, their history is much longer than that. While integrable non-autonomous recursion relations can be found already in the work of Laguerre [2], it is Shohat [3] who first derived a system that qualifies as a discrete Painlevé equation. Its identification had to wait for 50 years, when the same system was rediscovered by Brezin and Kazakov [4] who, by computing its continuum limit, identified it as a discrete Painlevé equation. While a few other forms of discrete Painlevé equations were also obtained [5,6,7], their systematic construction is due to the present authors in collaboration with J. Hietarinta [1], where the discrete analogues of equations $\mathrm{P_I}$ to $\mathrm{P_V}$ were derived using the deautonomisation method [8]. The derivation of a first discrete form for $\mathrm{P_{VI}}$ was given by Jimbo and Sakai

[9] who proved that what was called the asymmetric (in the QRT [10] sense) discrete P_{III} had indeed P_{VI} as continuum limit. The quest for a discrete P_{VI} was finally over when a QRT-symmetric form was proposed [11] by the present authors.

By that time it was clear that the discrete Painlevé equations were systems richer than their continuous brethren. The relation of discrete to continuous equations, through continuum limits, as a first attempt at a classification, proved quite unsatisfactory since instances of discrete Painlevé equations involving up to 8 parameters were already known. Taking the continuum limit transforms one of the parameters of the discrete Painlevé equation into the continuous variable but this leaves up to 7 genuine parameters to be matched to those of the continuous Painlevé equations which can have only up to four parameters. To put it in an oversimplified way, based on the continuum limit, most discrete Painlevé equations would be the discrete analogues of P_{VI}.

The solution to the question of classification of discrete Painlevé equations was provided by Sakai [12]. His approach consisted in studying rational surfaces in connection to extended Weyl groups. Discrete Painlevé equations are recovered as birational mappings corresponding to translations of an affine Weyl group. While the Sakai approach may seem somewhat abstract, it is quite useful for the understanding of various aspects of discrete Painlevé equations and discrete systems in general. Sakai himself provided the link between the property of singularity confinement [13] and the construction of the space of initial conditions. He has shown that all discrete Painlevé equations have a maximum of 8 confined singularities and can be described by a maximum of 8 blow-ups.

One important finding of Sakai concerns the equations related to the group $E_8^{(1)}$. Apart from the already well-known additive and multiplicative equations a new kind of discrete Painlevé equation did exist, one where the independent variable as well as the parameters enter through the arguments of elliptic functions. They are by now referred to as elliptic discrete Painlevé equations [14].

The discovery of Sakai led naturally to a redefinition of what a discrete Painlevé equation is, doing away with the restrictive definition as an integrable second-order non-autonomous mapping the continuum limit of which is a (differential) Painlevé equation. Instead, we now consider that a discrete Painlevé equation is a birational mapping on $\mathbb{P}^1 \times \mathbb{P}^1$ obtained by translations on the periodic repetition of a non-closed pattern on a lattice associated to one of the affine Weyl groups belonging to the degeneration cascade starting from $E_8^{(1)}$. The immediate consequence of this is that the number of discrete Painlevé equations is infinite [15]. In fact there exist infinitely many such equations for *each* of the affine Weyl groups in the Sakai degeneration cascade (except for the four 'no-parameter' groups $A_1^{(1)}$). In [16] we gave specific examples of construction of discrete Painlevé equations which, through their structure, illustrate the fact that we can have infinitely many discrete Painlevé equations associated to a given affine Weyl group.

From Sakai's classification it was clear that the richest, but also most complicated, family of discrete Painlevé equations is the one associated to $E_8^{(1)}$. A first step towards the systematic construction of equations belonging to this family was presented in [17] where we introduced the form we dubbed trihomographic. (In [18] we showed that all discrete Painlevé equations can be cast into a trihomographic form). Several examples of such systems were produced in [19] including instances of elliptic discrete Painlevé equations. A more systematic approach was made possible with the introduction of the representation in terms of an ancillary variable [20]. Thanks to this representation the study of the singularities of the system was greatly simplified allowing a straightforward application of the singularity confinement integrability criterion.

While the discrete Painlevé equations so discovered do possess the correct number of 8 parameters, their relation to the affine Weyl group $\mathrm{E}_8^{(1)}$ has never been established. To tell the truth, the only equations for which the precise trajectories in the lattice associated to $\mathrm{E}_8^{(1)}$ are known are the 'generic' discrete Painlevé equations, derived in [21] by the present authors in collaboration with Y. Ohta. Finding the proper embedding of the various discrete Painlevé equations of our list presented in [17] into the geometry of $\mathrm{E}_8^{(1)}$ has been a long-standing challenge. In this paper we intend to rise to this challenge and provide a method for the detailed geometrical description of these most interesting systems. The construction of the trajectory of these discrete Painlevé equations may appear somewhat involved but as we shall show, once the result is obtained, it is elementary to check its validity. We start by a brief reminder of the geometry of the $\mathrm{E}_8^{(1)}$ weight lattice and the bilinear identities obtained through the relations of the nonlinear variables to the corresponding τ functions. Then, we proceed to an outline of our method and the systematic construction of the trajectories of all trihomographic discrete Painlevé equations derived in [17, 19].

2 The geometry of the $\mathrm{E}_8^{(1)}$ weight lattice and the construction of trajectories

The geometry of the $\mathrm{E}_8^{(1)}$ weight lattice has been described at length in [21]. We will just give here a short summary.

Our basic result in [21] was that the τ-functions "live" on the points of the 8-dimensional weight lattice of $\mathrm{E}_8^{(1)}$. The coordinates of these points, in the orthonormal basis we considered are either *all* integer or *all* half-integer, with the additional constraint that the sum of all coordinates is even. In fact, one can reverse the direction of any number of the orthogonal directions. If an even number of such directions are inverted, nothing happens. If one inverts an odd number of directions, then for a point of integer coordinates the sum remains even, but for all points of half-integer coordinates the sum becomes odd. These two choices are perfectly equivalent, but for purely æsthetical reasons, we will choose the latter constraint, which we will henceforth call the alternate constraint, in some of the following sections and subsections. It goes without saying that the choice of constraint will be clearly announced.

This is not the only way the choices of this paper will differ from the original choices of [21]. For the points where τ-functions exist, the largest denominator of the coordinates is 2. But later in the paper we will consider other objects, namely the nonlinear variables. These variables "live" on points the coordinates of which can have a denominator as large as 4. In order to get rid of the all denominators we will multiply all coordinates by 4, with respect to the choices of [21], in addition to the change of constraint. To summarise, the τ-functions exist on points of coordinates either all multiple of 4 or all congruent to 2 modulo 4. In the first case the sum will always be a be multiple of 8. In the second case the sum will be either a multiple of 8, as per the original constraint, or congruent to 4 modulo

8, as per the alternate constraint, depending on the specific choice we make in a given subsection. For the remainder of the present section we will stick to the original choice of [21], up to the extra factor 4, i.e., a sum of coordinates multiple of 8 for coordinates either multiple of 4 or congruent to 2 modulo 4.

The origin obviously satisfies the requirements of the previous paragraph. The nearest-neighbours, NN, positions where τ's exist are at squared distance 32 of the origin. Some have six coordinates of value zero, the last ones being $a_i = \pm 4$, $a_j = \pm 4$, the other ones have all eight coordinates of absolute value 2, with sum multiple of 8. We will call NV's (for 'Nearest-neighbour-connecting Vectors') the vectors, of squared length 32, from the origin to any of its nearest-neighbours. Though the NV's, in this specific basis, seem to belong to two classes, this is not true; it is a pure artefact of choice of the basis. In fact the NV's correspond to each other by the symmetries of the underlying finite group $\mathrm{E}_8^{(1)}$. The entire lattice is invariant by translation by any NV.

Next we turn to the next-nearest-neighbours (NNN's) of a given τ. We can reach them by moving away from this τ by a vector that we will call an NNV (for 'Next-Nearest-neighbour-connecting Vectors') which is as short as possible a sum of NV's (which is not an NV itself). The squared length of an NNV is 64. All NNVs are fully equivalent, corresponding to each other through the symmetries of the finite group $\mathrm{E}_8^{(1)}$. In this basis they seem to come in three classes, but again it is purely an artefact of the choice of basis. There are NNVs with only one nonzero coordinate of value ± 8, some with four coordinates of value zero and four coordinates ± 4, and some with one coordinate ± 6, the seven other being ± 2, with total sum multiple of 8.

For convenience, in what follows and whenever there is no ambiguity, we will use a lower-case symbol for the name of a nonlinear variable and the same symbol in upper-case to mean the point where this variable is defined.

We have shown in [21] that the nonlinear variables, for which we will use the symbols x or y (and occasionally z, w and u) are defined at points of the lattice which are half-way from the position of a τ-function to one of its NNN's. For example, we have a nonlinear variable x defined at the point $X =$(4 0 0 0 0 0 0 0), midpoint between the origin and the point (8 0 0 0 0 0 0 0). It can be easily shown that X (and in fact any other such point) is at the midpoint not only of the original pair, but of exactly eight pairs of τ sites which are next-nearest-neighbours of each other, namely the original one {(0 0 0 0 0 0 0 0) (8 0 0 0 0 0 0 0)} and seven more of the form {(4 0 ... 4 ... 0), (4 0 ... −4 ... 0)}, etc, where the second non-vanishing coordinates is at any of the seven last positions. The vectors joining the two sites of each of these eight pairs are all distinct NNVs. One can easily see that any two of them are orthogonal. Note however that each of them is only defined up to a sign.

The nonlinear variable x at point X must now be related to the τ's. For each X we have eight NNVs $\overrightarrow{V_i}$s and we can introduce eight quantities C_i which are the scalar products of these vectors with the position vector $\overrightarrow{O'X}$. (Note here that the origin O' of this position vector need not coincide with the origin of coordinates: it may well be shifted by eight arbitrary numbers α_i). However, as we explained above, the orientations are not determined, consequently there exist an arbitrariness in the definition of the sign of each C_i. Next, we introduce the quantities ϕ_i which are the products of the two τ's at the ends of each NNV around X. One has:

$$x = \frac{f(C_j)\phi_i - f(C_i)\phi_j}{g(C_j)\phi_i - g(C_i)\phi_j} \tag{1}$$

where the $f(C_i)$'s and $g(C_i)$'s are as yet undetermined functions of their respective C_i. Note however that since the C_i's are not determined better than up to a sign, $f(C_i)$ and $g(C_i)$ must both be even functions. If one of the τ-functions at the end of one NNV $\overrightarrow{V_i}$ vanishes, then the quantity ϕ_i vanishes and the value of x is $f(C_i)/g(C_i)$, totally determined by this vanishing. This remark is the basis of the present paper: if one has identified an equation with a specific 'singularity pattern', one initial 'singular' value of the nonlinear variable x determining the value of the next one, for several steps, followed by a 'confinement' of the singularity, i.e. a free value of the variable after a given number of steps, it means that the positions of the respective variables which have a fixed value, i.e. 'within the singularity', are exactly half a NNV away from the τ-function that vanishes, that is, at squared distance 16 from it. Contrariwise, the 'free nonlinear variables', outside the singularity, are further away. This gives very severe restrictions on the 'trajectory' of the positions of the respective variables in the geometry of $\mathrm{E}_8^{(1)}$, depending on the equation at hand.

As explained at length in [21], there exist 28 different ways to write X in terms of the ϕ_i. By equating any two of these expressions we obtain equations for the ϕ_i's, i.e. for the product of the τ-functions:

$$(f(C_j)g(C_k) - f(C_k)g(C_j))\phi_i + (f(C_k)g(C_i) - f(C_i)g(C_k))\phi_j + (f(C_i)g(C_j) - f(C_j)g(C_i))\phi_k = 0 \tag{2}$$

The overdetermined (but consistent) system of equations (2) is a non-autonomous Hirota-Miwa system [22] which describes completely the evolution of the multivariable τ-function in $\mathrm{E}_8^{(1)}$. They are the bilinear forms of the various equations that "live" in $\mathrm{E}_8^{(1)}$.

Consider a given point like $X = (4\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$ where a nonlinear variable exists. There are eight pairs of opposite NNVs such that X is one-half of these vectors away from a τ-function. In the particular case of the X we have chosen, these pairs are along each of the elementary vectors of the basis. Let us choose an NV (of length 32) which is not orthogonal to any of these NNVs, for instance $\overrightarrow{T} = [2\ 2\ 2\ 2\ 2\ 2\ 2\ 2]$. We now consider the point $Y = (3\ {-1}\ {-1}\ {-1}\ {-1}\ {-1}\ {-1}\ {-1})$ such that the vector from it to the site of X is half the NV considered above, $\overrightarrow{YX} = \overrightarrow{T}/2$. This turns out to be a valid site where we can define a nonlinear variable Y. This was not *a priori* obvious. For instance, if we translate the site of X by half of one other NV's, say $[0\ 0\ 0\ 0\ 0\ 0\ 4\ 4]$ orthogonal to some of the NNVs above, we do not end up at a midpoint of two NNN's τ's, and no nonlinear variable can be defined there. Note that the squared distance from X to Y is 8.

Similarly to Y we can introduce $\overline{Y}$ corresponding to the point $(5\ 1\ 1\ 1\ 1\ 1\ 1\ 1)$ such that $\overrightarrow{X\overline{Y}} = \overrightarrow{T}/2$. Here, the overline symbol $\overline{}$ denotes a translation by the full NV, $\overrightarrow{T}$. Since the point $\overline{Y}$ is distant from the site of Y by a full NV, a translation which leaves the lattice invariant, all the τ's around $\overline{Y}$ are in the same positions with respect to it as those around Y. The same is also true as far as the τ's around X and $\overline{X}$ are concerned. (On the other hand, the τ's around X and Y are *not* the same). In fact, one can easily convince oneself that the eight NNVs around Y and $\overline{Y}$ are identical, and have all their coordinates 2, but for one coordinate equal to -6 at any of the eight positions, up to a global sign. They are symmetrical of the NNVs around X with respect to the hyperplane orthogonal to the YX line.

Instead of $\overrightarrow{T}$ we could have chosen any other NV with all coordinates of absolute value equal to 2. Let us consider a different one, forming an angle $\pi/3$ with $\overrightarrow{T}$, thus with scalar product with the latter equal to 16. To be specific let us choose the point Z such that the vector $\overrightarrow{ZX}$ is half the NV [2 -2 -2 2 2 2 2 2]. The point $Z =$(3 1 1 -1 -1 -1 -1 -1) forms an equilateral triangle with X and Y. The point $\widetilde{Z} =$(5 -1 -1 1 1 1 1 1)), symmetric of Z with respect to X is also a valid point to define a nonlinear variable, and forms an equilateral triangle with X and $\overline{Y}$.

In order to define a variable like x through (1) we need two products ϕ involving four τ's. It turns out that just six well chosen τ's suffice to define all three variables x, y and z: the two τ_{+-} and τ_{-+} at (2 2 -2 -2 -2 -2 -2 -2) and (2 -2 2 -2 -2 -2 -2 -2) (the indices refer to the signs of the second and third coordinates) and the four $\tau_{2,\epsilon}$ and $\tau_{3,\epsilon}$ ($\epsilon = \pm 1$) at the points (4 4ϵ 0 0 0 0 0 0) and (4 0 4ϵ 0 0 0 0 0) respectively. Indeed, X is the midpoint of the two pairs $\{\tau_{i+}, \tau_{i-}\}$ $i = 2, 3$ while Y is that of the pairs $\{\tau_{+-}, \tau_{2-}\}$, $\{\tau_{-+}, \tau_{3-}\}$ and Z that of the pairs $\{\tau_{+-}, \tau_{3+}\}$ and $\{\tau_{-+}, \tau_{2+}\}$.

We can obtain the value of x by specifying $i = 3, j = 2$ in (1)

$$x = \frac{f(C_2)\phi_3 - f(C_3)\phi_2}{g(C_2)\phi_3 - g(C_3)\phi_2} \tag{3}$$

with $\phi_i = \tau_{i+}\tau_{i-}$. Solving for the ratio of τ's we find:

$$\frac{\tau_{2+}\tau_{2-}}{\tau_{3+}\tau_{3-}} = \frac{g(C_2)x - f(C_2)}{g(C_3)x - f(C_3)} \tag{4}$$

Similarly we have

$$\frac{\tau_{+-}\tau_{2-}}{\tau_{-+}\tau_{3-}} = \frac{g(F_2)y - f(F_2)}{g(F_3)y - f(F_3)} \quad \text{and} \quad \frac{\tau_{-+}\tau_{2+}}{\tau_{+-}\tau_{3+}} = \frac{g(K_2)z - f(K_2)}{g(K_3)z - f(K_3)} \tag{5}$$

where the F_i and K_i are the analogs, for Y and Z respectively, of the C_i for X.

It is straightforward to eliminate all the τ's from (4-5) and find the contiguity relation on the equilateral triangle of side of squared length 8.

$$\frac{g(C_3)x - f(C_3)}{g(C_2)x - f(C_2)} \, \frac{g(F_2)y - f(F_2)}{g(F_3)y - f(F_3)} \, \frac{g(K_2)z - f(K_2)}{g(K_3)z - f(K_3)} = 1 \tag{6}$$

This relationship between x, y and z, which is linear separately in each of the nonlinear variables, is the basic one. We have dubbed relations of this form trihomographic.

The compatibility of the highly overdetermined (but consistent) system of the Hirota-Miwa equations (2), or rather, of the Miura relations (6) have been studied at length in [21]. In that paper the aim was to derive the generic equations satisfied by the nonlinear variables, starting from (6). What is important for the purpose of the present paper is the degrees of the equations obtained by eliminating some nonlinear variables from appropriate combinations of Miura relations. Still we will follow the steps of [21] where eliminating variables in order to obtain equations on triangles of various shapes was the way we obtained finally the equation "on the straight line" $YX\overline{Y}$.

Besides YXZ one can consider some more equilateral triangles, with one summit at X. Around such triangles we can get analogues of equation (6). In particular we are interested in the equilateral triangle WXZ where W has coordinates $(5\ 1\ 1\ 1\ {-1}\ {-1}\ {-1}\ {-1})$ so $\overrightarrow{XW}$ is orthogonal to $\overrightarrow{YX}$. Eliminating z between the Miura in these two triangles, one can obtain an equation in the isosceles right triangle YXW. One can easily convince oneself that this relation is still linear separately in y and w. In [21] we only mentioned that the degree in x was higher than one, but the fact is that this degree is exactly 2. On the other hand, the point U of coordinates $(5\ 1\ 1\ 1\ 1\ 1\ {-1}\ {-1})$ forms an equilateral triangle not only with X and $\overline{Y}$ but also with X and W. So just as in the above construction, one can obtain by eliminating the variable u an equation in the isosceles right triangle $WX\overline{Y}$, which is linear separately in w and $\overline{y}$ and, of course, also of the second degree in x. We have shown in [21] that eliminating w leads to a relation involving only y, x and $\overline{y}$, which is still linear separately in y and $\overline{y}$, and of the fourth degree in x.

The construction presented there allowed us to derive the nonlinear equation for x and y. It goes without saying that the bulk of computations was considerable and, as a matter of fact, in the case of the elliptic discrete Painlevé equation, prohibitively so. Thus we did not present its explicit form and limited ourselves in [21] to those of the multiplicative and additive equations.

Note that the degree of x is 1 in the Miura on the equilateral triangle YXZ, where the squared distance between the two other points is 8. The degree is 2 in the equation on the isosceles right triangles YXW and $WX\overline{Y}$ where the squared distance between the two other points is 16. On the segment of straight line $YX\overline{Y}$, where the squared distance between the two other points is 32, the degree of x in the equation is 4. The latter equation can also be obtained by eliminating z between the equation on equilateral triangle YXZ and an, as yet not discussed, equation on the triangle $ZX\overline{Y}$. It turns out that the degree of x in the latter equation is 3. And it is easy to check that the squared distance between Z and $\overline{Y}$ is 24. The computations to obtain these degrees, as mentioned above, are considerable but manageable, at least in the case of the multiplicative and additive equations (but the degrees will be the same for the elliptic equations, though the full expression becomes much longer) because all the τs involved are at distance 16 from X and belong to various instances of (1).

Going beyond, by direct calculation in the lattice would be totally prohibitive. So at this point all we can say is that equations relating three points, one being at squared distance 8 from both the others, is linear in the latter two and the degree in the 'central' one is 1, 2, 3 or 4 when the squared distance between the two others is 8, 16, 24 or 32, respectively. This will also be a very useful tool for what follows.

3 Outlining our method

In this section we shall present, in a general setting, the approach we shall follow in order to obtain the trajectories in the $\mathrm{E}_8^{(1)}$ weight space for the trihomographic discrete Painlevé equations obtained in [17] (and presented in a better organised form in [19]). We should point out from the outset that this constructive approach is rather involved and heuristic. However, once the result is obtained, it is straightforward to show that the geometry does indeed correspond to the equation under consideration.

As stated above, all the equations studied in the present paper are of the trihomographic form (6). But this relation must be reinterpreted. Instead of an intermediate step in the construction of the equation on the straight-line segment $YX\overline{Y}$ it is one instance of the equation itself.

In fact, we are dealing with two kinds of equations. The first kind, called "symmetric equations" (using the QRT terminology) are in terms of a single variable x and each instance is on an equilateral triangle $X_{n-1}X_nX_{n+1}$. The second kind, called "asymmetric equations" (always in the QRT sense) are in terms of two variables x, y and instances alternate on equilateral triangles $Y_{n-1}X_nY_n$ and $X_nY_nX_{n+1}$. In both cases, the equations depend on the variable n in two ways: there is a secular dependence, to which is superposed a periodic dependence.

As we stated above, for each x (resp. y) we have 8 NNVs $\overrightarrow{V_i}$s and we can introduce 8 quantities C_i (resp. F_i) which are the scalar products of these vectors and the position vector $\overrightarrow{O'X}$ (resp. $\overrightarrow{O'Y}$).

The point O' is a fixed point. But for convenience, as explained in the previous section, we will not attribute its 8 coordinates zero values, but rather 8 arbitrary numbers.

