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Derivation of Painlevé type system with $D_4^{(1)}$ affine Weyl group symmetry in a self-similarity limit

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Abstract

We show how the zero-curvature equations based on a loop algebra of D_4 with a principal gradation reduce via self-similarity limit to a polynomial Hamiltonian system of coupled Painlevé III models with four canonical variables and $D_4^{(1)}$ affine Weyl group symmetry.

1 Zero curvature derivation of the t_3 flow of the $D_4^{(1)}$ hierarchy

Our starting point will be an integrable hierarchy with commuting flows defined via the zero-curvature formalism based on a loop algebra of $\mathcal{G} = D_4$ endowed with a principal gradation. We will apply a conventional self-similarity limit on the t_3 flow of the hierarchy and show how through several explicit changes of variables the reduced model can be cast into the polynomial Hamiltonian system of four canonical variables invariant under $D_4^{(1)}$ affine Weyl group symmetry.

An unusual aspect of the t_3 flow of the D_4 hierarchy is that it is parametrized by two independent variables, $\varepsilon_1, \varepsilon_2$, reflecting the fact that a kernel of $E^{(1)}$, a central object of zero-curvature equations, turns out to have a two-dimensional kernel on the level of grade three. The presence of these parameters enriches the symmetry structure of the two-dimensional hierarchy of zero-curvature equations based on $D_4^{(1)}$ affine Weyl algebra and survives the self-similarity limit as shown in equations (19). Considering these equations with dependence on only one of these parameters or their linear combination we obtain, up to few normalization adjustments, and after several changes of variables, the model of reference [1], where it was proposed as a pair of coupled Painlevé III equations that form a Hamiltonian system invariant under $D_4^{(1)}$ affine Weyl group symmetry. To the best of our knowledge, an explicit derivation of this particular model as a reduction of two-dimensional hierarchy of zero-curvature equations based on $D_4^{(1)}$ affine Weyl algebra, has not been done previously.

To make the presentation self-contained we provide all the necessary algebraic background information in Appendix A and main expressions of the zero-curvature calculation in Appendix B.

For other derivations of integrable hierarchy based on $D_4^{(1)}$ affine algebra the reader can consult [2] and the references therein. There also exist in the literature other approaches to applying similarity reduction to integrable hierarchy of type $D_4^{(1)}$ [3] but the focus there was on deriving the sixth Painlevé equation.

As a starting point we consider the zero-curvature equation for the third flow in the setting of the affine algebra $D_4^{(1)}$:

$$[\partial_x + E^{(1)} + A_0\,,\, \partial_{t_3} + D^{(0)} + D^{(1)} + D^{(2)} + D^{(3)}] = 0 \tag{1}$$

with $D^{(i)} \in \mathcal{G}_i$ and $A_0 = \sum_{i=1}^{4} \phi_i H_i$ with $E^{(1)}, \mathcal{G}_i, H_i$ defined in Appendix A.

The highest grade-four component of equation (1), $[E^{(1)}, D^{(3)}] = 0$, is solved by

$$D^{(3)} = \varepsilon_1 V_1 + \varepsilon_2 V_2\,, \tag{2}$$

where V_1, V_2 are two matrices defined in (56) that span a basis for the two-dimensional kernel of $E^{(1)}$ in $\mathcal{G}^{(3)}$. The standard zero-curvature technique allows recursive derivation of the lower grade matrices $D^{(i)}, i = 0, 1, 2$ from appropriate grade projections of equation (1). Grade 2 element given by:

$$D^{(2)} = M_1 E^{(0)}_{e_1-e_3} + M_2 E^{(0)}_{e_2+e_4} + M_3 E^{(0)}_{e_2-e_4} + M_4 E^{(1)}_{-e_1-e_3}\,, \tag{3}$$

is solved for from the grade 3 equation

$$[E^{(1)}, D^{(2)}] + [A_0, D^{(3)}] = 0\,, \tag{4}$$

from which one obtains explicitly coefficients $M_i, i = 1, 2, 3, 4$ of $D^{(2)}$ given in equation (62).

The grade 2 component of the zero curvature equations (1) is

$$[E^{(1)}, D^{(1)}] + [A_0, D^{(2)}] + \partial_x D^{(2)} = 0\,, \tag{5}$$

where

$$D^{(1)} = d_1 E_1 + d_2 E_2 + d_3 E_3 + d_4 E_4 + d_5 E_5\,, \tag{6}$$

where we employed the basis elements $E_i, i = 1, \ldots, 5$ given in expressions (61). Equation (5) yields explicit expressions for $d_i, i = 2, \ldots, 5$ given in equation (63).