Let us discuss the symmetric case first. For every m, each vector $\overrightarrow{X_{m-1}X_m}$ is half an NV, of squared length 8. But contrary to the NV vector $\overrightarrow{Y\overline{Y}}$ which was of interest in [21], here the vector $\overrightarrow{X_{m-1}X_{m+1}}$ is *not* an NV (in fact, it is precisely half an NV), and therefore a translation by this vector does *not* leave the entire lattice invariant. This means that the "environment" of X_{m+1} in terms of τs is not the same as that of X_{m-1}: the 8 NNVs are different at each X. Hence the equation on triangle $X_{n-1}X_nX_{n+1}$ must be written as

$$\frac{g(C_1(n-1))x_{n-1}-f(C_1(n-1))}{g(C_2(n-1))x_{n-1}-f(C_2(n-1))}\,\frac{g(C_3(n+1))x_{n+1}-f(C_3(n+1))}{g(C_4(n+1))x_{n+1}-f(C_4(n+1))}\,\frac{g(C_5(n))x_n-f(C_5(n))}{g(C_6(n))x_n-f(C_6(n))}=1 \tag{7}$$

where the indices 1 to 6 are purely arbitrary, in as much as the set of C_i's is completely different for each value of n, until one has advanced by P steps, where P is the total period related of the equation at hand. In the triangle $X_{P+n-1}X_{P+n}X_{P+n+1}$, the equation is the same as in the triangle $X_{n-1}X_nX_{n+1}$ except for the *secular* dependence: the NNV vectors around each point are the same in both triangles. This means that the vector $\overrightarrow{S} = \overrightarrow{X_mX_{P+m}}$, the same for any m, leaves the lattice invariant and is a sum of NV's, and *a priori* we do not know its squared length. The P vectors $\overrightarrow{X_mX_{m+1}}$ "wind" around a straight line generated by the vector $\overrightarrow{S}$.

By symmetry, all of these P vectors have the same scalar product d with $\overrightarrow{S}$, and thus the squared length of $\overrightarrow{S}$ is Pd, but again, *a priori* we do not know what d is, except that it cannot be zero, lest the secular behaviour disappear. Thus the C_i's at X_n and X_{P+n} are the same up to the addition of the scalar product of the relevant NNV with $\overrightarrow{S}$.

The aim of this present paper is, for each of the 12 equations of [17] and [19] (since the geometry is the same for additive, multiplicative or elliptic equations), to present a consistent choice of coordinates for the sites of the non-linear variables.

For the 5 symmetric equations it means first finding the P vectors $\overrightarrow{X_mX_{m+1}}$ and then an appropriate "initial point" X_0 from which to find all the other points by adding the relevant vectors. The tools are the following ones. For each equation it is easy to compute the homogeneous degrees of the iterates, starting from initial conditions of degree 0 for x_m and degree 1 for x_{m+1}. As long as the degree of x_{m+q} is not larger than 4 one knows that the squared length of $\overrightarrow{X_mX_{m+q}}$ is exactly 8 times this degree, so we know the squared length of the sum of q consecutive vectors $\overrightarrow{X_mX_{m+1}}$ (starting from *any one of them*) until the degree of x_{m+q} becomes larger than 4. Moreover, though we do not *a priori* know $\overrightarrow{S}$ and the common value d of its scalar product with all the $\overrightarrow{X_mX_{m+1}}$, an educated guess can often be used, to be confirmed at the end of the calculations (or infirmed, in which case a different guess can be tried). Fixing the scalar product with a guessed $\overrightarrow{S}$ helps finding the solution if the guess is right. If not, it rapidly leads to contradictions, and one must start afresh. Of course the solution is far from being unique, as any transformation of the group $\mathrm{E}_8^{(1)}$ would give another solution.

Two rules of thumb did help us to find a solution. First, since all the periods P of the equations studied in the present paper are even, rather than guessing the sum of P vectors to get $\overrightarrow{S}$, we tried to guess the sum of only $P/2$ vectors $\overrightarrow{X_mX_{m+1}}$ as a vector such that its scalar product with the sum of the $P/2$ first ones would be the same (namely, $d/2$). This rule of thumb, combined with the main condition, namely that the partial sums $\overrightarrow{X_mX_{m+q}}$ are of squared length 8 times the degree of x_{m+q} in terms of x_{m+1} (for x_m of degree 0), as long as this degree does not exceed 4, allows the calculations to become manageable. In fact, in all cases, we did find such a vector which indeed turned out to be $\overrightarrow{S}/2$. So for each case there are only $P/2$ distinct vectors, but their sum $\overrightarrow{S}/2$ never turns out to be a sum of NV's and a translation by this vector does not leave the lattice invariant. The "environment" of $X_{P/2+m}$ in terms of τs is totally different from that of X_m, the NNVs not being the same. The same NNVs are only obtained after P steps, by repeating the same $P/2$ vectors in the same order.

The second rule of thumb was used only after we found the solution for the first few cases we are going to present. It became apparent that the squared distance between X_m and X_{m+q} was *always* 8 times the degree of x_{m+q} in terms of x_{m+1} (for x_m of degree 0). And this was true not only up to degree 4, but, in fact, for arbitrary degree. Since we have no *proof* that this property is true beyond degree 4, we only used it as an "accelerator" to find the vectors $\overrightarrow{X_mX_{m+1}}$ from a single starting point, much faster than if one had to limit oneself to degree 4 and keep changing the starting point when getting too far. However we always checked *a posteriori* that the results obtained through the "fast rule, beyond degree 4" never violated the "safe rule, up to 4 only" from every possible starting point.

Next we consider the first step in the case for asymmetric equations. In that case, one has a succession of equations on triangles $X_nY_nX_{n+1}$, $Y_nX_{n+}Y_{n+1}$. All the vectors $\overrightarrow{X_mY_m}$ and $\overrightarrow{Y_mX_{m+1}}$ are half-NV's of squared length 8. Before getting to the next triangles where the terms in the equation are just shifted by the secular evolution, there are altogether $2P$ such vectors. However, the situation is not that complicated. Since the vectors $\overrightarrow{X_mX_{m+1}}$ (or $\overrightarrow{Y_mY_{m+1}}$, for that matter), are also half-NV's, on can skip the points Y (resp. X) and get an equation for x (resp. y) only.

Granted, this equation is not *in general* trihomographic. But this is not essential for the discussion at this point. As long as we can compute the degrees of the iterates x_{m+q} in terms of x_{m+1} (for x_m of degree 0), we can use the strategy described above. Thus, knowing the equation for x only, we can find as in the previous case, the P vectors $\overrightarrow{X_mX_{m+1}}$, or rather, as it always turns out, the first $P/2$ such vectors, guess their sum $\overrightarrow{S}/2$ and use the fact that their scalar product with it is always the same as a guide. One can do it for

y as well as x but doing both does not help much. Indeed, the solution is far from being unique, there is a lot of freedom in the first steps. A specific, arbitrary, choice can be made without loss of generality. But there is no way to correlate such choices for two different calculations. So the strategy is, once the succession of the $\overrightarrow{X_m X_{m+1}}$ (or, equivalently, the $\overrightarrow{Y_m Y_{m+1}}$, but not both) is chosen, to "insert" the other variable in between.

All the tools we presented in the symmetric case can be used, and the calculations are typically rather faster. One point has to be made clear here: for all the equations *in the present paper*, the scalar products with $\overrightarrow{S}$ (or $\overrightarrow{S}/2$, for that matter) of the vectors $\overrightarrow{X_m Y_m}$ and $\overrightarrow{Y_m X_{m+1}}$ are equal. There is no deep reason for that. Symmetry imposes that all the $\overrightarrow{X_m Y_m}$ have the same scalar product with $\overrightarrow{S}$, and the same is true of all the $\overrightarrow{Y_m X_{m+1}}$, but these two quantities, of sum d, need *not* coincide. The reason they do coincide for the equations in the present paper is a consequence of the way these equations were found, by deautonomising QRT mappings. Equations presented in [23] do not suffer from this bias and some of them indeed do *not* share this property.

The last step, finding the appropriate initial point, is the same for symmetric and asymmetric equations. Presenting it in a general setting would make the presentation unnecessarily complicated, since the choice of the initial point in question relies on a heuristic approach. We feel that this step will be better explained in actual examples and thus we proceed, in the section that follows, to examine, one by one the discrete Painlevé equations of our list.

4 Working out specific examples

Before proceeding to the construction of the trajectories for specific discrete Painlevé equations it is useful to devote a few paragraphs to present the list thereof. The equations we are going to deal with were derived in [17] based on the trihomographic representation inspired by the form of the Miura (7). As shown there the generic form of a trihomographic equation, in the symmetric case, is

$$\frac{x_{n+1}-(z_n+z_{n-1}+k_n)^2}{x_{n+1}-(z_n+z_{n-1}-k_n)^2}\frac{x_{n-1}-(z_n+z_{n+1}+k_n)^2}{x_{n-1}-(z_n+z_{n+1}-k_n)^2}\frac{x_n-(2z_n+z_{n-1}+z_{n+1}-k_n)^2}{x_n-(2z_n+z_{n-1}+z_{n+1}+k_n)^2}=1 \tag{8}$$

where the functions z_n and k_n are specific to each equation.

Five symmetric discrete Painlevé equations were derived in [17]. In order to give the detailed n-dependence of their parameters we start by introducing the independent variable by $t_n \equiv \alpha n + \beta$, two constants γ and δ and two periodic functions $\phi_m(n)$ and $\chi_m(n)$. The first periodic function has period m, i.e. $\phi_m(n+m) = \phi_m(n)$, and is given by

$$\phi_m(n) = \sum_{l=1}^{m-1} \epsilon_l^{(m)} \exp\left(\frac{2i\pi ln}{m}\right). \tag{9}$$

Note that the summation starts at 1 instead of 0, since, given the expressions below, the constant term can be absorbed through a redefinition of β in t_n (or, depending on the case, of the constants γ and δ) and thus ϕ_m introduces $(m-1)$ parameters. The second periodic function χ_{2m} obeys the equation $\chi_{2m}(n+m) + \chi_{2m}(n) = 0$ and thus has period

$2m$ while involving only m parameters. It can be expressed in terms of roots of unity as

$$\chi_{2m}(n) = \sum_{\ell=1}^{m} \eta_\ell^{(m)} \exp\left(\frac{i\pi(2\ell-1)n}{m}\right). \tag{10}$$

(We should point out here that in the initial publications [17] and [19] the function χ was introduced with an index m instead of $2m$. This was amended in later publications [23] since it is more natural to have the index coincide with the periodicity of the function). Using the auxiliary functions just introduced we have for the five symmetric cases the expressions:

I (4, 5) $u_n = t_n + \phi_4(n) + \phi_5(n),\ z_n + z_{n+1} = u_{n+2} + u_n,\ k_n = u_{n+2} + u_{n-1}$

II (2, 3, 4) $u_n = t_n + \phi_2(n) + \phi_3(n),\ z_n + z_{n+1} = u_{n+1},\ k_n = \gamma + \phi_4(n)$

III (2, 3, 5) $u_n = t_n + \phi_2(n) + \phi_3(n) + \phi_5(n),\ z_n = u_n,\ k_n = u_{n+1} + u_n + u_{n-1}$

IV (2, 7) $u_n = t_n + \phi_2(n) + \phi_7(n),\ z_n = u_{n+1} - u_n + u_{n-1},\ k_n = u_{n+2} - u_n + u_{n-2}$

V (2, 3, 8) $u_n = t_n + \phi_2(n) + \phi_3(n),\ z_n + z_{n+1} = u_{n+1},\ k_n = \chi_8(n)$

where the numbers in the parentheses correspond to the periodicities of the parameters of each equation.

While the previous cases corresponds to equations symmetric in the QRT sense, trihomographic discrete Painlevé equations were also obtained in the asymmetric case. The general expression in this case is

$$\frac{x_{n+1} - (\zeta_n + z_n + k_n)^2}{x_{n+1} - (\zeta_n + z_n - k_n)^2}\,\frac{x_n - (\zeta_n + z_{n+1} + k_n)^2}{x_n - (\zeta_n + z_{n+1} - k_n)^2}\,\frac{y_n - (2\zeta_n + z_n + z_{n+1} - k_n)^2}{y_n - (2\zeta_n + z_n + z_{n+1} + k_n)^2} = 1 \tag{11a}$$

$$\frac{y_n - (\zeta_{n-1} + z_n + \kappa_n)^2}{y_n - (\zeta_{n-1} + z_n - \kappa_n)^2}\,\frac{y_{n-1}(\zeta_n + z_n + \kappa_n)^2}{y_{n-1} - \zeta_n + z_n - \kappa_n)^2}\,\frac{x_n - (\zeta_n + 2z_n + \zeta_{n-1} - \kappa_n)^2}{x_n - (\zeta_n + 2z_n + \zeta_{n-1} + \kappa_n)^2} = 1. \tag{11b}$$

and again the z, ζ, k and κ must be specified in each case. Seven discrete Painlevé equations were derived in [17]:

VI (8) $u_n = t_n + \phi_8(n),\ z_n = u_n - u_{n+1} + u_{n-1} - u_{n-2},\ \zeta_n = u_{n+2} - u_n + u_{n-2},$
$k_n = u_n,\ \kappa_n = u_{n+2} + u_{n-3}$

VII (4, 5) $u_n = t_n + \phi_4(n) + \phi_5(n),\ z_n = -u_n - u_{n-1},\ \zeta_n = u_{n+1} + u_n + u_{n-1},$
$k_n = u_n,\ \kappa_n = u_{n+1} + u_n + u_{n-1} + u_{n-2}$

VIII (4, 6) $u_n = t_n + \phi_3(n) + \phi_4(n),\ z_n = u_n + u_{n-1} + \psi_6(n) + \psi_6(n-1), \zeta_n = -u_n - 2\psi_6(n),$
$k_n = u_{n+1} + u_n + u_{n-1} + \psi_6(n),\ \kappa_n = \psi_6(n) + \psi_6(n-1)$

where $\psi_6(n) = \theta_1(-j)^n + \theta_2(-j^2)^n$, with $j^3 = 1$.

(Note that in [17, 19] the k and κ were permuted. Moreover the parametrisation chosen there was not the most convenient one since it introduced an extra, spurious, parameter. Using the periodic function $\psi_6(n)$ ensures that the number of parameters is the correct one).

IX $(2,3,4)$
$$u_n = t_n + \phi_3(n) + \phi_4(n),\ z_n + \zeta_n = u_{n-1} - \gamma,\ z_{n+1} + \zeta_n = u_{n+1} + \gamma,$$
$$k_n = u_{n+1} + u_n + u_{n-1} - \phi_2(n),\ \kappa_n = \gamma + \phi_2(n)$$

X $(2,2,4)$
$$u_n = t_n + \phi_4(n),\ z_n + \zeta_n = u_{n-1} - \gamma,\ z_{n+1} + \zeta_n = u_{n+1} + \gamma,$$
$$k_n = u_{n+1} - u_n + u_{n-1} + \phi_2(n),\ \kappa_n = \delta + \widetilde{\phi}_2(n)$$

where $\phi_2(n)$ and $\widetilde{\phi}_2(n)$ are two independent functions of period 2. (Note that in [17, 19] a misprint is present in the definition of $z_n + \zeta_n$ and $z_{n+1} + \zeta_n$. The expressions given above are the correct ones).

XI $(4,4)$
$$u_n = t_n + \phi_4(n),\ z_n + \zeta_n = u_{n-1} - \gamma,\ z_{n+1} + \zeta_n = u_{n+1} + \gamma,$$
$$k_n = u_{n+1} - u_n + u_{n-1} + \phi_2(n),\ \kappa_n = \chi_4(n)$$

and finally

XII $(2,3,4)$
$$z_n + \zeta_n = t_n - \gamma + \phi_3(n-1),\ z_{n+1} + \zeta_n = t_n + \gamma + \phi_3(n+1),$$
$$k_n = \chi_4(n),\ \kappa_n = \delta + \phi_2(n)$$

where, again, the numbers in the parentheses correspond to the periodicities of the parameters of the equation.

In the remainder of the article we shall refer very often to the 12 discrete Painlevé equations of this list. Each of these equations is to be understood as the generic symmetric (8) or asymmetric (11) equation complemented by the parameters as detailed in the preceding paragraphs. In order to avoid awkward formulations we have adopted the notation of a boldface roman numeral, **I**, **II**, $\cdots$, **XII** to denote the corresponding equation of the list of twelve, with the precise n-dependence of its parameters, given in the list.

In the following subsections we construct the trajectories of all 12 discrete Painlevé equations in the $\mathrm{E}_8^{(1)}$ weight space. Once the trajectory was constructed it turned out that in all cases we could define at least one evolution by skipping one intermediate point and, in some cases depending on the equation, even more evolutions skipping more (up to four) intermediate points. While the resulting equations are only in a few cases trihomographic, most of them still belong to a class of equations that we have previously studied. The publication [23] we are referring to was based on the introduction of what we called the ancillary variable. In fact, we showed there that using an ancillary variable ξ such that

$$x_n = \xi_n^2 \tag{12}$$

one can rewrite the general additive (difference) discrete Painlevé equation

$$\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)}$$

$$= 2\frac{x_n^4 + S_2 x_n^3 + S_4 x_n^2 + S_6 x_n + S_8}{S_1 x_n^3 + S_3 x_n^2 + S_5 x_n + S_7} \quad (13)$$

(were S_k are the elementary symmetric functions of the quantities $z_n + \kappa_n^i$, and κ^i eight parameters which are, generically, functions of the independent variable) as

$$\frac{x_{n+1} - (\xi_n - z_n - z_{n+1})^2}{x_{n+1} - (\xi_n + z_n + z_{n+1})^2}\,\frac{x_{n-1} - (\xi_n - z_n - z_{n-1})^2}{x_{n-1} - (\xi_n + z_n + z_{n-1})^2} = \frac{\prod_{i=1}^{8}(\kappa_n^i + z_n - \xi_n)}{\prod_{i=1}^{8}(\kappa_n^i + z_n + \xi_n)} \quad (14)$$

Note that while in the generic case the right-hand side of (14) is a ratio of 8 terms, corresponding to the quartic over cubic rational right-hand side of (13), simplifications may occur leading to right-hand sides involving ratios of six, four or even just two factors (the latter case being equivalent to a trihomographic one). In what follows we shall use an equivalent form of (14)

$$\frac{x_{n+1} - (\xi_n - Z_n)^2}{x_{n+1} - (\xi_n + Z_n)^2}\,\frac{x_{n-1} - (\xi_n - Z_{n-1})^2}{x_{n-1} - (\xi_n + Z_{n-1})^2} = \frac{\prod_{i=1}^{8}(\xi_n - A_n^i)}{\prod_{i=1}^{8}(\xi_n + A_n^i)}, \quad (15)$$

using the notations introduced in [23], where the ancillary parameters are $Z_n = z_n + z_{n+1}$ and $A_n^i = \kappa_n^i + z_n$.

As mentioned above, in the subsections that follow we shall present the construction of trajectories for the equations of the list. Rather than following the order of that list we shall proceed from the simplest cases to the more complex, which will, hopefully, make our presentation easier to follow.

4.1 The periods 2,7 case

In this section we shall illustrate the workings of the trajectory construction by applying them to the case of a discrete Painlevé equation with periods 2 and 7, case **IV** of the list given at the beginning of the section. In [23] it was listed as Class III of chapter 3. The periods superposed on the secular behaviour are 2 and 7, so the full period is 14. We compute the homogeneous degree growth of x starting from initial conditions where x_0 has degree 0 and x_1 has degree 1. We obtain thus the sequence of degrees $d_n = 0, 1, 1, 2, 3, 4, 6, 7, 10, 12, 15, 18, 21, 25, 28, \cdots$, a quadratic growth, as expected given the integrable character of the equation.

The normalisation chosen here is that for all τ-functions of coordinates congruent to 2 modulo 4, the sum of these coordinates will be congruent to 4 modulo 8. Now we introduce 7 vectors of squared length 8, each half of an NV (thus the double of each of the vectors of the list is indeed an NV, with components ± 2 with sum 4):

$$\begin{bmatrix} 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix}$$

and the list is understood as being repeated periodically. The squared length of the sum of any number m of consecutive ones is 8 times d_m.

The sum of all 7 vectors is [7 1 1 1 1 1 1 1]. We denote this vector as $\overrightarrow{S}/2$ and the scalar product of each of the elementary vectors with $\overrightarrow{S}/2$ is 8. The vector $\overrightarrow{S}$ is indeed a vector with components 14 and 2 (all congruent to 2 modulo 4) the sum of which is 28, congruent to 4 modulo 8 and a translation by this vector does leave the whole lattice invariant. The symmetry 7 can be easily seen: besides the first component which is always 1, three consecutive components -1 go around the seven last columns as around a torus, moving to the right by three positions from one line to the next.

In order to explain how to find the initial position of some X_m to which one has to add the above vectors to find all the other Xs one has to consider the singularity patterns of this equation. In order to obtain these patterns it suffices to work with a simplified form of the equation when one keeps the secular behaviour but ignores the periodicity. We have

$$\frac{x_{n+1}-(3t_n-\alpha)^2}{x_{n+1}-(t_n-\alpha)^2}\frac{x_{n-1}-(3t_n+\alpha)^2}{x_{n-1}-(t_n+\alpha)^2}\frac{x_n-9t_n^2}{x_n-25t_n^2}=1 \tag{16}$$

and the patterns are $\{x_{n-2}=25t_{n-2}^2, x_{n-1}=(3t_{n-2}-\alpha)^2, x_n=(t_{n-2}-4\alpha)^2, x_{n+1}=(t_{n+3}+4\alpha)^2, x_{n+2}=(3t_{n+3}+\alpha)^2, x_{n+3}=25t_{n+3}^2\}$ and $\{x_{n-1}=9t_{n-1}^2, x_n=(t_{n-1}-\alpha)^2, x_{n+1}=(t_{n+2}+\alpha)^2, x_{n+2}=9t_{n+2}^2\}$.

Throughout each singularity pattern, the variables have precise values which are exactly the square of some C_j defined in section 2. This means that they are exactly at squared distance 16 from a τ-function that happens to vanish.

So in order to reproduce the longest pattern, namely the one with five steps (involving thus 6 points) there must be a τ-function at distance exactly 16 from 6 consecutive X, while the X before and after these six ones are further away. To make this visually easy to follow, we choose to call the position of this τ-function the origin of coordinates. Since the point O' that we chose as the one from where all positions are measured has totally arbitrary coordinates, this is a freedom we can afford. Below are 8 points which are precisely separated by the seven vectors presented at the beginning of this section.

$$\begin{pmatrix} -4 & 2 & 0 & 2 & 0 & -2 & 0 & -2 \\ -3 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ -2 & 2 & 0 & 2 & 0 & -2 & 0 & 0 \\ -1 & 1 & -1 & 3 & 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 2 & 0 & -2 & 2 & 0 \\ 1 & 1 & 1 & 3 & 1 & -1 & 1 & -1 \\ 2 & 2 & 0 & 2 & 0 & 0 & 2 & 0 \\ 3 & 3 & 1 & 3 & 1 & -1 & 1 & -1 \end{pmatrix}$$

The first and last points are at a squared distance of 32 from the origin. The 6 intermediate ones are exactly at a squared distance of 16, and correspond to the "long" singularity pattern. The scalar products of $\overrightarrow{S}$ with the vectors joining the origin to each of those six points are (from top to bottom) -40, -24, -8, 8, 24 and 40 respectively, so eight times the the numbers $\{-5,-3,-1,1,3,5\}$. We remark that the squares of these numbers match the values of the coefficients of t_n present in the "long" pattern above. We shall refer to such a collection of numbers as the "schematic" singularity pattern.

For the "short" pattern with only three steps (and thus involving the four central points), one can easily check that the τ-function at the point $(0\ 0\ 0\ 4\ 0\ -4\ 0\ 0)$ is indeed at squared distance 16 of these points but at squared distance 32 of the two topmost and the two bottommost points. Moreover, the scalar product with $\overrightarrow{S}$ of the vectors joining it to the four central points are again -24, -8, 8, 24, that is, 8 times the coefficients of n, the square of which are present in this "short" pattern $\{-3,-1,1,3\}$.

One remark is in order here. Since the vector $\overrightarrow{X_n X_{n+2}}$ is also half of an NV, an equation can be found on the triangle $X_{n-2}X_nX_{n+2}$. Since the squared distance between X_{n-2} and X_{n+2} is 24 rather than 8, this triangle is not equilateral, and the equation is not trihomographic. Still, it is a valid equation which can be obtained by eliminating one x out of two. Clearly we can choose to eliminate the x with even or odd indices and the resulting values of the coefficients will be slightly different in order to reflect this choice. In what follows we shall present explicitly the case where the evolution is over even values $n = 2m$ (and the evolution over odd values can be obtained *mutatis mutandis*).

The equation thus obtained has been identified as Class II, case 7 in chapter 5, i.e. 5.2.7, in [23]. It assumes a simple form when written using the ancillary variable ξ where the right-hand side is a ratio of products of 6 terms. Written in the usual, "canonical" form, the equation has a right-hand side which is a ratio of a cubic polynomial in x over a quadratic one.

It is interesting at this point to express the quantities Z_m and A^i_m entering the equation in terms of the parameters appearing in **IV**. We introduce the quantity c which is equal to $-\phi_2(n) = -\phi_2(0)$, since we chose the evolution over even indices $n = 2m$ rather than odd, and find

$$\begin{aligned}
Z_m &= 4(2\alpha m+\beta)+4\alpha+\phi_7(2m+3)+\phi_7(2m+2)+\phi_7(2m)+\phi_7(2m-1)\\
A^1_m &= 3(2\alpha m+\beta)-4\alpha+c+\phi_7(2m+1)+\phi_7(2m)-\phi_7(2m-1)+2\phi_7(2m-3)\\
A^2_m &= 5(2\alpha m+\beta)-c+2\phi_7(2m+2)+\phi_7(2m+1)-\phi_7(2m)+\phi_7(2m-1)+2\phi_7(2m-2)\\
A^3_m &= 3(2\alpha m+\beta)+4\alpha+c+2\phi_7(2m+3)-\phi_7(2m+1)+\phi_7(2m)+\phi_7(2m-1)\\
A^4_m &= 2\alpha m+\beta-2\alpha-c-\phi_7(2m+1)+\phi_7(2m)+\phi_7(2m-1)\\
A^5_m &= 3(2\alpha m+\beta)+c+\phi_7(2m+1)+\phi_7(2m)+\phi_7(2m-1)\\
A^6_m &= 2\alpha m+\beta+2\alpha-c+\phi_7(2m+1)+\phi_7(2m)-\phi_7(2m-1)
\end{aligned}$$

As explained in [23], this equation has 6 distinct patterns. The patterns described there match the succession, i.e. what we called above "schematic singularity patterns" $\{-5,-1,3\}$, $\{-3,1,5\}$ for the two "long" ones and twice each of $\{-3,1\}$ and $\{-1,3\}$. Depending on which x we keep, the two "long" patterns of the equation 5.2.7 of [23] can be seen to come from the "long pattern" $\{-5,-3,-1,1,3,5\}$ of the equation under consideration (up to a factor 8), while one each of $\{-3,1\}$ and $\{-1,3\}$ come from the "short pattern" $\{-3,-1,1,3\}$. But what about the second "copy" of $\{-3,1\}$ and $\{-1,3\}$?

In [17] where our equation was identified for the first time, we only called "singularities" situations where one specific value of x_n, for generic x_{n-1}, would fix the value of x_{n+1}, or in other words, would create a blow-up. But we did not consider actions "at a distance". Just by looking at equation (16), we can see that if it happens, for generic x_n, that x_{n-1}

is equal to $(3t_n + \alpha)^2$, then this would imply that x_{n+1} is equal to $(t_n - \alpha)^2$, while if $x_{n-1} = (t_n + \alpha)^2$, then $x_{n+1} = (3t_n - \alpha)^2$. Since for this equation, this "interaction at distance two" did not cause a blow-up, we did not consider it. In some sense the blow-up-free patterns $\{-3, \star, 1\}$ an $\{-1, \star, 3\}$ (where the $\star$ stands for a free value) are "hidden" in equation (16). But when treating the equation "on every other point" this becomes a singularity, duly noted in 5.2.7 of [23]. The trajectory of the points, however, is the same for both equations, the only difference being in the points one is keeping. So what is the cause of these definite values of x at these precise points? What we can deduce from their existence is that there exists a τ in the neighbourhood of this trajectory that is exactly at distance 16 from the point "-3" at $(-2\ 2\ 0\ 2\ 0\ -2\ 0\ 0)$ and from the point "$+1$" at $(0\ 2\ 0\ 2\ 0\ -2\ 2\ 0\ )$ but further away from every other point on the trajectory, in particular from the point "-1" at $(-1\ 1\ -1\ 3\ 1\ -1\ 1\ -1)$, because the value of the x at the intermediate point is arbitrary: the vanishing of the τ in that position should determine only the points "-3" and "$+1$". Conversely there must be a τ at distance 16 from the point "-1" and also from the point "$+3$" at $(1\ 1\ 1\ 3\ 1\ -1\ 1\ -1)$, but further away from the point "$+1$" and all the others. And indeed this is the case. The first one is the τ at $(0\ 4\ 0\ 0\ 0\ -4\ 0\ 0)$ and the second one is at $(0\ 0\ 0\ 4\ 0\ 0\ 0\ -4)$. This explains the singularity patterns of both equations, the original one considered over every point X and the one obtained by skipping every other X.

4.2 The periods 2,3,5 case

The additive equation with period 2, 3 and 5 was presented in [17] and given as case **III** in [19] and in the list presented at the beginning of this section. This equation is listed in [23] as Class I of section 3. The overall period of the coefficients of the mapping is 30.

The homogeneous degree growth of x starting from initial conditions where x_0 has degree 0 and x_1 has degree 1 is given by the sequence of degrees d_m: 0, 1, 1, 1, 2, 2, 3, 4, 5, 6, 7, 9, 10, 12, 14, 15, $\cdots$.

Just as in subsection 4.1, we looked for 15, rather than 30, vectors of squared length 8, each half an NV.

$$\begin{bmatrix} 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

In the three leftmost columns, a single -1 goes around a torus, moving to the left, with the two other positions being $+1$. In the five rightmost columns, two consecutive -1 go around a torus, moving to the left by two positions down each line, with the three other positions being $+1$.

Again, the list of vectors must be understood not just a list of fifteen vectors but rather as an infinite repetition of the ones given above. As in the previous case, the constraint chosen for this section is that for all τ-functions of coordinates congruent to 2 modulo 4, the sum will be congruent to 4 modulo 8, rather than the original constraint of multiple of 8. The sum $\overrightarrow{S}/2$ of these fifteen vectors is [5 5 5 3 3 3 3 3] and the scalar product of each of the elementary vectors with $\overrightarrow{S}/2$ is 8. The vector $\overrightarrow{S}$ is indeed a vector with components 10 and 6, both congruent to 2 modulo 4, the total sum of which is 60, congruent to 4 modulo 8 and a translation by this vector does leave the whole lattice invariant.

To find an initial point, we must consider the singularity patterns of the equation. Again, we ignore the periodic functions and keep only the secular dependence. The equation becomes

$$\frac{x_{n+1}-(5t_n-\alpha)^2}{x_{n+1}-(t_n+\alpha)^2}\frac{x_{n-1}-(5t_n+\alpha)^2}{x_{n-1}-(t_n-\alpha)^2}\frac{x_n-t_n^2}{x_n-49t_n^2}=1 \tag{17}$$

There are two singularity patterns, a "long" one $\{x_{n-3}=49t_{n-3}^2, x_{n-2}=(5t_{n-3}-\alpha)^2, x_{n-1}=(3t_{n-3}-4\alpha)^2, x_n=(t_{n-3}-9\alpha)^2, x_{n+1}=(t_{n+4}+9\alpha)^2, x_{n+2}=(3t_{n+4}+4\alpha)^2, x_{n+3}=(5t_{n+4}+\alpha)^2, x_{n+4}=49t_{n+4}^2\}$ and a "short" one $\{x_n=t_n^2, x_{n+1}=t_{n+1}^2\}$. As already explained these two patterns can we written schematically as $\{-7,-5,-3,-1,1,3,5,7\}$ and $\{-1,1\}$.

The "long" pattern fixes the initial point, when we insist that it be generated by the vanishing of the τ-function at the origin of coordinates. For the sake of completeness, we present the full "half-period" pattern of points, although only eight of them (from the second one to the ninth, inclusive) belong to the "long" pattern. The list below has 16 points because we start with one point before entering the pattern.

$$\begin{pmatrix}
-2 & 0 & -4 & 0 & 2 & -2 & 0 & -2\\
-1 & -1 & -3 & -1 & 1 & -1 & 1 & -1\\
-2 & 0 & -2 & 0 & 2 & 0 & 0 & -2\\
-1 & 1 & -3 & 1 & 1 & -1 & 1 & -1\\
0 & 0 & -2 & 0 & 2 & 0 & 2 & -2\\
-1 & 1 & -1 & 1 & 3 & -1 & 1 & -1\\
0 & 2 & -2 & 0 & 2 & 0 & 2 & 0\\
1 & 1 & -1 & 1 & 3 & 1 & 1 & -1\\
0 & 2 & 0 & 2 & 2 & 0 & 2 & 0\\
1 & 3 & -1 & 1 & 3 & 1 & 3 & -1\\
2 & 2 & 0 & 2 & 4 & 0 & 2 & 0\\
1 & 3 & 1 & 1 & 3 & 1 & 3 & 1\\
-2 & 4 & 0 & 2 & 4 & 2 & 2 & 0\\
3 & 3 & 1 & 3 & 3 & 1 & 3 & 1\\
2 & 4 & 2 & 2 & 4 & 2 & 4 & 0\\
3 & 5 & 1 & 3 & 5 & 1 & 3 & 1
\end{pmatrix}$$

The scalar product of $\overrightarrow{S}$ with the vector from the origin to the topmost point is -72, and it increases by 16 down each line. From the second line to the ninth, these scalar products are 8 time the numbers $\{-7,-5,-3,-1,1,3,5,7\}$, as expected from the "long" schematic pattern. The corresponding points are indeed all at a squared distance 16 from the origin, while the topmost is at squared distance 32. The same is true for the tenth, and the squared distance never decreases further down.

The "short" pattern as defined above involves only one step $\{-1, 1\}$. With the hindsight of subsection 4.1, we decided to investigate whether more patterns might not be "hidden" by the absence of a blow-up, while some precise value of x_m might fully determine the value of an x_{m+q} with $q > 1$. Just by looking at equation (17) one can see easily that for x_n generic, a value $(5t_n + \alpha)^2$ for x_{n-1} determines the value of x_{n+1} as $(t_n + \alpha)^2$ that is, t^2_{n+1}. Granted, the patterns just after equation (17) only take into account the secular behaviour, not the periodic one, but the essential facts do survive with the periodic dependence added to the secular one. This "action at distance" looks like the one we found in subsection 4.1. But there is a major difference: the value $x_{n'} = t^2_{n'}$ for $n' = n+1$ is a very special one. It is the one that "opens" the "short" singularity pattern. The "short" singularity proceeds to $x_{n'+1} = t^2_{n'+1}$ and ends there, namely, $x_{n'+2}$ has a free value. But for $n'' = n' + 2 = n + 3$, it is the case that $x_{n''-1} = t^2_{n''-1}$, while $x_{n''}$ is generic. Another glance at (17) shows that in that case the value of $x_{n''+1}$ is not free but must be equal to $(5t_{n''} - \alpha)^2$: the pattern initiated by $x_{n-1} = (5t_n + \alpha)^2$ extends up to $x_{n+4} = (5t_{n+3} - \alpha)^2$. Only one blow-up appears in the pattern, but its total length extends over 6 points altogether, two of which have free values. Schematically the "dressed short pattern" is $\{-5, \star, -1, 1, \star, 5\}$ where $\star$ denotes a free value. One can infer that there must exist a τ in the neighbourhood of this trajectory that is exactly at distance 16 from the points "-5" at $(-2\ 0\ -2\ 0\ 2\ 0\ 0\ -2)$, "$-1$" at $(0\ 0\ -2\ 0\ 2\ 0\ 2\ -2)$, "$+1$" at $(-1\ 1\ -1\ 1\ 3\ -1\ 1\ -1)$ (as per the "short" pattern), and "$+5$" at $(1\ 1\ -1\ 1\ 3\ 1\ 1\ -1)$ but further away from all the others, and in particular from "-3" at $(-1\ 1\ -3\ 1\ 1\ -1\ 1\ -1)$ and from "$+3$" at $(0\ 2\ -2\ 0\ 2\ 0\ 2\ 0\ )$. One can check that a τ at $(0\ 0\ 0\ 0\ 4\ 0\ 0\ -4)$ does satisfy these conditions. Its vanishing is the cause of this "dressed short pattern".

Is this enough, or are there more "hidden patterns"? Since the degree of x_{m+2} in terms of x_{m+1} (for x_m of degree 0) is one, one can eliminate one x out of two, as in subsection 4.1. The resulting equation has been identified. It is Class III, case 4 in section 4 of [23]. When written using the ancillary variable ξ, the right-hand side is a ratio of products of 4 terms, meaning that in the usual, "canonical" form, the equation has a right-hand side which is a ratio of a quadratic polynomial in x over a linear one. We introduce the quantity c which is equal to $\phi_2(n) = \phi_2(0)$, since we chose the evolution over even indices $n = 2m$ rather than odd, and find

$$
\begin{aligned}
Z_m &= 4\alpha(2m+1) + 4\beta + \phi_3(2m+1) + \phi_5(2m+2) + 2\phi_5(2m+1) + \phi_5(2m)\\
A^1_m &= 5(2\alpha m + \beta) - 6\alpha - \phi_3(2m) + \phi_5(2m) + 2\phi_5(2m-1) + 2\phi_5(2m-2) + c\\
A^2_m &= 7(2\alpha m + \beta) + \phi_3(2m) + 2\phi_5(2m+1) + 3\phi_5(2m) + 2\phi_5(2m-1) - c\\
A^3_m &= 5(2\alpha m + \beta) + 6\alpha - \phi_3(2m) + 2\phi_5(2m+2) + 2\phi_5(2m+1) + \phi_5(2m) + c\\
A^4_m &= -(2\alpha m + \beta) - \phi_3(2m) - \phi_5(2m) - c
\end{aligned}
$$

It has four patterns, corresponding to $\{-7, -3, 1, 5\}$, $\{-5, -1, 3, 7\}$, $\{5, -1\}$ and $\{1, 5\}$. The first two clearly come from the "long pattern" $\{-7, -5, -3, -1, 1, 3, 5, 7\}$ of the equation we are discussing here, depending on which value one starts when skipping every other x. And the two last ones are clearly included in the "dressed short pattern" $\{-5, \star, -1, 1, \star, 5\}$. It has to be noted that after "-5" and "-1" and skipping "$+1$" one reaches a free value, and then another free one completely beyond the pattern, so the equation 4.3.4 of [23] does not "know" about the values "$+1$" and "$+5$" that exist in this particular pattern of our equation. Similarly, if starting from a free value one skips "-5" to some other free value, and again skips -1, then to find "$+1$" and further "$+5$", the equation 4.3.4 does not "know of" the "skipped" special values. So the patterns $\{-5, -1\}$ and $\{1, 5\}$ really are independent, as far as 4.3.4 of [23] is concerned, they do not come from some "dressed" pattern in that equation. It would seem that the two τs at the origin and at $(0\ 0\ 0\ 0\ 4\ 0\ 0\ {-4})$ do suffice.

But this is not yet the end. It is not just x_{m+2} that is of the first degree in terms of x_{m+1} (for x_m of degree 0). For this equation it is also the case of x_{m+3}. This means that we can even get an equation relating one x out of three, skipping the two intermediate ones.

This equation has also been identified. It is Class II, case 5 in section 5 , 5.2.5, of [23] and has the form (15) when written in terms of the ancillary variable. The quantities Z_n and A_n^i are now given by

$$
\begin{aligned}
Z_m &= 6(3\alpha m+\beta)+9\alpha+2\phi_5(3m+2)+2\phi_5(3m+1)+\phi_5(3m)+\phi_5(3m-2)\\
A_m^1 &= 5(3\alpha m+\beta)-6\alpha+\phi_2(m)+\phi_5(3m)+2\phi_5(3m-1)+2\phi_5(3m-2)+c\\
A_m^2 &= 7(3\alpha m+\beta)-\phi_2(m)+2\phi_5(3m+1)+3\phi_5(3m)+2\phi_5(3m-1)-c\\
A_m^3 &= 5(3\alpha m+\beta)+6\alpha+\phi_2(m)+2\phi_5(3m+2)+2\phi_5(3m+1)+\phi_5(3m)+c\\
A_m^4 &= (3\alpha m+\beta)+\phi_2(m)+\phi_5(3m)-c\\
A_m^5 &= 3(3\alpha m+\beta)-\phi_2(m)+2\phi_5(3m+2)-\phi_5(3m)+2\phi_5(3m-2)+d\\
A_m^6 &= 3(3\alpha m+\beta)-\phi_2(m)+2\phi_5(3m+2)-\phi_5(3m)+2\phi_5(3m-2)-d
\end{aligned}
$$

where we have introduced the quantities c and d which are equal to $c = -\phi_3(0)$ and $d = \phi_3(1) - \phi_3(-1)$ since we chose the evolution over indices $n = 3m$ rather than $3m \pm 1$. It has six patterns, two "long ones" $\{-7, -1, 5\}$ and $\{-5, 1, 7\}$ and four short ones, $\{-5, 1\}$, $\{-1, 5\}$ and twice $\{-3, 3\}$. The two long ones and one short pattern $\{-3, 3\}$ can be obtained from our "long pattern", which for clarity we repeat here: $\{-7, -5, -3, -1, 1, 3, 5, 7\}$, by skipping two values starting from $-7, -5$ and -3 respectively. Also the independent patterns $\{-5, 1\}$, $\{-1, 5\}$ (for equation 5.2.5 of [23]) come from our "dressed pattern" $\{-5, \star, -1, 1, \star, 5\}$, skipping two x starting from "-5" or "-1" respectively. But what about the second pattern $\{-3, 3\}$? It can only come from the vanishing of some other τ-function, one which is at a squared distance 16 from "-3" at $(-1\ 1\ {-3}\ 1\ 1\ {-1}\ 1\ {-1})$ and "$+3$" at $(0\ 2\ {-2}\ 0\ 2\ 0\ 2\ 0\ )$, but further away from anything else. The position of this τ is $(0\ 4\ {-4}\ 0\ 0\ 0\ 0\ 0)$ as one can easily check. This "action at distance three" is not as easily seen on the equation as the "action at distance two". Still, one can check that if x_{n-1} takes the value $(3t_n + 7\alpha)^2$ in (17), the relationship between x_n and x_{n+1} is the same as the relationship between $x_{n'-1}$ and $x_{n'}$ in (17) written for $n' = n + 1$ if $x_{n'+1} = x_{n+2}$ takes the value $(3t_{n'} - 7\alpha)^2$, thus establishing the "action at distance three" of the value $(3t_m + 10\alpha)^2$ for x_m, with consequence the value of $x_{m'} = x_{m+3}$ being $(3t_{m'} - 10\alpha)^2$. This corresponds indeed to a "hidden pattern" $\{-3, \star, \star, 3\}$ of equation (17).

4.3 The period 8 case

In this subsection, we will again use the alternate constraint. The sum of the coordinates of a τ with coordinates multiple of 4 is a multiple of 8 but if a τ has coordinates congruent to 2 modulo 4 the sum of these coordinates is congruent to 4 modulo 8.

This asymmetric equation was numbered **VI** in [19] and in the list at the beginning of the section. Being asymmetric, this equation is outside the scope of [23]. Since the equation is asymmetric, we have to consider the degrees of x and y starting both from x_0, y_0 and y_0, x_1, the degrees being in terms of the second variable. We find the degrees (d^x_m, d^y_m)=(0,1), (1,1), (2,2), (3,4), (4,6), (7,8), (10,11), (13,15), (16,19) $\ldots$ and (d^y_m, d^x_{m+1})=(0,1), (1,1), (2,2), (3,4), (5,6), (7,8), (10,11), (13,15), (16,19), $\ldots$. In fact the degrees obtained starting from (d^x_0, d^y_0)=(0,1) and (d^y_0, d^x_1)=(0,1) are the same *except* for the degree d^x_{8k+4} in the first choice of initial data which is equal to $(4k+2)^2$ while the degree d^y_{8k+4} for the second choice of initial data which is equal to $(4k+2)^2+1$. (In the case $k=0$ shown above we have 4 and 5 respectively),

The easiest way to find the vectors is to consider the sequence of degrees of one variable. Indeed, if one expresses y_0 in terms of x_1, the degrees in x_1 of all subsequent objects will be the same as in terms of y_0, because y_0 is of the first degree of x_1 and the precise dependance on x_0, of degree zero, does not matter. So we have for x only $d^x_0 = 0, d^x_1 = 1, d^x_2 = 2, d^x_3 = 3, d^x_4 = 4, d^x_5 = 7, d^x_6 = 10, d^x_7 = 13, d^x_8 = 16, \ldots$. Again, there is no proof that beyond degree 4, the squared length of $\overrightarrow{X_0X_m}$ is really 8 times the degree of x_m in terms of x_1, but it is an assumption of high heuristic power which has not been disproved in all cases studied up to now.