The grade one component of (1) reads as

$$\partial_x D^{(1)} + [E^{(1)}, D^{(0)}] + [A_0, D^{(1)}] = 0\,, \tag{7}$$

with $D^{(0)} = \sum_i v_i H_i$. The advantage of using the basis (61) is that

$$\begin{aligned}[E^{(1)}, D^{(0)}] &= (v_2 - v_1)E^{(0)}_{e_1-e_2} + (v_3 - v_2)E^{(0)}_{e_2-e_3} + (v_4 - v_3)E^{(0)}_{e_3-e_4} \\ &- (v_3 + v_4)E^{(0)}_{e_3+e_4} + (v_1 + v_2)E^{(1)}_{-e_1-e_2} \\ &= v_1 E_2 + v_2 E_3 + v_3 E_4 + v_4 E_5 \,,\end{aligned} \tag{8}$$

where $E_i, i = 1, \dots E_5$ are the basis elements given in expressions (61).

Solving the grade one equation (7) in direction of E_1 yields

$$\partial_x d_1 = -\frac{1}{3}\phi_2 d_4 - \frac{1}{3}\phi_3 d_3 + \frac{1}{3}\phi_1 d_2 + \frac{1}{3}\phi_4 d_5 + \frac{2}{3}\phi_3 d_4 + \frac{2}{3}\phi_2 d_3, \tag{9}$$

plugging expressions (63) and taking out the total derivative gives

$$d_1 = \frac{\varepsilon_1 + \varepsilon_2}{12}\left(-\phi_1^2 - \phi_4^2 + \phi_2^2 + \phi_3^2 + 2\phi_2\phi_3\right) + \frac{1}{6}(\varepsilon_1 - \varepsilon_2)\phi_1\phi_4 \,. \tag{10}$$

Solving the grade one equation (7) in directions of $E_i, i = 2, 3, 4, 5$ yields

$$v_i = -\partial_x d_{i+1} - C_{i+1}, \qquad i = 1, 2, 3, 4 \,, \tag{11}$$

with $C_i, i = 2, 3, 3, 5$ given in (64) and with d_1 in given in expression (10) while expressions $d_i, i = 2, .., 5$ are given in (63).

Inserting these values of d_i and C_i into (11) we obtain v_i given in expression (65).

The grade zero component of (1) is

$$\partial_x D^{(0)} - \partial_{t_3} A_0 + [A_0, D^{(0)}] = 0 \,. \tag{12}$$

Since $[A_0, D^{(0)}] = 0$ the equation (12) reduces to

$$\partial_{t_3} A_0 = \sum_i \partial_{t_3}\phi_i H_i = \partial_x D^{(0)} = \sum_i \partial_x v_i H_i \,,$$

that in components gives the t_3 flows written as

$$\partial_{t_3}\phi_i = \partial_x v_i, \; i = 1, 2, 3, 4 \,. \tag{13}$$

When on the right hand side we insert values of v_i from equation (65) we find the t_3-flow explicitly written in equation (66) with their symmetries listed below in equations (16).

With definitions

$$u = \phi_1 + \phi_4, \; v = \phi_1 - \phi_4, \; f = \phi_2 + \phi_3, \; g = \phi_2 - \phi_3 \,. \tag{14}$$

Equations (66) can be conveniently rewritten in this notation as

$$
\begin{aligned}
\partial_{t_3} u &= \varepsilon_1 \partial_x (\frac{1}{4} u v^2 - \frac{1}{4} u f^2 + \frac{1}{2} v \partial_x f - \frac{1}{2} f \partial_x v) \\
&\quad + \varepsilon_2 \partial_x (\frac{1}{4} u v^2 - \frac{1}{4} u g^2 + v \partial_x f + \frac{1}{2} u \partial_x g + \frac{1}{2} f \partial_x v + \partial_x^2 u)\,, \\
\partial_{t_3} v &= \varepsilon_1 \partial_x (\frac{1}{4} v u^2 - \frac{1}{4} v g^2 + u \partial_x f + \frac{1}{2} v \partial_x g + \frac{1}{2} f \partial_x u + \partial_x^2 v) \\
&\quad + \varepsilon_2 \partial_x (\frac{1}{4} v u^2 - \frac{1}{4} v f^2 + \frac{1}{2} u \partial_x f - \frac{1}{2} f \partial_x u)\,, \\
\partial_{t_3} f &= \varepsilon_1 \partial_x (\frac{1}{4} f g^2 - \frac{1}{4} f u^2 - u \partial_x v - \frac{1}{2} v \partial_x u - \frac{1}{2} f \partial_x g - \partial_x^2 f) \\
&\quad + \varepsilon_2 \partial_x (-\frac{1}{4} f v^2 + \frac{1}{4} f g^2 - v \partial_x u - \frac{1}{2} u \partial_x v - \frac{1}{2} f \partial_x g - \partial_x^2 f)\,, \\
\partial_{t_3} g &= \varepsilon_1 \partial_x (\frac{1}{4} g f^2 - \frac{1}{4} g v^2 + \frac{1}{2} f \partial_x f - \frac{1}{2} v \partial_x v) \\
&\quad + \varepsilon_2 \partial_x (\frac{1}{4} g f^2 - \frac{1}{4} g u^2 + \frac{1}{2} f \partial_x f - \frac{1}{2} u \partial_x u)\,.
\end{aligned}
\tag{15}
$$