The vector $\overrightarrow{X_0X_8}$ is certainly $\overrightarrow{S}$ and we can reasonably assume it has squared length 128. Conversely, the vector $\overrightarrow{X_0X_4}$ has certainly squared length 32, because up to the degree 4 included, the squared length can be proven to be 8 times the degree. It is therefore reasonable to *assume* that $\overrightarrow{X_0X_4} = \overrightarrow{S}/2$ and try this as a heuristic tool to find the solution. Note that of course $\overrightarrow{Y_0Y_8} = \overrightarrow{S}$, and the degree of y_8 in terms of x_1 (or y_1, for that matter) for y_0 of degree zero, is indeed 16, but $\overrightarrow{Y_0Y_4}$ will be found to obey one of our heuristic rules (its squared length is indeed 40, 8 times the degree of y_4 in terms of y_1 or x_1, for y_0 of degree 0) but not the other one, obviously it is not $\overrightarrow{S}/2$.

Guided by the degrees of the X_m and the assumption that $\overrightarrow{X_0X_4} = \overrightarrow{S}/2$, we found the consistent choice which a posteriori was verified

$\overrightarrow{X_0X_1} = [2,0,0,0,2,0,0,0]$
$\overrightarrow{X_1X_2} = [0,0,0,2,0,0,0,2]$
$\overrightarrow{X_2X_3} = [0,0,2,0,0,0,2,0]$
$\overrightarrow{X_3X_4} = [0,2,0,0,0,2,0,0]$.

The sum $\overrightarrow{S}/2$ of these consecutive vectors is [2 2 2 2 2 2 2 2] and the scalar product of each of the $\overrightarrow{X_mX_{m+1}}$ vectors with $\overrightarrow{S}/2$ is 8. Note that the sum of the components of $\overrightarrow{S}/2$ is 16, a multiple of 8 and thus this vector does *not* satisfy the alternate constraint we have chosen. It does *not* define an invariant translation of the whole lattice. This is consistent with all the previous examples. Its double, namely $\overrightarrow{S}$, of course does define an invariant translation.

We must now turn to the vectors $\overrightarrow{X_mY_m}$ and $\overrightarrow{Y_mX_{m+1}}$. As we said earlier, in all the equations in this paper, they have the same scalar product with $\overrightarrow{S}/2$, thus 4 each. Though four vectors are enough for the X only, there is no shortcut for the Y. One has to find eight different "splitting" of the above vectors in two half NV's, with an eye on the strange value, namely 40, of the squared length of $\overrightarrow{Y_m Y_{m+4}}$ for all m. To make a long story short, here is the list of the $\overrightarrow{X_m Y_m}$ and $\overrightarrow{Y_m X_{m+1}}$ starting from the splitting of $\overrightarrow{X_0 X_1} = [2\,0\,0\,0\,2\,0\,0\,0]$.

$$\begin{bmatrix}
1 & -1 & -1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 \\
-1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\
-1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 \\
-1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 \\
-1 & 1 & 1 & -1 & -1 & 1 & 1 & 1
\end{bmatrix}$$

Having reached the period, this sequence now repeats indefinitely.

In order to find the initial point to which to add these vectors to find the positions of all Xs and Ys we must consider the singularity patterns. A simplified expression, ignoring the periodic function is

$$\frac{x_{n+1} - 4t_n^2}{x_{n+1}} \frac{x_n - 4t_n^2}{x_n} \frac{y_n - t_n^2}{y_n - 9t_n^2} = 1 \tag{18a}$$

$$\frac{y_n - (3t_n - 2\alpha)^2}{y_n - t_n^2} \frac{y_{n-1} - (3t_n - \alpha)^2}{y_{n-1} - (t_n - \alpha)^2} \frac{x_n}{x_n - (4t_n - 2\alpha)^2} = 1 \tag{18b}$$

Four distinct singularity patterns do exist. A "long" one

$$\{x_{n-2} = (4t_{n-2} - 2\alpha)^2, y_{n-2} = (3t_{n-2} - 2\alpha)^2, x_{n-1} = 4t_{n-3}^2, y_{n-1} = t_{n-5}^2, x_n = 16\alpha^2, y_n = t_{n+4}^2,$$
$$x_{n+1} = 4t_{n+2}^2, y_{n+1} = (3t_{n+2} - \alpha)^2, x_{n+2} = (4t_{n+2} - 2\alpha)^2\}$$

a "medium" one $\{y_{n-1} = 9t_{n-1}^2, x_n = 4t_{n-1}^2, y_n = t_{n-2}^2, x_{n+1} = 4\alpha^2, y_{n+1} = t_{n+3}^2, x_{n+2} = 4t_{n+2}^2, y_{n+2} = 9t_{n+2}^2\}$ and two "short" ones $\{x_n = 0, y_n = t_n\}$, $\{y_n = t_n, x_{n+1} = 0\}$. Schematically these patterns are $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$, $\{-3, -2, -1, 0, 1, 2, 3\}$ and $\{0, 1\}$, $\{-1, 0\}$.

Now driven by previous experience, we look for "hidden patterns". "Action at distance two" can be directly seen on the equation. Looking at (18b) we can see that, for generic x_n, if y_{n-1} takes the value $(3t_n - \alpha)^2$, which is not a remarkable value in (18a) (unless $\alpha = 0$, but in that case the equation would have no secular evolution), then y_n does take the remarkable value t_n^2. For generic x_n, (18) implies that x_{n+1} takes a zero value. This is precisely the second "short pattern", schematically $\{-1, 0\}$. If we now look at

(18b) evaluated at $n' = n+1$ we have $x_{n'} = x_{n+1} = 0$ while $y_{n'-1} = y_n = t_n^2$. But this value is precisely the quantity $(t_{n'} - \alpha)^2$ so the denominator of the second factor vanishes. At this point the value of $y_{n'} = y_{n+1}$ is not fixed by the singularity pattern. If we now consider (18b) evaluated at $n' = n+1$, since $y_{n'}$ does not take a remarkable value, the zero value of $x_{n'}$ implies that $x_{n'+1} = x_{n+2}$ takes the value $4t_{n'}^2$. At this point the pattern ends, because that value for $x_{n'+1}$ is not remarkable, neither in (18b) evaluated at $n' = n+1$ nor in (18a) evaluated at $n'' = n+2$ because the remarkable value for $x_{n+2} = x_{n''}$ is $4t_{n''}^2$. So schematically this pattern is $\{-3, \star, -1, 0, \star, 2\}$ where, as already explained, the $\star$ represents a free value. The second "short pattern" is in fact "dressed" in a not so short pattern involving six points, but two of them are free and only one blow-up is involved. A similar reasoning starting from a generic y_n and $x_n = 4t_n^2$ implies zero value for x_{n+1}. Going into (18b) evaluated at $n' = n+1$ one gets $y_{n+1} = y_{n'} = t_{n'}^2$, as per the first "short pattern", which is finally "dressed" as $\{-2, \star, 0, 1, \star, 3\}$. Thus the four patterns the positions for X and Y must support are really the "long" one $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$, the "medium" one $\{-3, -2, -1, 0, 1, 2, 3\}$ and the two now "dressed", not-so-short anymore, patterns.

Imposing that the "long" singularity pattern be caused by the vanishing of the τ-function at the origin means that there must be a sequence of positions for X and Y that extends on five Xs and four Ys in between at squared distance exactly 16 from the origin. We present here a somewhat longer sequence, starting with one pair of X, Y at squared distance 32 from the origin, just before the "long" singularity pattern and continuing with one single Y followed by three extra pairs (X, Y) at squared distance at least 32 from the origin, eighteen points altogether. The bottommost X and Y are exactly $\overrightarrow{S}$ away from the topmost X and Y respectively. All Xs have even coordinates, while all Ys have odd coordinates.

$$\begin{pmatrix}
-4 & -2 & -2 & -2 & -2 & 0 & 0 & 0 \\
-3 & -3 & -3 & -1 & -1 & -1 & 1 & 1 \\
-2 & -2 & -2 & -2 & 0 & 0 & 0 & 0 \\
-3 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\
-2 & -2 & -2 & 0 & 0 & 0 & 0 & 2 \\
-3 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
-2 & -2 & 0 & 0 & 0 & 0 & 2 & 2 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & 3 \\
-2 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\
-1 & -1 & 1 & 1 & 1 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\
-1 & -1 & 1 & 1 & 1 & 3 & 3 & 3 \\
0 & 0 & 0 & 2 & 2 & 2 & 2 & 4 \\
1 & 1 & 1 & 1 & 1 & 3 & 3 & 3 \\
0 & 0 & 2 & 2 & 2 & 2 & 4 & 4 \\
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 \\
0 & 2 & 2 & 2 & 2 & 4 & 4 & 4 \\
1 & 1 & 1 & 3 & 3 & 3 & 5 & 5
\end{pmatrix}$$

The scalar product of $\overrightarrow{S}$ with the vectors $\overrightarrow{OX}$ start from -24 for the topmost X and increase by 4 from one line to the next one. On the third line, the pattern begins with $(-2$

$-2\ -2\ -2\ 0\ 0\ 0\ 0)$, with scalar product with $\overrightarrow{S}$ equal to -16, four times the value of the label, namely "-4", of this X which opens the "long" pattern. The scalar products keep increasing by 4 from one line to the next one, labelling the points as integers from "-3" to "$+4$", odd for Ys, even for Xs, till the last X $(0\ 0\ 0\ 0\ 2\ 2\ 2\ 2)$ in the "long" pattern.

It is easy to see that the τs at the point $(-4\ 0\ 0\ 0\ 0\ 0\ 0\ 4)$ is at squared distance 32 of the first and the last Xs of the "long pattern" but at squared distance 16 of all the others, namely, all the points of the "medium pattern". Thus it is precisely the τ the vanishing of which causes the latter pattern.

The "dressed (not so) short patterns" $\{-3, \star, -1, 0, \star, 2\}$ and $\{-2, \star, 0, 1, \star, 3\}$ are caused by the vanishing of τs at $(-4\ 0\ 0\ 0\ 0\ 0\ 4\ 0)$ and $(0\ -4\ 0\ 0\ 0\ 0\ 0\ 0\ 4)$ respectively, as one can easily check, the points mentioned in each pattern being at squared distance 16 of the relevant τ, while the free value indicated by $\star$ appears when the point is too far from that τ.

Since the vectors $\overrightarrow{X_m X_{m+1}}$, on the one hand, and $\overrightarrow{Y_m Y_{m+1}}$ on the other hand, are both half-NV's, one can eliminate either variable to get an equation in the other variable alone. None of these equations are trihomographic, but they have both been identified earlier.

The "x-only" reduction is case 2 of Class IV in section 4, 4.4.2, of [23]. Written in the usual, "canonical" form, it has a right-hand side which is a ratio of a quadratic polynomial in x over a linear one. Written in terms of the ancillary variable it has the same form as (15) with a right-hand side being a ratio of a product of four terms. Starting form the t_n and $\phi_8(n)$ of **VI** we obtain for 4.4.2 the quantities Z_n and A_n^i

$$
\begin{aligned}
Z_n &= 2(\alpha n+\beta)+\phi_8(n+2)+\phi_8(n-2)\\
A_n^1 &= 4(\alpha n+\beta)-2\alpha+2\phi_8(n+2)-\phi_8(n+1)+\phi_8(n)+\phi_8(n-1)-\phi_8(n-2)+2\phi_8(n-3)\\
A_n^2 &= 2(\alpha n+\beta)+\phi_8(n+1)+\phi_8(n)-\phi_8(n-1)+\phi_8(n-2)\\
A_n^3 &= \phi_8(n+1)-\phi_8(n)-\phi_8(n-1)+\phi_8(n-2)\\
A_n^4 &= 2(\alpha n+\beta)-2\alpha+\phi_8(n+1)-\phi_8(n)+\phi_8(n-1)+\phi_8(n-2)
\end{aligned}
$$

The equation has four patterns, one "long" pattern of the form $\{-4, -2, 0, 2, 4\}$, one "medium" patterns of the type $\{-2, 0, 2\}$, and two "short" patterns $\{-2, 0\}$ and $\{0, 2\}$. One can immediately see the "long pattern" and the "medium pattern" of 4.4.2 in the "long pattern" and "medium" patterns of the equation considered here, by keeping only the x. Each of the "short pattern" of 4.4.2 is visible in one of the "dressed short patterns", though the "dressing" is indeed needed, the "short patterns" as identified in [17] and [19] were not sufficient.sign of the each

The "y-only" reduction is the case 2 of Class III in section 4 of [23], also with the right-hand side as a quadratic polynomial over a linear one in canonical from. Similarly, starting from the parameters of **VI** we obtain the quantities Z_n and A_n^i, which appear in the equation when written in a form involving the ancillary variable, as

$$\begin{aligned}
Z_n &= 2(\alpha n+\beta)+\alpha+\phi_8(n+3)-\phi_8(n+2)+\phi_8(n+1)+\phi_8(n)-\phi_8(n-1)+\phi_8(n-2)\\
A_n^1 &= 3(\alpha n+\beta)+\phi_8(n+2)+\phi_8(n)+\phi_8(n-2)\\
A_n^2 &= 3(\alpha n+\beta)-2\alpha+\phi_8(n+2)+\phi_8(n)-\phi_8(n-2)+2\phi_8(n-3)\\
A_n^3 &= 3(\alpha n+\beta)+2\alpha+2\phi_8(n+3)-\phi_8(n+2)+\phi_8(n)+\phi_8(n-2)\\
A_n^4 &= -(\alpha n+\beta)-\phi_8(n+2)+\phi_8(n)-\phi_8(n-2)
\end{aligned}$$

The equation has four patterns. Two of them are "long" pattern of the type $\{-3,-1,1,3\}$. One can immediately see that both our "long pattern" and our "medium pattern" contain one such pattern when we keep only the y. The other two patterns are short, of type $\{-3,-1\}$ and $\{1,3\}$ respectively. Again, they appear in the "dressed not so short patterns" of (18), but not in the "short patterns" as identified in [17] and [19].

It would seem we have everything, but it not really the case. If we look at the degrees, we can see that the degree of y_{m+1} in terms of x_m and y_m is one, so this triangle is equilateral, $\overrightarrow{X_mY_{m+1}}$ is half an NV for any m. Since the degree of x_{q+2} in terms of y_q and x_{q+1} is 1, it follows that $\overrightarrow{Y_qX_{q+2}}$ is half an NV for any q. Taking first $q=m+1$ we see that there is an equation in the triangle $X_mY_{m+1}X_{m+3}$. Then using $m'=m+3$ in place of m, one sees there is also an equation in the triangle $Y_{m+1}X_{m+3}Y_{m+4}$. So one can write a succession of equations from X to Y to X but skipping two points taking only one point in three. The resulting equation is not trihomographic, so out of the scope of [17] and [19], and not symmetric since it involves both x and y, so out of the scope of [23], but one can derive it by appropriate eliminations. To our knowledge, it has not been identified before. Deriving it in its full freedom involving all 8 parameters would lead to an expression too long to be explicitly given.

On the other hand obtaining its autonomous form is perfectly manageable. Thus after some moderately lengthy calculations, with $t_n\equiv\beta$ for all n, we obtain the mapping

$$\frac{(y_{m+1}-x_{m+3}+9\beta^2)(y_{m+1}-x_m+9\beta^2)+36y_{m+1}\beta^2}{(y_{m+1}-x_{m+3}+9\beta^2)+(y_{m+1}-x_m+9\beta^2)}=\frac{3}{2}\,\frac{y_{m+1}^3+57\beta^2y_{m+1}^2+171\beta^4y_{m+1}+27\beta^6}{3y_{m+1}^2+34\beta^2y_{m+1}+27\beta^4}$$

$$\frac{(x_{m+3}-y_{m+1}+9\beta^2)(x_{m+3}-y_{m+4}+9\beta^2)+36x_{m+3}\beta^2}{(x_{m+3}-y_{m+1}+9\beta^2)+(x_{m+3}-y_{m+4}+9\beta^2)}=\frac{3}{2}\,\frac{x_{m+3}(x_{m+3}^2+56\beta^2x_{m+3}+144\beta^4)}{3x_{m+3}^2+32\beta^2x_{m+3}+16\beta^4}$$

One can obtain many of its patterns from the patterns of our equation. The "long pattern" provides 3 patterns for the new equation, by skipping two points to reach the third one, namely $\{-4,-1,2\}$, $\{-3,0,3\}$ and $\{-2,1,4\}$. The "medium pattern" also provides 3 patterns for the new equation $\{-3,0,3\}$, $\{-2,1\}$ and $\{-1,2\}$. The "dressed short patterns" provide two patterns each, $\{-3,0\}$, and $\{-1,2\}$ for one, $\{-2,1\}$ and $\{0,3\}$ for the other. This amounts to 10 patterns.

In fact, this new equation has a total of 12 patterns, including three of each type $\{-2,1\}$ and $\{-1,2\}$ when we only found two of each. The third pattern of each type cannot involve a blow-up, nor an "action at distance two", or we would have found it earlier; our search for such patterns was exhaustive. They must be "action at distance three" of the type $\{-2,\star,\star,1\}$ and $\{-1,\star,\star,2\}$. Indeed, one can also check that a value

$(t_n+\alpha)^2$ for y_{n-1} in (18b) and $4(t_n-\alpha)^2$ for x_{n+1} in (18b) lead to the same relationship between x_n and y_n. In other words, if it happens that y_m takes the value t_{m+2}^2 then three position later, at $m'=m+2$, $x_{m'}$ will take the value $4t_{m'-2}^2$. Similarly, a value $4t_{n+1}^2$ for x_n in (18) induces the same relationship between y_n and x_{n+1} as a value t_{n-1}^2 for y_{n+1} in (18b) evaluated at $n'=n+1$, in other words $x_n=4t_{n+1}^2$ implies $y_{n'}=t_{n'-2}^2$ three positions later, at $n'=n+1$.

So do we see this "action at distance three" in our trajectory ? Yes, indeed. These "elongated" patterns are caused, respectively, by the vanishing of the τs at (0 0 −4 0 0 0 0 4) and (−4 0 0 0 0 4 0 0) respectively, which are at squared distance 16 of the relevant points, and at least at squared distance 32 from all the others.

4.4 The period 4 case (and its ghost)

In [17] two slightly different deautonomisations are proposed for the additive asymmetric system of period 4. And two different numberings **X** and **XI** are given in [19] and in the list at the beginning of section 4. However, these two equations are in fact just one and the same. The general form of an asymmetric equation is (11). Exchanging numerator and denominator of each factor of (11b) obviously leaves the equation invariant, but changes κ_n into its opposite.

In this case we have for both equations $t_n=\alpha n+\beta$ and $u_n=t_n+\phi_4(n)$, $k_n=u_{n+1}-u_n+u_{n-1}+\phi_2(n)$, supplemented by $\kappa_n=\delta+\eta(-1)^n$ for the first one, where we have written explicitly the period 2 function rather than calling it a different ϕ_2, and $\kappa_n=\chi_4(n)$ for the second one. The first expression means the period 2 series of values $(\dots a,b,a,b,a,b,\dots)$ with $a=\delta+\eta, b=\delta-\eta$. Since one is free at any time to change κ_n into its opposite, we can do so twice in a row every four steps. The series of values becomes $(\dots a,b,-a,-b,a,b,-a,-b,\dots)$, which is just what $\chi_4(n)$ means. So there is no real difference between the two equations. (The same equivalence will be discussed in subsection 4.6). The overall period is four in both cases, due to the ϕ_4 term in u_n. We note that $\phi_4(n)$ can be rewritten $\tilde{\phi}_2(n)+\tilde{\chi}_4(n)$, which will be useful later.

We will study the first choice for κ, but the trajectory is obviously the same for the second one. Since the equation is asymmetric, we have to consider the degrees of x and y starting both from x_0, y_0 and y_0, x_1, the degrees being in terms of the second variable. We find (d_m^x, d_m^y)=(0,1), (1,2), (2,4), (5,7), (8,11), … and (d_m^y, d_{m+1}^x)=(0,1), (1,2), (3,4), (5,7), (8,11), …

As we explained in section 2, the easiest way to find the vectors is to eliminate one variable to get a symmetric equation in terms of the other one, even if the resulting equation is not trihomographic. But one need not even compute this equation. It is enough to consider the sequence of degrees of one variable. For instance, since for x_0 of degree zero, the degree of x_1 in terms of y_0 is one, it means that if one expresses y_0 in terms of x_1, the degrees in x_1 of all subsequent objects will be the same as in terms of y_0. The dependence in terms of x_0 will be different, but this is not essential. So we have for x only $d_0^x=0, d_1^x=1, d_2^x=2, d_3^x=5, d_4^x=8,\dots$.

In this subsection, we will take the original constraint that the sum of the components of every NV vector must be a multiple of 8. So we take the half-NV's

$\overrightarrow{X_0X_1}=[2\,0\,2\,0\,0\,0\,0\,0]$

$\overrightarrow{X_1X_2}=[0\,2\,0\,2\,0\,0\,0\,0]$ and again

$\overrightarrow{X_2X_3} = [2\,0\,2\,0\,0\,0\,0\,0]$
$\overrightarrow{X_3X_4} = [0\,2\,0\,2\,0\,0\,0\,0]$ etc.

The sum $\overrightarrow{S}/2$ of any two consecutive vectors is $[2\,2\,2\,2\,0\,0\,0\,0]$ and the scalar product of each of the $\overrightarrow{X_mX_{m+1}}$ vectors with $\overrightarrow{S}/2$ is 8.

We must now turn to the vectors $\overrightarrow{X_mY_m}$ and $\overrightarrow{Y_mX_{m+1}}$. As we said earlier, in all the equations in this paper, they have the same scalar product with $\overrightarrow{S}/2$, 4 each. This leads us to
$\overrightarrow{X_0Y_0} = [2\,0\,0\,0\,0\,0\,0\,2]$
$\overrightarrow{Y_0X_1} = [0\,0\,2\,0\,0\,0\,0\,-2]$ and by analogy
$\overrightarrow{X_1Y_1} = [0\,2\,0\,0\,0\,0\,0\,2]$
$\overrightarrow{Y_1X_2} = [0\,0\,0\,2\,0\,0\,0\,-2]$

At this point we see that the degree of y_2 is 3, hence we expect $\overrightarrow{Y_0Y_2}$ to have squared length 24, and not 16. This can be arranged if the splitting of $\overrightarrow{X_2X_3}$ with respect to Y_2 is not the same as that of the equal vector $\overrightarrow{X_0X_1}$ but rather as
$\overrightarrow{X_2Y_2} = [0\,0\,2\,0\,0\,0\,0\,2]$
$\overrightarrow{Y_2X_3} = [2\,0\,0\,0\,0\,0\,0\,-2]$

Then $\overrightarrow{Y_0Y_2} = [0\,2\,4\,2\,0\,0\,0\,0]$ is indeed of squared length 24. The sequence ends with
$\overrightarrow{X_3Y_3} = [0\,0\,0\,2\,0\,0\,0\,2]$
$\overrightarrow{Y_3X_4} = [0\,2\,0\,0\,0\,0\,0\,-2]$

Having reached the period, this sequence now repeats indefinitely.

Note that though $\overrightarrow{X_mX_{m+2}}$ is indeed equal to $\overrightarrow{S}/2$, for all m, this is never the case for $\overrightarrow{Y_mY_{m+2}}$. That equality is a rule of thumb, useful to guess the results, and it did work for X. But we never claimed this heuristic rule to be a theorem. What is true, of course, is that $\overrightarrow{Y_mY_{m+4}}$ is always equal to $\overrightarrow{S}$.

In order to find the initial point to which to add these vectors to find the positions of all Xs and Ys we must consider the singularity patterns. As usual, we shall work with a simplified form of (11) where we ignore the periodic functions. Though taking $\kappa \equiv 0$ in $\mathbf{X}$ looks absurd at first glance, in fact choosing $\tilde{\phi}_2 \equiv 0$ and taking the limit $\delta \to 0$ is meaningful, and the resulting equation is just a subcase of the initial equation and the degrees remain the same. However, here, for simplicity, we keep a non zero δ.

$$\frac{x_{n+1}-(2t_n-\alpha-\gamma)^2}{x_{n+1}-(\alpha+\gamma)^2}\,\frac{x_n-(2t_n+\alpha+\gamma)^2}{x_n-(\alpha+\gamma)^2}\,\frac{y_n-t_n^2}{y_n-9t_n^2}=1 \tag{19a}$$

$$\frac{y_n-(t_n+\gamma+\delta)^2}{y_n-(t_n+\gamma-\delta)^2}\,\frac{y_{n-1}-(t_{n-1}-\gamma+\delta)^2}{y_{n-1}-(t_{n-1}-\gamma-\delta)^2}\,\frac{x_n-(2t_n-\alpha-\delta)^2}{x_n-(2t_n-\alpha+\delta)^2}=1 \tag{19b}$$

The four singularity patterns are a “long” one of the form $\{-3,-2,-1,0,1,2,3\}$

$$\{y_{n-1}=9t_{n-1}^2, x_n=(2t_{n-1}-\alpha-\gamma)^2, y_n=t_{n-2}^2, x_{n+1}=(3\alpha+\gamma)^2, y_{n+1} = t_{n+3}^2, x_{n+2}=(2t_{n+2}+\alpha+\gamma)^2, y_{n+2}=9t_{n+2}^2\}$$

two “medium” ones, both of the form $\{-2,-1,0,1,2\}$

$$\{x_{n-1}=(2t_{n-1}-\alpha\pm\delta)^2, y_{n-1}=(t_{n-1}+\gamma\pm\delta)^2, x_n=(\alpha\mp\delta)^2, y_n=(t_n-\gamma\mp\delta)^2, x_{n+1}=(2t_{n+1}-\alpha\mp\delta)^2\}$$

and a “short” one of the form $\{-1,0,1\}$

$$\{y_n = t_n^2, x_{n+1} = (\alpha+\gamma)^2, y_{n+1} = t_{n+1}^2\}.$$

Imposing that the “long” singularity pattern be caused by the vanishing of the τ-function at the origin means that there must be a sequence of positions for X and Y that extends on four Ys and three Xs in between at squared distance exactly 16 from the origin. We present here a slightly longer sequence, starting from an X at squared distance 32 from the origin, just before the “long” singularity pattern and continuing with one extra X and one extra Y both also at squared distance 32 from the origin. The bottommost X and Y are exactly $\overrightarrow{S}$ away from the topmost X and Y respectively. All X have their rightmost coordinate equal to -2, while all Y have a zero rightmost coordinate.

$$\begin{pmatrix} -4 & -2 & -2 & 0 & 0 & 0 & 2 & -2 \\ -2 & -2 & -2 & 0 & 0 & 0 & 2 & 0 \\ -2 & -2 & 0 & 0 & 0 & 0 & 2 & -2 \\ -2 & -2 & 0 & 2 & 0 & 0 & 2 & 0 \\ -2 & 0 & 0 & 2 & 0 & 0 & 2 & -2 \\ -2 & 0 & 2 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 2 & -2 \\ 0 & 2 & 2 & 2 & 0 & 0 & 2 & 0 \\ 0 & 2 & 2 & 4 & 0 & 0 & 2 & -2 \\ 2 & 2 & 2 & 4 & 0 & 0 & 2 & 0 \end{pmatrix}$$

The topmost X is such that the scalar product of $\overrightarrow{S}/2$ with the vectors $\overrightarrow{OX}$ is -16. The scalar product increases by 4 from one point to the next one, but, of course, every other point is a Y. This X, being at squared distance 32 from the origin is outside the singularity pattern. The latter begins with the topmost Y, $(-2\ -2\ -2\ 0\ 0\ 0\ 2\ 0)$, at squared distance 16 from the origin, and such that the scalar product of $\overrightarrow{S}/2$ with the $\overrightarrow{OY}$ is -12, i.e. 4 times the value of the index “-3” of this point. The last Y in the pattern, “3”, is the point $(0\ 2\ 2\ 2\ 0\ 0\ 2\ 0)$ such that the scalar product of $\overrightarrow{S}/2$ with the vectors $\overrightarrow{OX}$ is 12.

It is easy to see that the τs at the points $(-2\ -2\ 2\ 2\ \pm2\ \mp2\ 2\ -2)$, (which correctly have the sum of their coordinates equal to 0, a multiple of 8) are at squared distance 32 from the two Y points “-3” and “3” but at squared distance 16 from the three X points “-2”, “0” and “2” as well as the intermediate Ys. Therefore their respective vanishings cause one “medium pattern” each. And similarly, the equally acceptable τ at $(-4\ 0\ 0\ 4\ 0\ 0\ 0\ 0)$ is at squared distance 16 of the Ys at “-1” and “1” and the X at “0” but no other point. Its vanishing causes the “short” pattern.

Since the vectors $\overrightarrow{X_m X_{m+1}}$, on the one hand, and $\overrightarrow{Y_m Y_{m+1}}$ on the other hand, are both half-NV’s, one can eliminate either variable to get an equation in the other variable alone. None of these equations are trihomographic, but they have both been identified earlier.

The “x-only” reduction is the unique case of Class I in section 4, namely 4.1, of [23]. One could write it in terms of the ancillary variables in form (15) using instead of A^i the following B^i (and we remind the reader here, that we can rewrite $\phi_4(n)$ as $\tilde{\phi}_2(n)+\tilde{\chi}_4(n)$):

$$
\begin{aligned}
Z_n &= 2(\alpha n+\beta) - 2\tilde{\phi}_2(n)\\
B_n^1 &= 2(\alpha n+\beta) - \alpha - \tilde{\chi}_4(n) - \tilde{\chi}_4(n-1) + 4\tilde{\phi}_2(n) - \phi_2(n) - \gamma - 2\alpha\\
B_n^2 &= 2(\alpha n+\beta) - \alpha - \tilde{\chi}_4(n) - \tilde{\chi}_4(n-1) - (4\tilde{\phi}_2(n) - \phi_2(n) - \gamma - 2\alpha)\\
B_n^3 &= 2(\alpha n+\beta) - \alpha + \tilde{\chi}_4(n) + \tilde{\chi}_4(n-1) + \kappa_n\\
B_n^4 &= 2(\alpha n+\beta) - \alpha + \tilde{\chi}_4(n) + \tilde{\chi}_4(n-1) - \kappa_n
\end{aligned}
$$

These expressions are valid for both cases **X** and **XI**, provided one uses the appropriate κ_n, namely $\kappa_n = \delta + \eta(-1)^n$ for the former and $\kappa_n = \chi_4(n)$ for the latter.

A singularity entered at point n at a B_n^i exits at point $n+2$ at B_{n+2}^j, with j not necessarily equal to i. In principle this is perfectly acceptable. In fact, this is the case for many equations of [23]. It turns out, however that precisely for 4.1 of [23] (and a few others) the singularity entered at each A_n^i exists at A_{n+2}^i, with the same i. Therefore, in order to make the connection with the unique case of Class I in section 4, 4.1 of [23], clearer, it is more appropriate to redefine the A^is in terms of B^js in such a way that the same holds true. This means that we must have $A_n^i + A_{n+2}^i = Z_n + Z_{n+1}$ for $i = 1, 2, 3, 4$ and all n. So we must take $A_n^1 = B_n^1$, $A_n^2 = B_n^2$ for $n = 4N$ and $n = 4N+1$ and $A_n^1 = B_n^2$, $A_n^2 = B_n^1$ for $n = 4N+2$ and $n = 4N+3$.

In the case of equation X similar relations between A^3, A^4 and B^3, B^4 are needed for the singularity to enter and exit at A^i with same indices. In the case of equation **XI**, since κ_n is already a χ_4, one should take $A^3 = B^3$ and $A^4 = B^4$ for all n. Note that in all cases the sum of the periodic parts of the A^i and B^i cancels at every n, as expected. Thus equation (17) of [23] is satisfied $\frac{1}{2}\sum_{i=1}^4 A_n^i = \frac{1}{2}\sum_{i=1}^4 B_n^i = Z_n + Z_{n-1}$.

The equation has four patterns of the type $\{-2, 0, 2\}$. One can immediately see that within the "long pattern" as well as both of the "medium" ones, keeping only the x gives one such pattern each. But the fourth one is missing. Since we have not identified it in the previous papers concerning equation (19ab), it means it must be a "hidden" pattern, causing no blow-up. Rather, it must come from an "action at distance". Just a glance at equation (19a) shows that, for generic y_n, if $x_n = (2t_n + \gamma)^2$ then $x_{n+1} = \gamma^2$. Note that this value of x_n is not remarkable in equation (19b) for generic values of α, γ and δ. Only the coefficient of n is the same as for the values "of interest" there. Thus one expect y_{n+1} to be generic, and for $n' = n+1$ the fact that $x_{n'} = \gamma^2$ means that $x_{n'+1}$ should be equal to $(2t_{n'} - \gamma)^2$. The two consecutive "actions at distance two" in equation (19ab), i.e., **X** or equivalently **XI**, constitute a blow-up-free pattern $\{-2, \star, 0, \star, 2\}$, which was not detected in [4, 7], which only dealt with blow-ups. This pattern is caused by the vanishing of the τ-function at (0 0 0 0 0 0 4 −4) which is at squared distance 16 of the x "−2", "0" and "2" and no other points. And indeed, it induces an extra singularity of type $\{-2, 0, 2\}$ for its x-only reduction.

The "y-only" reduction is the case 1 of Class I in section 5, i.e. 5.1.1, of [23]. Starting from (11) we obtain the quantities Z_n and A_n^i entering the equation

$$\begin{aligned}
Z_n &= \alpha(2n+1) + 2\beta + \tilde{\chi}_4(n) + \tilde{\chi}_4(n+1)\\
A_n^1 &= 3(\alpha n + \beta) - \tilde{\chi}_4(n) - 5\tilde{\phi}_2(n) + \phi_2(n)\\
A_n^2 &= \alpha n + \beta + \tilde{\chi}_4(n) + \tilde{\phi}_2(n) - \phi_2(n)\\
A_n^3 &= \alpha n + \beta + \tilde{\chi}_4(n) + \tilde{\phi}_2(n) + (\gamma + \delta + \eta)(-1)^n\\
A_n^4 &= \alpha n + \beta + \tilde{\chi}_4(n) + \tilde{\phi}_2(n) + (\gamma - \delta - \eta)(-1)^n\\
A_n^5 &= \alpha n + \beta + \tilde{\chi}_4(n) + \tilde{\phi}_2(n) - (\gamma - \delta + \eta)(-1)^n\\
A_n^6 &= \alpha n + \beta + \tilde{\chi}_4(n) + \tilde{\phi}_2(n) - (\gamma + \delta - \eta)(-1)^n
\end{aligned}$$

The equation has six singularity patterns. One is a "long" pattern of the type $\{-3,-1,1,3\}$. One can immediately see that pattern within our "long pattern". The five other ones are all of the type $\{-1,1\}$. Each of our "medium pattern" contains one such pattern for the y and so does our "short" one. But two more similar patterns are missing. They must also be "hidden" in our equation for causing no blow-ups (otherwise we would not have missed them in our previous papers concerning the equation at hand, that concentrated in patterns causing blow-ups), therefore they must be "actions at distance two". Again, looking at (19b), we see that, for x_n generic, if y_{n-1} takes any of the two values $(t_{n-1} - \gamma \pm \delta)^2$, then y_n takes the value $(t_n + \gamma \mp \delta)^2$. Again, neither of these values of y_n is remarkable in equation (19) for generic values of α, γ and δ. Only the coefficient of n coincides with that of the "value of interest" t_n^2. The two "hidden" patterns "at distance two" we have just identified are of the form $\{-1, \star, 1\}$. The last thing to do to complete this subsection is to find the two positions where the vanishing of a τ-function would cause such a behaviour, at distance 16 from the Ys at "-1" an "$+1$". The positions are in fact (-2 -2 2 2 ±2 ±2 2 2). Indeed, the sum of the coordinates of these points, depending on the common sign of the fifth and sixth coordinates, are 8 and 0 respectively as per the original constraint.

4.5 The periods 4,5 case

This section will consider two equations of periods 4 and 5. One is an asymmetrical one, described in [17], numbered **VII** in [19]. The other one, symmetrical, is nothing but the "x-only" reduction of the first one. Since it happens to be trihomographic too, it also appears in [17] and in [19], numbered **I**. Being symmetrical it is mentioned in [23] in section 3 Class II. The "y-only" reduction also exists and has been identified. It is not trihomographic, and we will come back to it later.

Since the equation is asymmetric, we have to consider the degrees of x and y starting both from x_m, y_m and y_m, x_{m+1}, the degrees being in terms of the second variable. We find (d_m^x, d_m^y)=(0,1), (1,1), (1,1), (2,2), (2,3), (3,4), (4,5), (6,6), (7,8), (9,10), (10,12),… and (d_m^y, d_{m+1}^x)=(0,1), (1,1), (2,1), (2,2), (2,3), (3,4), (5,5), (6,6), (7,8), (9,10), (11,12),….

We can remark that the degree of x_{10} in terms of y_0 (for x_0 of degree 0) is 10, while that of y_{10} in terms of x_1 (for y_0 of degree 0) is 11. Differences already appeared at x_2 from x_0 (degree 1) versus y_2 from y_0 (degree 2) and x_6 from x_0 (degree 4) versus y_6 from y_0 (degree 5). This phenomenon already appeared in the asymmetrical equations of subsection 4.3 and 4.4. In both cases, though $\overrightarrow{Y_0Y_P}$ ($P = 8$ and 4 respectively) was always the corresponding vector $\overrightarrow{S}$, the value of the degree of $y_{P/2}$ was clearly incompatible with a value $\overrightarrow{S}/2$ for $\overrightarrow{Y_0Y_{P/2}}$. But in both equations it was still true that $\overrightarrow{X_0X_{P/2}}$ was indeed equal to $\overrightarrow{S}/2$. Hoping (with *a posteriori* confirmation) that it would still be the case here, we were able to find first 10 vectors $\overrightarrow{X_mX_{m+1}}$ for m from 0 to 9, then thanks to the behaviour of y_2, y_6 and y_{10}, a consistent way to "split" them.

We present here the ten first $\overrightarrow{X_mX_{m+1}}$. (In this section we will use the original constraint, namely that the sum of the coordinates of the site of a τ-function is always a multiple of 8, whether the coordinates are multiples of 4, or congruent to 2 modulo 4).

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 2 & -2 & 0 & 0\\
0 & 0 & 2 & 0 & 0 & 2 & 0 & 0\\
2 & 0 & 0 & 0 & 0 & -2 & 0 & 0\\
0 & 0 & 0 & 2 & 0 & 2 & 0 & 0\\
0 & 2 & 0 & 0 & 0 & -2 & 0 & 0\\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 0\\
0 & 0 & 2 & 0 & 0 & -2 & 0 & 0\\
2 & 0 & 0 & 0 & 0 & 2 & 0 & 0\\
0 & 0 & 0 & 2 & 0 & -2 & 0 & 0\\
0 & 2 & 0 & 0 & 0 & 2 & 0 & 0
\end{bmatrix}$$

On the five first columns, all coordinates take the value 0 except one with value +2 that goes round to the left by two units down each line around these five columns as around a torus. (Granted the skip by two instead of one seems arbitrary, and even bizarre, but there are æsthetical reasons for that when the y are introduced). On the sixth one, values of 2 and -2 alternate. The seventh and last columns stay blank.

The sum $\overrightarrow{S}/2$ of any consecutive 10 vectors is [4 4 4 4 4 0 0 0] and the scalar product of each of the $\overrightarrow{X_mX_{m+1}}$ vectors with $\overrightarrow{S}/2$ is 8. Note that the sum of the components of $\overrightarrow{S}/2$ is 20, not a multiple of 8, and thus this vector does *not* satisfy the original constraint we have chosen in this chapter. As in all the previous examples, it does not define an invariant translation of the whole lattice. Again $\overrightarrow{S}$, of course does define an invariant translation. The eleven first pairs of vectors $\{\overrightarrow{X_mY_m}, \overrightarrow{Y_mX_{m+1}}\}$ are

$$
\begin{bmatrix}
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\
-1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\
-1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\
-1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 \\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & -1
\end{bmatrix}
$$

Note that the ten first pairs represent only one half-period. Indeed, though it is true that $\overrightarrow{X_{m+10}X_{m+11}} = \overrightarrow{X_mX_{m+1}}$, contrariwise $\overrightarrow{X_{m+10}Y_{m+10}}$ and $\overrightarrow{X_mY_m}$ do not coincide. Their six first components are equal but the last two change sign. The same holds for $\overrightarrow{Y_{m+10}X_{m+11}}$ and $\overrightarrow{Y_mX_{m+1}}$. Therefore $\overrightarrow{X_mX_{m+10}}$ is equal to $\overrightarrow{S}/2$, of squared length 80, but $\overrightarrow{Y_mY_{m+10}}$ is of squared length 88, eight times the degree, namely 11, of y_{10} in terms of x_1 for y_0 of degree 0.

In order to find the initial point to which to add these vectors in order to find the positions of all Xs and Ys we must consider the singularity patterns. A simplified expression for the asymmetric equation, ignoring the periodic functions, is:

$$\frac{x_{n+1} - (2t_n + \alpha)^2}{x_{n+1} - \alpha^2} \frac{x_n - (2t_n - \alpha)^2}{x_n - \alpha^2} \frac{y_n - t_n^2}{y_n - 9t_n^2} = 1 \tag{20a}$$

$$\frac{y_n - (5t_n - 4\alpha)^2}{y_n - 9t_n^2} \frac{y_{n-1} - (5t_{n-1} + 4\alpha)^2}{y_{n-1} - 9t_{n-1}^2} \frac{x_n - (2t_n - \alpha)^2}{x_n - (6t_n - 3\alpha)^2} = 1 \tag{20b}$$

There are four singularity patterns, a "long" one, $\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$

$$\{x_{n-3} = (6t_{n-3} - 3\alpha)^2, y_{n-3} = (5t_{n-3} - 4\alpha)^2, x_{n-2} = (4t_{n-3} - 3\alpha)^2, y_{n-2} = (3t_{n-3} - 5\alpha)^2,$$
$$x_{n-1} = (2t_{n-3} - 5\alpha)^2, y_{n-1} = (t_{n-3} - 8\alpha)^2, x_n = 81\alpha^2, y_n = (t_{n+3} + 7\alpha)^2, x_{n+1} = (2t_{n+3} + 3\alpha)^2,$$
$$y_{n+1} = (3t_{n+3} + 2\alpha)^2, x_{n+2} = (4t_{n+3} - \alpha)^2, y_{n+2} = (5t_{n+3} - \alpha)^2, x_{n+3} = (6t_{n+3} - 3\alpha)^2\}$$

a "medium" one $\{-1,0,1\}$, $\{y_{n-1}=t_{n-1}^2, x_n=\alpha^2, y_n=t_n^2\}$ and two "short" ones $\{-3,-2\}$ and $\{2,3\}$, $\{y_n=9t_n^2, x_{n+1}=(2t_n+\alpha)^2\}$, $\{x_n=(2t_n-\alpha)^2, y_n=9t_n^2\}$.

We now look for "action at distance two". From (20b) we can see that, for generic x_n, if y_{n-1} takes the value $(5t_{n-1}+4\alpha)^2$, which is not a remarkable value in (20a), then y_n does take the remarkable value $9t_n^2$, the starting value of the first "short pattern", schematically $\{-3,-2\}$. We exit it through $x_{n+1}=(2t_n+\alpha)^2$ which means that y_{n+1} recovers a degree of freedom. Indeed in (20b) evaluated at $n'=n+1$, $x_{n'}=x_{n+1}=(2t_n+\alpha)^2=(2t_{n'}-\alpha)^2$ while $y_{n'-1}=9t_{n'-1}^2$ and the equation is satisfied for any $y_{n'}$. Now let us consider (20a) evaluated at n'. In the third factor, $y_{n'}$ is not remarkable, so the value $(2t_{n'}-\alpha)^2$ for $x_{n'}$ implies a value α^2 for $x_{n'+1}=x_{n+2}$. Back to (20b), now evaluated at $n''=n+2$. The value α^2 of $x_{n''}$ is not remarkable, and $y_{n''-1}=y_{n'}$ is a free quantity. Hence the same is true of $y_{n''}=y_{n+2}$. But in (20a) evaluated at n'' a free value for $y_{n''}$ and a value α^2 of $x_{n''}$ imply a value $(2t_{n''}+\alpha)^2$ for $x_{n''+1}$, which can be written $x_{n'''}=(2t_{n'''}-\alpha)^2$ for $n'''=n+3$. This means $x_{n'''}$ has just the value to enter the second "short" pattern, thus followed by $y_{n'''}=9t_{n'''}^2$. Exiting this "short" pattern, the next value of x is free, and in the next instance of (20b) the vanishing of the denominator of the second factor implies that of the numerator of the first factor, and thus fixes the value of the next y at $n''''=n+4$ as $(5t_{n''''}-4\alpha)^2$ through a final "action at distance two". So the "action at distance two" initiated by $y_{n-1}=(5t_{n-1}+4\alpha)^2)$ leads in fact to a rather long "dressed pattern", that involves both the "short patterns", and can be schematically written as $\{-5,\star,-3,-2,\star,0,\star,2,3,\star,5\}$. The value of $x_{n+2}=x_{n''}$ is α^2 and not actually 0, but there is no n^2 dependence, and this is what the schematic value 0 means.

Imposing that the "long" singularity pattern be caused by the vanishing of the τ-function at the origin means that there must be a sequence of positions for X and Y that extends on seven Xs and the six Ys in between at squared distance exactly 16 from the origin. We present here a somewhat longer sequence, starting with one pair of X,Y both at squared distance 32 from the origin, just before the "long" singularity pattern and continuing with one single Y followed by two extra pairs (X,Y), all at squared distance 32 from the origin, and a last pair at squared distance 48 from the origin, twenty-two points altogether. The last X is exactly $\overrightarrow{S}/2$ away from the first one, but this is not so for the corresponding Ys. The period is 20 and one has to wait for Y_{n+20} to find $\overrightarrow{Y_nY_{n+20}}=\overrightarrow{S}$. All Xs have even coordinates, while all Ys have odd coordinates. Below we give the transposed matrix of the X and Y.

$$\begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 2 & 3 & 4 & 3 & 4 & 3 & 4 & 5\\
0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 2 & 3 & 4 & 3\\
-2 & -1 & -2 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 2 & 1 & 2 & 3 & 2 & 3\\
-2 & -3 & -2 & -1 & -2 & -1 & -2 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & 2 & 1\\
-4 & -3 & -2 & -3 & -2 & -3 & -2 & -1 & -2 & -1 & -2 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1\\
2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 0 & 1 & 2 & 1\\
2 & 3 & 2 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 2 & 1\\
0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1
\end{pmatrix}$$

The singularity begins with the second X, "-6", at $(0\ 0\ -2\ -2\ -2\ 0\ 2\ 0)$ such that the scalar product of $\overrightarrow{S}/2$ with the vectors $\overrightarrow{OX}$ is -24. This scalar product increases by 4 from one point to the next one (but of course every other point is a Y). This X is the first point of the "long pattern". The last one is the point X, "6", $(2\ 2\ 2\ 0\ 0\ 0\ 2\ 0)$ such that the scalar product of $\overrightarrow{S}/2$ with the vectors $\overrightarrow{OX}$ is 24. The labelling of the intermediate points, even Xs and odd Ys is obvious.

One can convince oneself that the τs at the point $(4\ 0\ 0\ 0\ {-4}\ 0\ 0\ 0)$ is at squared distance 16 of points "0", "2" and "−2", "3" and "−3" and "5" and "−5" but at squared distance at least 32 of all the others, in particular "1" and "−1", "4" and "−4" and "6" and "−6". Thus it is precisely the τ the vanishing of which causes the rather long "dressed pattern" $\{-5,\star,-3,-2,\star,0,\star,2,3,\star,5\}$ we discovered by looking at "action at distance two".

Trying to find a τ the vanishing of which causes the "medium pattern" of (20) we encounter a surprise. It has to be at squared distance exactly 16 from "0", "1" and "−1", but certainly further than that from "2" and "−2", because these points are definitely outside the "medium pattern", defined by its entry and exit points. The only solution is at $(0\ 0\ 0\ 0\ 0\ 4\ 4\ 0)$. But in addition to "0", "1" and "−1", this point is also at squared distance exactly 16 of "4" and "−4", though farther from every other point, in particular, all those of the "dressed pattern" except, of course, "0". So its vanishing creates an "extended pattern" $\{-4,\star,\star,-1,0,1,\star,\star,4\}$ rather than just the "medium pattern" we previously knew of. Since the extension is by "action at distance three", there is no paradox about us not finding it earlier. Since "1" means that $y_m = t_m^2$ we can try and check whether the relationship between x_n and y_n induced by a value $y_{n-1} = t_{n-1}^2$ in (20b) does fix the value of x_{n+1} in (20a). As stated before, it is tricky to guess where "actions at distance three" exist, but it is easy to check if one knows where to look for it. And indeed, this relationship implies $x_{n+1} = (4t_n - 3\alpha)^2$, a good value for a point "4". A calculation "backwards" would also allow to find a good point "−4".

Since the vectors $\overrightarrow{X_mX_{m+1}}$, on the one hand, and $\overrightarrow{Y_mY_{m+1}}$, on the other hand, are both half-NV's, one can eliminate either variable to get an equation in the other variable alone. As we remarked earlier, this is just the trihomographic equation **I** for x. Its purely secular additive form can easily be obtained.

$$\frac{x_{n+1}-(4t_n-3\alpha)^2}{x_{n+1}-\alpha^2}\,\frac{x_{n-1}-(4t_n-\alpha)^2}{x_{n-1}-\alpha^2}\,\frac{x_n-(2t_n-\alpha)^2}{x_n-(6t_n-3\alpha)^2}=1 \tag{21}$$

with singularity patterns

$$\{x_{n-3}=(6t_{n-3}-3\alpha)^2, x_{n-2}=(4t_{n-3}-\alpha)^2, x_{n-1}=(2t_{n-3}-5\alpha)^2, x_n=81\alpha^2,$$
$$x_{n+1}=(2t_{n+3}+3\alpha)^2, x_{n+2}=(4t_{n+3}-\alpha)^2, x_{n+3}=(6t_{n+3}-3\alpha)^2\}$$

and

$$\{x_{n-1}=(2t_{n-1}-\alpha)^2, x_n=\alpha^2, x_{n+1}=(2t_{n+1}-\alpha)^2\}.$$

The long pattern $\{-6,-4,-2,0,2,4,6\}$ of this equation is just the x part of the long pattern of (20). As for the short pattern $\{-2,0,2\}$, it is to be found in the "dressed pattern" $\{-5,\star,-3,-2,\star,0,\star,2,3,\star,5\}$ we constructed earlier. The x part is of course limited to the $\{-2,\star,0,\star,2\}$ part. More precisely, we found there $x_{n+1} = (2t_n + \alpha)^2$, $x_{n+2} = \alpha^2$ and $x_{n+3} = (2t_{n+3} - \alpha)^2$. These values perfectly coincide with those given here, when evaluated at the appropriate n.

Since this equation is trihomographic, one can consider the equation obtained by keeping only one x out of two. This equation has also been identified. It is case 1 of Class II in section 4, i.e. 4.2.1, of [23]. Written in terms of the ancillary variable it has the same form as equation (15) given at the beginning of this section. Starting from the t_n and $\phi_4(n)$, $\phi_5(n)$ of (11), for an evolution over x_n with even indices $n = 2m$ rather than odd ones, we obtain the quantities Z_m and A^i_m entering the equation

$$
\begin{aligned}
Z_m &= 4(2m\alpha+\beta) - \phi_5(2m-2)\\
A^1_m &= 6(2\alpha m+\beta) - 6\alpha - \phi_4(2m) - \phi_4(2m-1) - \phi_5(2m+2) + \phi_5(2m+1) + \phi_5(2m-2)\\
A^2_m &= 4(2\alpha m+\beta) + 6\alpha + \phi_4(2m+2) - \phi_4(2m+1) + \phi_5(2m+2) - \phi_5(2m+1) - \phi_5(2m-2)\\
A^3_m &= 4(2\alpha m+\beta) - 14\alpha - \phi_4(2m+2) + \phi_4(2m+1) + \phi_5(2m+2) - \phi_5(2m+1) - \phi_5(2m-2)\\
A^4_m &= 2(2\alpha m+\beta) - 2\alpha + \phi_4(2m) + \phi_4(2m-1) + \phi_5(2m) + \phi_5(2m-1)
\end{aligned}
$$

This equation has four singularity patterns, a “long” one that can be schematised as $\{-6,-2,2,6\}$. This is present in the “long pattern” of (21), starting at “-6” and skipping every other x. There is one “short pattern” $\{-2,2\}$ that comes from the short pattern of (21) but also two “medium patterns” both of form $\{-4,0,4\}$. One of them is clearly in the long pattern of (21), starting at “-4”. But where is the second one? Well, it is in the extended pattern $\{-4,\star,\star,-1,0,1,\star,\star,4\}$ that we have identified earlier. In the “x-only” reduction, equation (21), this extended pattern becomes $\{-4,\star,0,\star,4\}$. It does not contain a blow-up, and therefore was not considered before. Though we might have noted in (21) that for generic x_n a value $(4t_n+\alpha)^2$ for x_{n-1} leads to a value α^2 for $x_{n+1} = x_{n'-1}$ for $n' = n+2$, and then to $(4t_{n'}-\alpha)^2$ for $x_{n'+1} = x_{n+3}$ in (21) evaluated at n'. In that equation one can see in $\{-4,\star,0,\star,4\}$ twice an “action at distance two”. The “0” here really means a value α^2 and if different from the “0” of the $\{-4,0,4\}$ within the long pattern, the value of which is $81\alpha^2$. All the patterns are thus accounted for, at this point.

Similarly we can consider the “y-only” reduction. The resulting equation has been identified in [23]: it is case 2 of class V of Section 4, 4.5.2, (and thus it has a right-hand side that is the ratio of a quadratic over a linear polynomial). The quantities Z_n and A^i_n entering the equation written in terms of the ancillary variable are

$$
\begin{aligned}
Z_n &= (2n+1)\alpha + \beta + \phi_4(n+2) + \phi_4(n-1) + \phi_5(n+2) + \phi_5(n-1)\\
A^1_n &= 5(\alpha n+\beta) - 4\alpha + \phi_4(n-2) + \phi_5(n-2) - \phi_5(n+2)\\
A^2_n &= 5(\alpha n+\beta) + 4\alpha + \phi_4(n+2) + \phi_5(n+2) - \phi_5(n-2)\\
A^3_n &= -3(\alpha n+\beta) + \phi_4(n+2) + \phi_5(n+2) + \phi_5(n-2)\\
A^4_n &= \alpha n + \beta + \phi_4(n+1) - \phi_4(n) + \phi_4(n-1) + \phi_5(n+1) - \phi_5(n) + \phi_5(n-1)
\end{aligned}
$$

This equation has four singularity patterns when we only consider patterns entirely consisting in blow-ups. The “long” one that can be schematised as $\{-5,-3,-1,1,3,5\}$ is present in the long pattern of (20ab), starting at “-5” and taking only the y. There are two “short pattern” $\{-5,-3\}$ and $\{3,5\}$ which in fact belong to a single “reconstructed pattern” $\{-5,-3,\star,\star,3,5\}$ with action at distance three. It can be seen in the “dressed pattern” $\{-5,\star,-3,-2,\star,0,\star,2,3,\star,5\}$ of (20ab). A final short pattern $\{-1,1\}$ can be seen in the extended pattern $\{-4,\star,\star,-1,0,1,\star,\star,4\}$ of (20ab) and in fact is already present in the "medium pattern" $\{-1,0,1\}$ with only blow-ups.

Because both $\overrightarrow{X_mY_{m+1}}$ and $\overrightarrow{Y_mX_{m+2}}$ are half NV's there are equations on the triangles $X_mY_{m+1}X_{m+3}$ and $Y_mX_{m+2}Y_{m+3}$ and thus there is an equation from X to Y to X skipping two points. This equation is asymmetric, involving x and y, and not trihomographic, and had not been identified before. Because the degree of x_{m+3} in terms of y_m and thus y_{m+1} is two for zero degree x_m, and the value of the degree of y_{m+3} for zero degree y_{m+1} or equivalently y_{m+2} is also two, we expect eight singularity patterns, four starting with an x (even) and four starting with a y (odd).

Skipping two points, the "long pattern" gives us $\{-6,-3,0,3,6\}$, $\{-5,-2,1,4\}$ and $\{-4,-1,2,5\}$, depending on the starting point. The pattern $\{-5,\star,-3,-2,\star,0,\star,2,3,\star,5\}$ gives $\{-5,-2\}$, $\{-3,0,3\}$ and $\{2,5\}$. Finally the pattern $\{-4,\star,\star,-1,0,1,\star,\star,4\}$ gives $\{-4,-1\}$ and $\{1,4\}$, a total of eight patterns, four starting by x and four starting by y (and also four ending in x and four ending in y, with various combinations) so all patterns are accounted for.

We shall not attempt to present the equation in its full freedom, involving all 8 parameters but content ourselves with exhibiting its autonomous form. Thus, with $t_n \equiv \beta$ for all n, we obtain the mapping

$$\frac{(y_{m+1}-x_{m+3}+9\beta^2)(y_{m+1}-x_m+9\beta^2)+36\beta^2y_{m+1}}{(y_{m+1}-x_{m+3}+9\beta^2)+(y_{m+1}-x_m+9\beta^2)} = \frac{3}{2}\frac{y_{m+1}^2+42\beta^2y_{m+1}-75\beta^4}{3y_{m+1}+5\beta^2}$$

$$\frac{(x_{m+3}-y_{m+1}+9\beta^2)(x_{m+3}-y_{m+4}+9\beta^2)+36\beta^2x_{m+3}}{(x_{m+3}-y_{m+1}+9\beta^2)+(x_{m+3}-y_{m+4}+9\beta^2)} = \frac{3}{2}\frac{x_{m+3}^2+36\beta^2x_{m+3}-192\beta^4}{3x_{m+3}-8\beta^2}$$

But for equation **VII**, $\overrightarrow{X_mY_{m+2}}$ and $\overrightarrow{Y_mX_{m+3}}$ are also half NV's, which means that there are equations on both triangles $X_mY_{m+2}X_{m+5}$ and $Y_mX_{m+3}Y_{m+5}$. The asymmetric equation from X to Y to X skipping four points to reach the fifth one has never been identified before. But because the degree of X_{m+5} in terms of Y_m and thus Y_{m+2} (for X_m of zero degree) is three, and the same is true of the degree of Y_{m+5} in terms of X_{m+1} and thus X_{m+3} (for Y_m of zero degree), we expect the number of patterns to be twelve. Again we present only the autonomous form, i.e.with $t_n \equiv \beta$ for all n:

$$\frac{(y_{m+2}-x_{m+5}+25\beta^2)(y_{m+2}-x_m+25\beta^2)+100\beta^2y_{m+2}}{(y_{m+2}-x_{m+5}+25\beta^2)+(y_{m+2}-x_m+25\beta^2)} = \frac{5}{2}\frac{y_{m+2}^3+161\beta^2y_{m+2}^2+1467\beta^4y_{m+2}+675\beta^6}{5y_{m+2}^2+166\beta^2y_{m+2}+405\beta^4}$$

$$\frac{(x_{m+5}-y_{m+2}+25\beta^2)(x_{m+5}-y_{m+7}+25\beta^2)+100\beta^2x_{m+5}}{(x_{m+5}-y_{m+2}+25\beta^2)+(x_{m+5}-y_{m+7}+25\beta^2)} = \frac{5}{2}\frac{x_{m+5}^3+160\beta^2x_{m+5}^2+1456\beta^4x_{m+5}+768\beta^6}{5x_{m+5}^2+164\beta^2x_{m+5}+416\beta^4}$$

Skipping four points in the "long pattern" gives us $\{-6,-1,4\}$, $\{-5,0,-5\}$, $\{-4,-1,6\}$, $\{-3,2\}$ and $\{-2,3\}$. The pattern $\{-5,\star,-3,-2,\star,0,\star,2,3,\star,5\}$ gives $\{-5,0,5\}$, $\{-3,2\}$ and $\{-2,3\}$. Finally the pattern $\{-4,\star,\star,-1,0,1,\star,\star,4\}$ gives $\{-4,1\}$ and $\{-1,4\}$, a total of ten patterns, five starting by x and five starting by y (and also five ending in x and five ending in y). Two patterns are still missing.

A more careful examination of the equation given above shows that it has three different patterns of each of the forms $\{-3,2\}$ and $\{-2,3\}$, while we accounted for only two of each. Since our search for "action at distance two" was exhaustive, and though we might have missed some "action at distance three" they could not have combined to the patterns we are looking for. This means that in the original equation these patterns must be of the form $\{-3, \star, \star, \star, \star, 2\}$ and $\{-2, \star, \star, \star, \star, 3\}$. We are not going to compute how this "action at distance five" propagates from step to step. But there must be τ-functions the vanishing of which causes these two patterns, so one τ at distance exactly 16 from "-3" and "2" and no other, and similarly for "-2" and "3". It turns out that these τ-functions do exist. Their positions are $(0\ 4\ 0\ 0\ -4\ 0\ 0\ 0)$ and $(4\ 0\ 0\ -4\ 0\ 0\ 0\ 0)$ respectively.

4.6 The periods 2,3,4, 2,3,8 and 4,6 case

In this subsection we choose the original constraint, i.e. that the sum of all coordinates of a τ function be always a multiple of 8.

First we will clarity a few ambiguities. The equations we will discuss in this section have, in the list of [19] the numbers **II**, **V**, **VIII**, **IX** and **XII**. However, they are not five different equations. We begin with the asymmetric equation **IX** described in [17], namely our equation (11) supplemented by

$$t_n = \alpha n + \beta,\ u_n = t_n + \phi_3(n) + \phi_4(n)$$

$$\kappa_n = \gamma + \phi_2(n), \quad k_n = u_{n+1} + u_n + u_{n-1} - \phi_2(n), \quad z_n + \zeta_n = u_{n-1} - \gamma, \quad z_n + \zeta_{n-1} = u_n + \gamma$$

Eliminating y from **IX** we obtain the "x-only" reduction, which is trihomographic, of the form (8) supplemented by

$$u_n = (2n-1)\alpha + \beta - \phi_3(n-1) + \phi_4(n-2) + \phi_4(n)$$

$$z_n + z_{n+1} = u_{n+1},\ k_n = 3\alpha + \gamma - \phi_2(n) + \phi_4(n+1) - \phi_4(n+2)$$

Note that this is identical to equation **II** of (19), given at the beginning of section 4, up to a reinterpretation of the parameters. In fact, the quantity $\phi_4(n-2) + \phi_4(n)$ is a quantity with period 2 and the non-constant term in k_n has a period 4. So indeed u_n is of the form $T_n + \Phi_2(n) + \Phi_3(n)$ where $T_n = (2n-1)\alpha + \beta$ while k_n has the form $\Gamma + \Phi_4(n)$.

As we have already remarked in subsection 4.4, exchanging numerator and denominator of each factor of (8) leaves the equation invariant, but changes k_n into its opposite. The function k_n of **II** is of period 4, $(\cdots, a, b, c, d, a, b, c, d, \cdots)$. Changing the sign of four consecutive instances out of eight we get for a new choice of function k_n the sequence $(\cdots, a, b, c, d, -a, -b, -c, -d, a, b, \cdots)$, which is just what we call χ_8. We thus recover the equation given as **V** in [19] and given again at the beginning of section 4.

Now let us artificially rewrite equation **II** by separating even and odd indices and replacing x_{2n}, z_{2n}, k_{2n}, x_{2n+1}, z_{2n+1} k_{2n+1} by x_n, z_n, κ_n, y_n, ζ_n, k_n respectively. The periodic function Φ_2 in u_n of equation **II**, rewritten as $G(-1)^n$, contributes $-G$ to the quantity $z_n + \zeta_n$ of **XII** which is u_{2n+1} of **II** and G to $z_{n+1} + \zeta_n$ which is u_{2n+2} while Φ_3 remains a period-3 function φ_3, upon reinterpretation. The equation **II** is now split into a system of the form (11). At this point k and κ are of the form $(\cdots, a, c, a, c, \cdots)$ and $(\cdots, b, d, b, d, \cdots)$, namely both of the general form "$D + \varphi_2(n)$".

Now we leave (11b) unchanged and thus κ_n remains as is, but we invert two out of four instances of (11a). Then k_n becomes the sequence $(\cdots, a, c, -a, -c, \cdots)$ which is just what is called χ_4.

We have thus reconstructed the equation called **XII** of [19] and given again at the beginning of section 4.

A single trajectory will suffice to describe the evolution all these four equations, **IX**, a genuinely asymmetric equation, and **II**, **V**, **XII** which are essentially one and the same, symmetric, equation, written in three different ways, one of which is artificially asymmetric.

Contrariwise, the asymmetric equation described in [17] and denoted by **VIII** in [19] (and at the beginning of section 4) is a distinct equation. In particular, it does have an "x-only" reduction, but the latter is *not* a trihomographic equation but rather the equation case 4, Class IV in section 4, i.e. 4.4.4, of [23]. The two asymmetric equations, **VIII** and **IX** however, are so intimately related that we will study both together rather than dedicate a separate sub-section to each of them.

In order to get the vectors relating both the Xs and the Ys for **IX**, the easiest way is to find first the vectors relating the X for **II**. The latter has period 12, and the degrees d_q of the x_{m+q}, starting from degree zero for x_m and one for x_{m+1} are $d_m = 0, 1, 1, 2, 3, 5, 6, 9, 11, 14, 17, 21, 24, \cdots$, and thus $d_{12} = 24$.

Because the degree of x_{m+12} is four times that of x_{m+6} and $\overrightarrow{X_m X_{m+12}} = \overrightarrow{S}$ we can expect $\overrightarrow{X_m X_{m+6}} = \overrightarrow{S}/2$ so we look for only six vectors $\overrightarrow{X_m X_{m+1}}$

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & -2 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

and the sum of these six vectors is $\overrightarrow{S}/2$ =[4 4 4 0 0 0 0 0]. This pattern has to be repeated before we get to the actual period, which is 12. The scalar product of each vector $\overrightarrow{X_m X_{m+1}}$ with $\overrightarrow{S}/2$ is 8.

To get the $\overrightarrow{X_m Y_m}$ and $\overrightarrow{Y_m X_{m+1}}$ we need the degrees for the asymmetric equation **IX** starting both from x_0 of degree 0, y_0 of degree 1 and y_0 of degree 0, x_1 of degree 1. We find (d^x_n, d^y_n) =(0,1), (1,1), (1,2), (2,3), (3,4), (5,6), (6,8), (9,10), (11,13), (14,16), (17,19), (21,23), (24,27),$\cdots$ which means that $d^x_{12} = 24$ and (d^y_n, d^x_{n+1}) =(0,1), (1,1), (2,2), (2,3), (3,4), (5,6), (7,8), (9,10), (11,13), (14,16), (18,19), (21,23), (24,27)$\cdots$ i.e., $d^y_{12} = 24$.

We see here that $\overrightarrow{Y_m Y_{m+2}}$ has squared length 16 and not 8 like $\overrightarrow{X_m X_{m+2}}$. Again according to our rule of thumb, $\overrightarrow{Y_m Y_{m+6}}$ and $\overrightarrow{Y_m Y_{m+10}}$ also differ in squared length from the X analogs. In particular $\overrightarrow{Y_m Y_{m+6}}$ is *not* $\overrightarrow{S}/2$. The degrees of x_{12} on the first sequence, of y_{12} on the second one, are 24, consistent with the squared length 192 of $\overrightarrow{S}$ =[8 8 8 0 0 0 0 0].

We must now turn to the vectors $\overrightarrow{X_m Y_m}$ and $\overrightarrow{Y_m X_{m+1}}$. As we said earlier, in all the equations in this paper, they have the same scalar product with $\overrightarrow{S}/2$, thus 4 each. This led us to

$$
\begin{bmatrix}
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1\\
1 & -1 & 1 & 1 & -1 & -1 & -1 & 1\\
-1 & 1 & 1 & 1 & 1 & 1 & 1 & -1\\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1\\
1 & -1 & 1 & 1 & -1 & 1 & -1 & -1\\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & 1\\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1\\
1 & -1 & 1 & 1 & 1 & -1 & 1 & 1\\
-1 & 1 & 1 & -1 & 1 & 1 & -1 & -1\\
1 & 1 & -1 & 1 & -1 & -1 & -1 & 1\\
1 & -1 & 1 & 1 & 1 & 1 & 1 & -1\\
-1 & 1 & 1 & -1 & -1 & -1 & 1 & 1\\
1 & 1 & -1 & 1 & -1 & 1 & -1 & -1\\
1 & -1 & 1 & -1 & 1 & -1 & -1 & 1\\
-1 & 1 & 1 & -1 & -1 & 1 & 1 & -1\\
1 & 1 & -1 & 1 & 1 & -1 & 1 & 1\\
1 & -1 & 1 & -1 & 1 & 1 & -1 & -1\\
-1 & 1 & 1 & 1 & -1 & -1 & -1 & 1\\
1 & 1 & -1 & 1 & 1 & 1 & 1 & -1\\
1 & -1 & 1 & -1 & -1 & -1 & 1 & 1\\
-1 & 1 & 1 & 1 & -1 & 1 & -1 & -1\\
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1\\
1 & -1 & 1 & -1 & -1 & 1 & 1 & -1\\
-1 & 1 & 1 & 1 & 1 & -1 & 1 & 1
\end{bmatrix}
$$

In order to find the initial point to which to add these vectors to obtain the positions of all Xs and Ys we must consider the singularity patterns.

We consider the simplified expression, ignoring the periodic functions but keeping a nonzero γ, to lift ambiguities

$$\frac{x_{n+1}-(4t_n-\alpha-\gamma)^2}{x_{n+1}-(2t_n+\alpha+\gamma)^2}\frac{x_n-(4t_n+\alpha+\gamma)^2}{x_n-(2t_n-\alpha-\gamma)^2}\frac{y_n-t_n^2}{y_n-25t_n^2}=1 \tag{22a}$$

$$\frac{y_n-(t_n+2\gamma)^2}{y_n-t_n^2}\frac{y_{n-1}-(t_{n-1})^2}{y_{n-1}-(t_{n-1}-2\gamma)^2}\frac{x_n-(2t_n-\alpha-\gamma)^2}{x_n-(2t_n-\alpha+\gamma)^2}=1 \tag{22b}$$

In [17] we presented four distinct singularity patterns, a “long” one, schematically $\{-5,-4,-3,-2,-1,0,1,2,3,4,5\}$,

$$\{y_{n-2}=25t_{n-2}^2, x_{n-1}=(4t_{n-2}-\alpha-\gamma)^2, y_{n-1}=(3t_{n-2}-\alpha)^2, x_n=(2t_{n-2}-3\alpha-\gamma)^2, y_n=(t_{n-2}-4\alpha)^2,$$
$$x_{n+1}=(\gamma+7\alpha)^2, y_{n+1}=(t_{n+3}+4\alpha)^2, x_{n+2}=(2t_{n+3}+3\alpha+\gamma)^2, y_{n+2}=(3t_{n+3}+\alpha)^2,$$
$$x_{n+3}=(4t_{n+3}+\alpha+\gamma)^2, y_{n+3}=25t_{n+3}^2\}$$

a “medium” one $\{-2,-1,0,1,2\}$

$$\{x_{n-1}=(2t_{n-1}-\alpha+\gamma)^2, y_{n-1}=(t_{n-1}+2\gamma)^2, x_n=(\gamma-\alpha)^2, y_n=(t_{n+1}-2\gamma)^2, x_{n+1}=(2t_{n+1}-\alpha-\gamma)^2\}$$

and two short ones $\{-2-1\}$ and $\{1,2\}$, $\{x_n=(2t_n-\alpha-\gamma)^2, y_n=t_n^2\}$, $\{y_n=t_n^2, x_{n+1}=(2t_{n+1}-\alpha+\gamma)^2\}$ because we were only looking for patterns composed of blow-ups. But by now we are used to the fact that patterns may well be longer, when "actions at distance" are taken into account.

In fact the "medium pattern" is somewhat longer when "dressed", but without reaching the length of the "long" one. In (22a) one sees that an initial value $x_n=(4t_n+\alpha+\gamma)^2$ for y_n arbitrary imposes $x_{n+1}=(2t_n+\alpha+\gamma)^2$ which means $x_{n'-1}=(2t_{n'-1}-\alpha+\gamma)^2$ for $n'=n+2$. This value is the entry to the "medium pattern" for n' that exists with $x_{n'+1}=(2t_{n'+1}-\alpha-\gamma)^2$. After this exit, the next value of y, namely $y_{n'+1}$ is arbitrary. Therefore, in (22a) evaluated for $n''=n'+1=n+3$ one has $x_{n''}=(2t_{n''}-\alpha-\gamma)^2$ and $y_{n''}$ arbitrary we have $x_{n''+1}-(4t_{n''}-\alpha-\gamma)^2$. This value is not remarkable otherwise, therefore the pattern ends there. Schematically, it has the form $\{-4,\star,-2,-1,0,1,2,\star,4\}$, because the values y_n and $y_{n'+1}=y_{n+3}$ are arbitrary and thus represented by stars.

The value $(2t_n-\alpha-\gamma)^2$ for x_n, entry to the first short pattern does not have an "antecedent". After exiting this pattern through $y_n=t_n^2$, the value of x_{n+1} is arbitrary. In (22a) evaluated at $n'=n+1$, $x_{n'}$ is arbitrary while $y_{n'-1}=y_n=t_n^2=t_{n'-1}^2$ and this implies that $y_{n'}=t_{n'}^2$. Thus after $x_{n'}$ we enter the second "short" pattern at $n'=n+1$. When exiting, the value $(2t_{n'+1}-\alpha+\gamma)^2$ is not remarkable and does not have consequences "at distance two". Still, the two "short patterns" are in fact not independent but form a single "reconstructed" pattern of the form $\{-2,-1,\star,1,2\}$, still the shortest of the three patterns that remain when "actions at distance two" are considered.

Finally, while for *arbitrary* x_n the value t_{n-1}^2 for y_{n-1} in (22b) can only be the exit of the first "short pattern" and will lead to $y_n=t_n^2$ and the second "short pattern" the value $(t_{n-1}-2\gamma)^2$ does not have any antecedent. It leads to $y_n=(t_n-2\gamma)^2$, but this value is not otherwise remarkable. So this blow-up-free pattern stops there. Its schematic form is just $\{-1,\star,1\}$.

Imposing that the "long" singularity pattern be caused by the vanishing of the τ-function at the origin means that there must be sequence of positions for X and Y that extends on six Ys and five intervening Xs at squared distance exactly 16 from the origin. We present here a slightly longer sequence

$$\begin{pmatrix}
-4 & -2 & 0 & 0 & -2 & 0 & 2 & 2\\
-3 & -1 & -1 & -1 & -1 & 1 & 1 & 1\\
-2 & -2 & 0 & 0 & -2 & 0 & 0 & 2\\
-3 & -1 & 1 & 1 & -1 & 1 & 1 & 1\\
-2 & 0 & 0 & 0 & -2 & 0 & 2 & 2\\
-1 & -1 & 1 & 1 & -3 & 1 & 1 & 1\\
-2 & 0 & 2 & 0 & -2 & 0 & 0 & 2\\
-1 & 1 & 1 & -1 & -3 & 1 & 1 & 1\\
0 & 0 & 2 & 0 & -2 & 0 & 2 & 2\\
-1 & 1 & 3 & -1 & -1 & 1 & 1 & 1\\
0 & 2 & 2 & 0 & -2 & 0 & 0 & 2\\
1 & 1 & 3 & 1 & -1 & 1 & 1 & 1\\
0 & 2 & 4 & 0 & -2 & 0 & 2 & 2\\
1 & 3 & 3 & 1 & -3 & 1 & 1 & 1
\end{pmatrix}$$

starting from an X at squared distance 32 from the origin, just before the "long" singularity pattern and continuing with one extra X and one extra Y both also at squared distance 32 from the origin.

We have chosen in this subsection the original contraint that the sum of the coordinates of every site of τ-functions is a multiple of 8. All X have all even coordinates while all Y have all odd coordinates. We are showing fourteen points altogether, but this is much less than the twelve Xs and twelve Ys needed for a whole period. The bottommost X is exactly $\overrightarrow{S}/2$ away from the topmost X, but this is *not* the case for the corresponding Ys. One needs a whole period for the Ys to be translated by $\overrightarrow{S}$.

The scalar products of $\overrightarrow{S}$ with the vectors from the second to the twelfth point increase by steps of 4 from -20 to 20, which is exactly the long pattern for points schematically represented by one quarter of these respective quantities, i.e. from "-5" to "$+5$". The "medium, dressed pattern" $\{-4,\star,-2,-1,0,1,2,\star,4\}$ is realised when the τ-function at the point $(0\ 0\ 0\ 0\ -4\ 0\ 0\ 4)$ vanishes. Indeed this point is at squared distance 32 from the points "± 5" and "± 3" (and at least as much from of all points beyond the "long pattern") but still at squared distance 16 from the other points of the "long pattern". In the same way, the points "± 2"and "± 1" of the "short reconstructed pattern" $\{-2,-1,\star,1,2\}$ are the only ones at exactly squared distance 16 from the point $(0\ 0\ 0\ 0\ -4\ 0\ 4\ 0)$ and this pattern is realised when the τ-function there vanishes.

The very short, blow-up-free pattern $\{-1,\star,1\}$ is related with the vanishing of the τ-function at the point $(0\ 0\ 0\ 0\ -4\ 4\ 0\ 0)$ which is at squared distance 16 of "± 1" and at lest 32 from every other point.

Having finished with the full equation **IX** we turn to its "x-only" and "y-only" reductions. For the former we obtained, in [17], the simplified form of equation, **II**, when the periodicity is omitted but a non vanishing γ is kept. We obtain the simplified form

$$\frac{x_{n+1}-(2t_n-\alpha+\gamma)^2}{x_{n+1}-(2t_n-\alpha-\gamma)^2}\frac{x_{n-1}-(2t_n+\alpha+\gamma)^2}{x_{n-1}-(2t_n+\alpha-\gamma)^2}\frac{x_n-(4t_n-\gamma)^2}{x_n-(4t_n+\gamma)^2}=1 \tag{23}$$

In the two singularity patterns, both of the schematical form $\{-4,-2,0,2,4\}$ we can recognise the x only part of the "long" and "medium dressed" patterns respectively for γ and $-\gamma$. What we call here t_n,γ can be expressed in terms of the ones of equation (22) as $t_n-\alpha/2,\gamma+3\alpha$.

No blow-up in **II** comes from the "reconstructed, short pattern" $\{-2,-1,\star,1,2\}$ of **IX**. However, there is an "action at distance two" pattern $\{-2,\star,2\}$, caused by the vanishing of the τ-function at $(0\ 0\ 0\ 0\ -4\ 0\ 4\ 0)$, as noted above. But by examining equation (23) one can see that there are *two* such patterns, because for x_n arbitrary, both choices of values $(2t_n+\alpha\pm\gamma)^2$ for x_{n-1} lead to a value $(2t_n-\alpha\mp\gamma)^2$ for x_{n+1}. This means that there should exist a τ-function the vanishing of which creates, for equation **IX** an "action at distance four" pattern $\{-2,\star,\star,\star,2\}$, thus at squared distance 16 of just the two points in this pattern and none other. One can check that it is indeed the case for the point $(0\ 0\ 0\ 0\ 0\ 0\ 4\ 4)$.

Since equation **II** is trihomographic, one can skip every other x to have an equation for one x out of two. This equation is not trihomographic but has been identified. It is the case 1 of Class II in the Section 5, i.e. 5.2.1, of [23]. It has six patterns, two "long" ones of the schematic form $\{-4,0,4\}$ and four "short" ones of the form $\{-2,2\}$. Each of the two patterns of equation **II** of the form $\{-4,-2,0,2,4\}$ mentioned just after

(23) contains one "long" and one "short" pattern of this non-trihomographic equation, depending on which point one starts. The other two "short" patterns come from blow-up-free, "hidden" patterns for the equation **II**. One of them comes from the "reconstructed" pattern $\{-2,-1,\star,1,2\}$ of equation (22a,b) as described in the paragraphs that follow the latter equation. The last one comes from the second pattern $\{-2,\star,2\}$ we just found from a search of "action at distance two" in **II**, namely $\{-2,\star,\star,\star,2\}$ in **IX**. We have thus found all the τs that explain all the patterns of this equation.

The corresponding Z_n and A_n^i are now

$$\begin{aligned}
Z_m &= 2(2m\alpha+\beta)+3\alpha-\phi_3(2m)\\
A_m^1 &= 2(2m\alpha+\beta)+\alpha-\phi_3(2m-1)+\gamma+\phi_4(2m)\\
A_m^2 &= 2(2m\alpha+\beta)+\alpha-\phi_3(2m-1)-\gamma-\phi_4(2m)\\
A_m^3 &= 2m\alpha+\beta-\alpha+\phi_3(2m-1)-h-\gamma-\phi_4(2m-1)\\
A_m^4 &= 2m\alpha+\beta+2\alpha+\phi_3(2m-1)+h+\gamma+\phi_4(2m+1)\\
A_m^5 &= 2m\alpha+\beta-\alpha+\phi_3(2m-1)-h+\gamma+\phi_4(2m-1)\\
A_m^6 &= 2m\alpha+\beta+2\alpha+\phi_3(2m-1)+h-\gamma-\phi_4(2m+1)
\end{aligned}$$

where $h=\phi_2(0)$.

The "y-only" reduction is equation case 1 of Class V in section 4, i.e. 4.5.1, of [23]. It has four patterns. One is a "long" pattern of the type $\{-5,-3,-1,1,3,5\}$. One can immediately see that pattern within our "long pattern". The three other ones are all of the type $\{-1,1\}$. One is visible in the "medium dressed pattern" $\{-4,\star,-2,-1,0,1,2,\star,4\}$ and another one in the "short reconstructed pattern" $\{-2,-1,\star,1,2\}$. The last one comes from the pattern $\{-1,\star,1\}$ we discussed last after (22ab).

Starting from the parameters u_n, k_n and κ_n of **IX** given at the beginning of this subsection we can obtain the parameters Z_n, A_n^i of 4.5.1. We start from $u_n=t_n+\phi_3(n)+\phi_4(n)$ (where $t_n=\alpha n+\beta$) and introduce the auxiliary quantity: $w_n=u_n+\phi_2(n)/3$. We find:

$$\begin{aligned}
Z_n &= u_n+u_{n+1}=t_n+t_{n+1}-\phi_3(n-1)+\phi_4(n)+\phi_4(n+1)=w_n+w_{n+1}\\
A_n^1 &= 5t_n-\phi_3(n)+2\phi_4(n-1)+\phi_4(n-1)+2\phi_4(n-1)-\phi_2(n)=2w_{n-1}+w_n+2w_{n+1}\\
A_n^2 &= t_n+\phi_3(n)+\phi_4(n)-\phi_2(n)=w_n-4\phi_2(n)/3
\end{aligned}$$

and, in order to proceed, define two auxiliary quantities:

$$\begin{aligned}
B_n^3 &= t_n+\phi_3(n)+\phi_4(n)+2\gamma+\phi_2(n)\\
B_n^4 &= t_n+\phi_3(n)+\phi_4(n)-2\gamma+\phi_2(n)
\end{aligned}$$

Just as we did in subsection 4.4, we introduce A^3, A^4 by identifying them to B^3, B^4 for even values of n and to B^4, B^3 for odd values. We are thus led to

$$\begin{aligned}
A_n^3 &= t_n+\phi_3(n)+\phi_4(n)+2\gamma(-1)^n+\phi_2(n)=w_n+2\gamma(-1)^n+2\phi_2(n)/3\\
A_n^4 &= t_n+\phi_3(n)+\phi_4(n)-2\gamma(-1)^n+\phi_2(n)=w_n-2\gamma(-1)^n+2\phi_2(n)/3
\end{aligned}$$

Thus, formally the Z_n, A_n^i obtained above are identical to those of 4.5.1, up to some re-interpretations. In fact interpreting w_n as the quantity called z_n in 4.5.1 of [23], i.e. $w_n = t_n + \phi_3(n) + \Phi_4(n)$, where $\Phi_4(n)$ is the period-4 function appearing in 4.5.1, we must obtain the latter from the periodic functions of **IX** as $\Phi_4(n) = \phi_4(n) + \phi_2(n)/3$. Then the expressions of Z_n and A_n^1 in **IX** and 4.5.1 of [23] coincide, and the three other A^is are indeed obtained as the sum of w_n (resp. z_n) with three period-2 functions of zero sum.

As both $\overrightarrow{X_mY_{m+1}}$ and $\overrightarrow{Y_mX_{m+2}}$ are of squared length 8, for any m there is an equation in each of the triangles $X_mY_{m+1}X_{m+3}$ and $Y_{m+1}X_{m+3}Y_{m+4}$, that is, an equation that propagates along the trajectory we have constructed, but skipping two points to reach the third one. This equation is asymmetrical, and thus outside the scope of [23]. Since the degrees of x_{m+3} (for x_m of degree 0) and y_{m+3} (for y_m of degree 0) are both equal to two, the equation is not trihomographic, but has right-hand sides which are a ratio of quadratic over linear polynomials. We give below its autonomous form:

$$\frac{(y_{m+1}-x_{m+3}+9\beta^2)(y_{m+1}-x_m+9\beta^2)+36y_{m+1}\beta^2}{(y_{m+1}-x_{m+3}+9\beta^2)+(y_{m+1}-x_m+9\beta^2)} = \frac{3}{2}\frac{y_{m+1}^2+50y_{m+1}\beta^2+45\beta^4}{y_{m+1}+7\beta^2} \tag{24a}$$

$$\frac{(x_{m+3}-y_{m+1}+9\beta^2)(x_{m+3}-y_{m+4}+9\beta^2)+36x_{m+3}\beta^2}{(x_{m+3}-y_{m+1}+9\beta^2)+(x_{m+3}-y_{m+4}+9\beta^2)} = \frac{3}{2}\frac{x_{m+3}^2+52x_{m+3}\beta^2+64\beta^4}{x_{m+3}+8\beta^2} \tag{24b}$$

This equation has eight distinct "connected" singularity patterns, four starting with an x and four starting with a y (and similarly four ending with a y and four ending with an x, but with any combinations). The "long pattern" $\{-5,-4,-3,-2,-1,0,1,2,3,4,5\}$, when skipping two points leads to three patterns for this equation, namely $\{-5,-2,1,4\}$, $\{-4,-1,2,5\}$ and $\{-3,0,3\}$. The "medium dressed pattern" $\{-4,\star,-2,-1,0,1,2,\star,4\}$ leads to two more, namely, $\{-4,-1,2\}$, $\{-2,1,4\}$ while the "short reconstructed pattern" $\{-2,-1,\star,1,2\}$ also leads to two more, namely $\{-2,1\}$ and $\{-1,2\}$. One pattern is missing, and it should start and end with y. By symmetry, it can only be another pattern of the form $\{-3,0,3\}$. Since it was not found before, when we looked for blow-ups and "action at distance two" it must be of "action at distance three" form $\{-3,\star,\star,0,\star,\star,3\}$ in the original equation. And indeed, the τ-function at the point $(-4\ 0\ 4\ 0\ 0\ 0\ 0\ 0)$ is exactly at squared distance 16 from "-3", "0" and "3" and at least 32 from any other point.

This concludes the study of the equations living on the trajectory of equation **IX**.

We now turn to the asymmetric equation denoted by **VIII**. This is generically a distinct equation. If however we take ψ_6 identically zero, then **VIII** is nothing but **IX** with $\gamma \equiv 0$ and $\phi_2 \equiv 0$. As in the case of equation **X** in subsection 4.4, choosing $\phi_2 \equiv 0$ and taking the limit $\gamma \to 0$ is meaningful, and the resulting equation is just a subcase of the initial one. The degrees are exactly the same as in the generic case. However, in this limit, there is no difference anymore between **VIII** and **IX**. In order to compute the degrees of the iterates of **VIII** on must keep a nonzero $\psi_6(n)$.

In that case, the degrees become (d_n^x, d_n^y) =(0,1), (1,1), (2,2), (3,3), (4,4), (5,6), (6,8), (9,10), (12,13), (15,16), (18,19), (21,23), (24,27),$\cdots$ and so $d_{12}^x = 24$. And (d_n^y, d_{n+1}^x) =(0,1), (1,1), (2,2), (2,3), (3,4), (5,6), (7,8), (9,10), (11,13), (14,16), (18,19), (21,23), (24,27),$\cdots$ and thus $d_{12}^y = 24$. We remark that the degrees (d_n^y, d_{n+1}^x) are identical to those obtained for **IX**. In particular, since the degree of y_1 in terms of y_0 and x_1 is one in that case, if one expresses x_1 in terms of y_0 and y_1 in the expression of all the y, the degrees do not change. The dependence in y_0 is different, of course, but this is of no matter. This shows that the degrees in y_1 of all the y in terms of y_0 and y_1 is the same as for equation **IX** and thus that the trajectory of the Ys is the same.

The only changes are for the degrees (d_n^x, d_n^y) which are modified for x of index $(6n+2)$, $(6n+3)$ and $(6n+4)$ which increase by one unit. Still, this shows that the trajectory of the Xs is definitely not the same as for IX, though it is very close to the latter. In particular the value 2 of the degree of x_2 for x_0 of degree 0 and y_0, or x_1 for that matter, of degree one, shows that the "x-only" reduction is not trihomographic contrary to the "x-only" reduction of **IX**, namely **II**. In fact this equation has been identified. It is case 4 of Class IV of section 4, i.e. 4.4.4 in [23].

Expressing the quantities Z_m and A_m^i entering the equation in terms of the parameters appearing in **VIII** we find

$$\begin{aligned}
Z_n &= 2t_n - \phi_3(n) + \phi_4(n-1) + \phi_4(n+1) - \psi_6(n) \\
A_n^1 &= 4t_n - 5\alpha + \phi_3(n+1) + \phi_4(n-2) - \phi_4(n+1) - \psi_6(n+1) \\
A_n^2 &= 4t_n + \alpha + \phi_3(n+1) - \phi_4(n-2) + \phi_4(n+1) + \psi_6(n+1) \\
A_n^3 &= -2t_n + \alpha + \phi_3(n+1) - \phi_4(n-1) - \phi_4(n) - \psi_6(n-1) - \psi_6(n) \\
A_n^4 &= 2t_n - \alpha - \phi_3(n+1) + \phi_4(n-1) + \phi_4(n) - \psi_6(n-1) - \psi_6(n)
\end{aligned}$$

There are four patterns for this equation, 4.4.4 of [23], namely the "x-only" reduction of equation **VIII**. The "long" one, $\{-4,-2,0,2,4\}$ and the "medium" one, $\{-2,0,2\}$ are present in the respective patterns of equation **VIII** as given in [17] and [19]. The other two patterns $\{-4,-2\}$ and $\{2,4\}$ will be explained in what follows.

Since 4.4.4 is not trihomographic, there is no equation skipping every other x. Contrary to the case of the "x-only" reduction, as we have remarked above, the degrees for the "y-only" reduction are unchanged, so the trajectory for the Ys is exactly the same as for **IX**. So it remains true that $\overrightarrow{Y_mY_{m+12}} = \overrightarrow{S} = [8\,8\,8\,0\,0\,0\,0\,0]$. One has to find how the Xs are affected without changing the Ys. Though some of the Xs are indeed affected it remains true that $\overrightarrow{X_mX_{m+6}} = \overrightarrow{S}/2$ for all m. So we only need six vectors $\overrightarrow{X_mX_{m+1}}$ for $m = 1$ to 6, but for lengths different for $\overrightarrow{X_mX_{m+2}}$ and $\overrightarrow{X_mX_{m+4}}$ from the case of **IX**. It turns out that the following sequence

$$\begin{bmatrix}
2 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 2 & 0 & 0 & 0 & -2 & 0 \\
2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & -2 \\
0 & 0 & 2 & 0 & 0 & 0 & 2 & 0
\end{bmatrix}$$

not only satisfies the correct lengths for all the $\overrightarrow{X_0X_q}$, but also neatly fits with the positions of the Ys which are common with equation **IX**.

The intervals between consecutive X and Y have period 12, thus we consider 24 intervals altogether.

$$\begin{bmatrix}
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1\\
1 & -1 & 1 & 1 & -1 & -1 & 1 & -1\\
-1 & 1 & 1 & 1 & 1 & 1 & -1 & 1\\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1\\
1 & -1 & 1 & 1 & -1 & 1 & -1 & -1\\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & 1\\
1 & 1 & -1 & -1 & -1 & 1 & 1 & -1\\
1 & -1 & 1 & 1 & 1 & 1 & -1 & 1\\
-1 & 1 & 1 & -1 & 1 & -1 & 1 & -1\\
1 & 1 & -1 & 1 & -1 & 1 & -1 & -1\\
1 & -1 & 1 & 1 & 1 & -1 & 1 & 1\\
-1 & 1 & 1 & -1 & -1 & 1 & 1 & -1\\
1 & 1 & -1 & 1 & -1 & -1 & -1 & 1\\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1\\
-1 & 1 & 1 & -1 & -1 & 1 & -1 & 1\\
1 & 1 & -1 & 1 & 1 & -1 & 1 & 1\\
1 & -1 & 1 & -1 & 1 & 1 & -1 & -1\\
-1 & 1 & 1 & 1 & -1 & -1 & -1 & 1\\
1 & 1 & -1 & 1 & 1 & 1 & 1 & -1\\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1\\
-1 & 1 & 1 & 1 & -1 & -1 & 1 & -1\\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1\\
1 & -1 & 1 & -1 & -1 & -1 & 1 & 1\\
-1 & 1 & 1 & 1 & 1 & 1 & 1 & -1
\end{bmatrix}$$

Imposing that the "long" singularity pattern be caused by the vanishing of the τ-function at the origin means that there must be a sequence of positions for X and Y that extends on six Ys and five Xs in between at squared distance exactly 16 from the origin. We present here a slightly longer sequence starting from an X at squared distance 32 from the origin, just before the "long" singularity pattern and continuing with one extra X and one extra Y both also at squared distance 32 from the origin.

$$\begin{pmatrix}
-4 & -2 & 0 & 0 & -2 & 2 & 2 & 0\\
-3 & -1 & -1 & -1 & -1 & 1 & 1 & 1\\
-2 & -2 & 0 & 0 & -2 & 0 & 2 & 0\\
-3 & -1 & 1 & 1 & -1 & 1 & 1 & 1\\
-2 & 0 & 0 & 0 & -2 & 0 & 2 & 2\\
-1 & -1 & 1 & 1 & -3 & 1 & 1 & 1\\
-2 & 0 & 2 & 0 & -2 & 0 & 0 & 2\\
-1 & 1 & 1 & -1 & -3 & 1 & 1 & 1\\
0 & 0 & 2 & 0 & -2 & 2 & 0 & 2\\
-1 & 1 & 3 & -1 & -1 & 1 & 1 & 1\\
0 & 2 & 2 & 0 & -2 & 2 & 0 & 0\\
1 & 1 & 3 & 1 & -1 & 1 & 1 & 1\\
0 & 2 & 4 & 0 & -2 & 2 & 2 & 0\\
1 & 3 & 3 & 1 & -3 & 1 & 1 & 1
\end{pmatrix}$$

The singularity patterns of equation **VIII**, presented in [17], are the same as those of **IX**, namely the long one $\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$, the medium one $\{-2, -1, 0, 1, 2\}$ and the two short ones $\{-2, -1\}$, $\{1, 2\}$ but they are not rearranged in the same way. In particular, we have yet to find rearranged patterns that explain the patterns $\{-4,-2\}$, $\{2,4\}$ of the "x-only" reduction of **VIII**. How can we find these rearrangements? In all the other cases studied in this paper, we considered the simplified equation where one keeps just the secular terms discarding all periodicities and looked for action at distance-two for this simplified equation. However here if we consider a purely secular equation, without the periods, we simply recover equation (22ab), which is *not* what we are looking for (and keeping the ψ_6 term would lead to prohibitively long expressions). So one has to look at the possible positions of the τ-functions to check that all patterns can be accounted for.

As usual, the "long pattern" is caused by the vanishing of the τ-function at the origin, as all points are at squared distance 16 from it. The "medium pattern" $\{-2,-1,0,1,2\}$ as originally found is now caused by vanishing of the τ-function at the point (0 0 0 0 -4 0 0 4), but here the points "-4" and "4" are at squared distance 32 from the latter point, so the "medium pattern" is not "dressed". The two short patterns are not combined into a longer one. Rather, each of them is independently "dressed". The vanishing of the τ-function at the point (0 0 0 0 -4 0 4 0) now causes a first "dressed short pattern" of the schematic form $\{-4,\star,-2,-1,\star,1\}$ being at squared distance 16 of these four points but at squared distance at least 32 from all other points, while the schematic form $\{-1,\star,1,2,\star,4\}$ containing the second originally found "short pattern" is caused by the vanishing of the τ-function at the point (0 0 0 0 -4 4 0 0). Each of these two patterns contains one of the two, as yet unaccounted for, patterns of the "x-only" reduction of **VIII**.

Note that these two τs, the vanishing of which cause these "dressed short patterns" of equation **VIII**, are exactly those who cause the "short reconstructed pattern" and the "very short blow-up-free pattern" of equation **IX**. Indeed, as we have already mentioned, the "y-only" reduction of **VIII** is the same as for the case of **IX**. Thus the Ys are on the same trajectory and consequently the τs, which cause these patterns through their vanishing, must perforce be the same. Indeed the "long pattern" of the "y-only" reduction involving all six odd points from "-5" to "5" is in the "long pattern" of **VIII** and its three "short patterns" of schematic form $\{-1,1\}$ can be found, one at a time in each of the other three patterns of the latter equation.

Even for non-vanishing ψ_6 this equation is identical to the equation 4.5.1 of [23]. This means that Z_n and the A_n^i have the same expressions. But the identification we did of the equation 4.5.1 of [23] with the "y-only" reduction of **IX** contains γ and ϕ_2, which are meaningless for **VIII**. It is not necessary, however, to put to zero ψ_6 in **VIII** and γ, ϕ_2 in **IX** in order to see the equivalence. One can indeed show, with some care, how the "y-only" reduction of **VIII** and **IX** lead to the same auxiliary parameters Z and A^i.

Starting from the values for u_n, k_n, κ_n, z_n and ζ_n for **VIII** a naive calculation gives

$$
\begin{aligned}
Z_n &= u_n + u_{n+1} = t_{n+1} - \phi_3(n-1) + \phi_4(n) + \phi_4(n+1)\\
A_n^1 &= 2u_{n-1} + u_n + 2u_{n+1}\\
B_n^2 &= u_n + 2\psi_6(n)\\
B_n^3 &= u_n - 2\psi_6(n-1)\\
B_n^4 &= u_n - 2\psi_6(n+1)
\end{aligned}
$$

where with hindsight we introduced B^i instead of A^i.

In this case the singularities that start at step n with (B^2, B^3, B^4) respectively exit at step $n+1$ with (B^3, B^4, B^2), and not with the B^i of same index. As mentioned in [23] this is the reason for which the spurious period 6 appears. The period 6 is indeed spurious for 4.5.1 whether it is the "y-only" reduction of **IX** or **VIII**, since it is exactly the *same* equation. One can eliminate this period by redefining the A^i from the B^i in the following way. Taking (A^2, A^3, A^4) as (B^2, B^3, B^4) for $n = 3m$, (B^3, B^4, B^2) for $n = 3m+1$ and (B^4, B^2, B^3) for $n = 3m+2$ then singularities that start at step n with A^i exits at step $n+1$ by the same A^i for $i = 1, 2, 3, 4$ and all n.

With this choice the expression of the A^is are

$$
\begin{aligned}
A^1_n &= 2u_{n-1} + u_n + 2u_{n+1}\\
A^2_n &= u_n + 2(-1)^n\psi_6(0)\\
A^3_n &= u_n - 2(-1)^n\psi_6(-1)\\
A^4_n &= u_n - 2(-1)^n\psi_6(1)
\end{aligned}
$$

which has exactly the form 4.5.1 since the sum of the three period 2 functions is indeed zero.

There is still an equation skipping two points to land on the third one, alternating X and Y, because as for **IX** the vectors $\overrightarrow{X_mY_{m+1}}$ and $\overrightarrow{Y_mX_{m+2}}$ are of squared length 8. But here we find a situation that was never encountered before. While the vector $\overrightarrow{Y_mY_{m+3}}$ is still of squared length 32, as for y_0 of degree zero, the degree of y_3 in x_1 (or for that matter x_2), is still two, the vector $\overrightarrow{X_mX_{m+3}}$ is *not* of squared length 32 but rather 48. Indeed for x_0 of degree zero x_3 is of degree 3 in y_0 or equivalently y_1. So we expect this equation, which has not been identified earlier, to have 10 patterns, four starting with x and six starting with y, and also four ending in x and six ending in y, but in any combination.

The long pattern of **VIII** gives the schematic forms $\{-5, -2, 1, 4\}$, $\{-4, -1, 2, 5\}$ and $\{-3, 0, 3\}$ as before. The "medium pattern", not being "dressed", still gives two patterns, but which are now shorter: $\{-1, 2\}$ and $\{-2, 1\}$. Each "dressed short pattern" gives two skipping two points patterns,$\{-4, -1\}$ and $\{-2, 1\}$ for the first one and $\{-1, 2\}$ and $\{1, 4\}$ for the second one. This amounts to 9 patterns, only five starting in y and similarly only five ending in y. Again one expects one extra pattern $\{-3, 0, 3\}$ which has to come from an "action at distance three" pattern $\{-3, \star, \star, 0, \star, \star, 3\}$ of **VIII**. This pattern is indeed caused by the vanishing of the τ-function at the point $(-4\ 0\ 4\ 0\ 0\ 0\ 0\ 0)$, which, again, is also the one the vanishing of which causes a similar pattern for equation **IX**.

This system is unique among those presented in this paper inasmuch as the equation relating two y in terms of x has a right-hand side quadratic over linear just as (24b), while the equation relating two x in terms of y does *not* have a right-hand side of the same form as (24a) but rather cubic over quadratic.

As before, if we consider a purely secular equation, we simply recover equation (24ab), while keeping the ψ_6 term would lead to prohibitively long expressions. If, however one takes the limit $\psi_6 \to 0$ *separately* in the numerator and the denominator one finds

$$\frac{(y_{m+1}-x_{m+3}+9\beta^2)(y_{m+1}-x_m+9\beta^2)+36y_{m+1}\beta^2}{(y_{m+1}-x_{m+3}+9\beta^2)+(y_{m+1}-x_m+9\beta^2)} = \frac{3}{2}\frac{y_{m+1}^3+49y_{m+1}^2\beta^2-5y_{m+1}\beta^4-45\beta^6}{y_{m+1}^2+6y_{m+1}\beta^2-7\beta^4} \tag{25a}$$

$$\frac{(x_{m+3}-y_{m+1}+9\beta^2)(x_{m+3}-y_{m+4}+9\beta^2)+36x_{m+3}\beta^2}{(x_{m+3}-y_{m+1}+9\beta^2)+(x_{m+3}-y_{m+4}+9\beta^2)} = \frac{3}{2}\frac{x_{m+3}^2+52x_{m+3}\beta^2+64\beta^4}{x_{m+3}+8\beta^2} \tag{25b}$$

However the numerator and denominator of (25a) have a common factor $(y_{m+1}-\beta^2)$ and simplifying by it leads back to (24a).

A last remark is in order at this point. We have mentioned above that, for every one of the equations we have studied, except **VIII**, the degrees of the iterates are the same whether one keeps the periodicity or ignores it. This is true even if one also sets to zero the parameter γ in **IX** and both parameters γ and δ in (19) in subsection 4.4. And this makes sense when one remembers that all the parameters of the equation, the β in t_n, the constants γ and δ discussed above for **IX** and (19) and all the parameters that enter in the periodic functions are determined by the scalar product with appropriates NV around each X or Y of the vector $\overrightarrow{O'X}$ (or $\overrightarrow{O'Y}$) where O' is some point with eight arbitrary coordinates corresponding to the total of eight degrees of freedom of each equation. It follows from this remark that, for each specific equation, there are choices of these arbitrary coordinates so that all the parameters vanish, with the exception of β which keeps changing from point to point. This does not affect the geometry of the equations, nor the degrees of the iterates.

For equation **VIII**, however, something unique takes place. If we choose the coordinates of the point O' in such a way that the coefficients θ_1 and θ_2 of $\psi_6(n)$ vanish, the geometry of the X and Y is the same, the squared distances are the same, but the equation becomes indistinguishable from **IX** for $\gamma=0, \phi_2=0$, for which the degrees are the same as for its generic case. For some of the Xs the degrees of the x at those points are smaller than what is expected for the generic case of **VIII**. This means that the relationship between degrees and squared distance that we used as a "rule of thumb" is violated for these choices for O'. Some simplification occurs in the expressions of the values at these points, due to the vanishing of the parameters θ_1 and θ_2, that decreases the degree with respect to the value expected from the squared distance.

5 Conclusion

In this paper we set out to study in detail discrete Painlevé equations associated with the affine Weyl group $\mathrm{E}_8^{(1)}$. The existence of these equations was first shown in the work of Sakai [12], which provided the tools for the classification of discrete Painlevé equations. Sakai's work had three most important consequences.

First, it showed the existence of a third kind of discrete Painlevé equation on top of the two already known, additive and multiplicative, namely one where the independent variable as well as the parameters enter through the arguments of elliptic functions. Two of the present authors, in collaboration with Y. Ohta, presented in [14] the first tangible examples of elliptic discrete Painlevé equations. In subsequent studies we explained how one can construct elliptic (and multiplicative) systems once the additive one is known. The second important contribution of Sakai's work was that one could at last dispense with the link, through continuum limits, to continuous Painlevé equations in order to characterise the discrete ones. A discrete Painlevé equation is the mapping obtained by translations on the periodic repetition of a non-closed pattern on a lattice associated to one of the affine Weyl groups belonging to the degeneration cascade starting from $\mathrm{E}_8^{(1)}$. The third consequence, stemming directly from this definition, is that, contrary to the continuous case, the number of discrete Painlevé equations is infinite. In [16] two of the present authors presented explicit examples and argued that the construction method can be extended to any of the affine Weyl groups of the degeneration cascade of $\mathrm{E}_8^{(1)}$ (except for the four parameter-less $\mathrm{A}_1^{(1)}$).

This paper was motivated by the second point mentioned above, related to the definition of what is a discrete Painlevé equation. Since the latter is described as a translation on a trajectory on the lattice of an affine Weyl group, it appeared challenging, once a discrete Painlevé equation is given, to find the corresponding trajectory. And we focused on known examples of equations associated with $\mathrm{E}_8^{(1)}$. In [17] and [19] we derived all the additive equations which can be written in trihomographic form. Twelve discrete Painlevé equations of this type were identified, which could be regrouped into 6 distinct families as far as the trajectories were concerned. (Considering additive discrete Painlevé equations is in no way a restriction, the extension of our approach to multiplicative and elliptic being straightforward). As explained in the body of the paper, we introduced a heuristic approach which allowed us to construct in detail the trajectory for each of the studied cases. The main ingredient for this was the singularity structure of the equation, leading to the introduction the corresponding τ-functions in an appropriate way.

In many instances, given the form of the initial equation, it was possible to define also an evolution when one or more intermediate points are skipped. This resulted, in some cases, into equations that were already obtained, in particular in [23], while in other cases the resulting equations were never encountered before. In the latter cases only the autonomous form of the mapping was given. Obtaining the detailed forms of the ancillary parameters for these asymmetric equations will be the object of some future work of ours.

Speaking of future directions of our work, the phenomenon of a degree smaller than the one expected from the value of the squared distance is worth special study. Among all equations studied in the present paper, the case of **VIII** is unique. However more such cases do exist. In fact, in [24], there are quite a few equations that contain periodic functions which similarly to the case of **VIII** are essential to them. If all the parameters of these periodic functions are set to zero, the equation completely changes and becomes indistinguishable from a different one, with lower degrees of growth, where the same number of parameters were set to zero (but without any effect to the degree growth). Of course the zero value of these parameters could be obtained by an appropriate choice of the coordinates of the point O' that causes extra simplifications in the expressions of the values of the x and y, but the trajectories remain different, and thus the rule of thumb of equating the squared distance with the degree is violated. We plan to return to this fascinating subject in a future publication.

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References

[1] A. Ramani, B. Grammaticos and J. Hietarinta, *Discrete versions of the Painlevé equations*, Phys. Rev. Lett. 67 (1991) 1829.

[2] E. Laguerre, *Sur la réduction en fractions continues d'une fraction qui satisfait à une équation différentielle linéaire du premier ordre dont les coefficients sont rationnels*, J. Math. Pures Appl. 1 (1885) 135.

[3] J.A. Shohat, *A differential equation for orthogonal polynomials*, Duke Math. J. 5 (1939) 401.

[4] E. Brézin and V.A. Kazakov, *Exactly solvable field theories of closed strings*, Phys. Lett. 236B (1990) 144.

[5] M. Jimbo and T. Miwa, *Monodromy perserving deformation of linear ordinary differential equations with rational coefficients*, Physica 2D (1981) 407.

[6] V. Periwal and D. Shevitz, *Unitary-matrix models as exactly solvable string theories*, Phys. Rev. Lett. 64 (1990) 1326.

[7] F.W. Nijhoff and V. Papageorgiou, *Similarity reductions of integrable lattices and discrete analogues of the Painlevé II equation*, Phys. Lett. A 153 (1991) 337.

[8] T. Mase, R. Willox, B. Grammaticos and A. Ramani, *Deautonomisation by singularity confinement: an algebro-geometric justification*, Proc. Roy. Soc. A 471 (2015) 20140956.

[9] M. Jimbo and H. Sakai, *A q-analog of the sixth Painlevé equation*, Lett. Math. Phys. 38 (1996) 145.

[10] G.R.W. Quispel, J.A.G. Roberts and C.J. Thompson, *Integrable mappings and soliton equations II*, Physica D34 (1989) 183.

[11] B. Grammaticos and A. Ramani, *On a q-discrete analogue of the Painlev 'e VI equation*, Phys. Lett. A 257 (1999) 288.

[12] H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Commun. Math. Phys. 220 (2001) 165.

[13] B. Grammaticos, A. Ramani and V. Papageorgiou, *Do integrable mappings have the Painlevé, property?*, Phys. Rev. Lett. 67 (1991) 1825.

[14] Y. Ohta, A. Ramani and B. Grammaticos, *Elliptic discrete Painlevé equations*, J. Phys. A 35 (2002) L653.

[15] B. Grammaticos and A. Ramani, *Generating discrete Painlevé equations from affine Weyl groups*, Reg. Chao. Dyn. 10 (2005) 145.

[16] A. Ramani and B. Grammaticos, *The number of discrete Painlevé equations is infinite*, Phys. Lett. A 373 (2009) 3028.

[17] A. Ramani and B. Grammaticos, *Discrete Painlevé equations associated with the affine Weyl group E8*, J. Phys. A 48 (2015) 355204.

[18] B. Grammaticos and A. Ramani, *On a novel representation of discrete Painlevé equations*, J. Math. Phys. 56 (2015) 083507.

[19] B. Grammaticos and A. Ramani, *Elliptic and multiplicative discrete Painlevé equations from the affine Weyl group E8*, J. Phys. A 48 (2015) 16FT02.

[20] B. Grammaticos and A. Ramani, *From trihomographic to elliptic Painlevé equations*, J. Phys. A 49 (2016) 45LT02.

[21] Y. Ohta, A. Ramani and B. Grammaticos, *An affine Weyl group approach to the 8-parameter discrete Painlevé equation*, J. Phys. A 34 (2001) 10523.

[22] A. Ramani, Y. Ohta, J. Satsuma and B. Grammaticos, *Self-duality and Schlesinger chains for the asymmetric d-PII and q-PIII equations*, Comm. Math. Phys. 192 (1998) 67.

[23] A. Ramani and B. Grammaticos, *Singularity analysis for difference Painlevé equations associated with the affine Weyl group E8*, J. Phys. A 50 (2017) 055204.

[24] A. Ramani, R. Willox, J. Satsuma, *Discrete Painlevé equations from singularity patterns: the asymmetric trihomographic case*, J. Math. Phys. 61 (2020) 033503.