These equations are invariant under:

$$
\begin{aligned}
&F_1 : u \leftrightarrow -v,\ \varepsilon_1 \to \varepsilon_2,\ g \to g, f \to f\,, \\
&F_4 : u \leftrightarrow v,\ \varepsilon_1 \to \varepsilon_2,\ g \to g, f \to f\,.
\end{aligned}
\tag{16}
$$

In addition for $\varepsilon_2 = 0$ these equations are invariant under:

$$
F_2 : f \leftrightarrow v,\ \varepsilon_1 \to -\varepsilon_1,\ g \to g, u \to u \tag{17}
$$

while for $\varepsilon_1 = 0$ they are invariant under:

$$
F_3 : f \leftrightarrow u,\ \varepsilon_2 \to -\varepsilon_2,\ g \to g, v \to v\,.
$$

Obviously, for one of the parameters ε_1 or ε_2 being zero the remaining parameter can be absorbed by redefining t_3.

The above operations satisfy $F_2^2 = I = F_3^2$ and

$$
F_1 F_4 = F_4 F_1 = TG = GT : u \to -u,\ v \to -v,\ g \to g, f \to f\,,
$$

where T and G are automorphisms :

$$
T : u \leftrightarrow u,\ v \to -v,\ g \to g, f \to -f\,,
$$

$$
G : u \leftrightarrow -u,\ v \to v,\ g \to g, f \to -f\,,
$$

with $F_4 T = T F_1$. All the above automorphisms of equations (15) are “mirror automorphisms”, meaning that they square to one.

2 Self-similarity reduction for the t_3 flow

We will look at the self-similar reduction of equation (15) with

$$\phi(x,t) = t^{-\frac{1}{3}}\varphi(z),\ z = \frac{x}{t^{\frac{1}{3}}} = xt^{-1/3}\,, \tag{18}$$

and correspondingly

$$\frac{d}{dx} = \frac{d}{dz}\frac{dz}{dx} = t^{-1/3}\frac{d}{dz}, \quad \frac{d}{dt} = \frac{d}{dz}\frac{dz}{dt} = -\frac{1}{3}t^{-1}\frac{d}{dz}z\,,$$

such that the KdV type of expression :

$$\frac{d}{dt}\phi(x,t) + \beta_1\frac{d}{dx}(\phi(x,t)\frac{d}{dx}\phi(x,t)) + \beta_2\frac{d}{dx}\phi^3(x,t) + \beta_3\frac{d^3}{dx^3}\phi(x,t) = 0\,,$$

is transformed to an equation fully expressible in terms of functions of z:

$$\frac{d}{dz}(z\varphi(z)) - 3\beta_1\frac{d}{dz}(\varphi(z)\frac{d}{dz}\varphi(z)) - 3\beta_2\frac{d}{dz}\varphi^3(z) - 3\beta_3\frac{d^3}{dz^3}\varphi(z) = 0\,.$$

Following these rules we are now able to take self-similarity limit of equations (15) to obtain:

$$\begin{aligned}
-\frac{z}{3}u + C_1 &= \varepsilon_1(\frac{1}{4}uv^2 - \frac{1}{4}uf^2 + \frac{1}{2}v\partial_z f - \frac{1}{2}f\partial_z v)\\
&\quad + \varepsilon_2(\frac{1}{4}uv^2 - \frac{1}{4}ug^2 + v\partial_z f + \frac{1}{2}u\partial_z g + \frac{1}{2}f\partial_z v + \partial_z^2 u)\\
-\frac{z}{3}v + K_2 &= \varepsilon_1(\frac{1}{4}vu^2 - \frac{1}{4}vg^2 + u\partial_z f + \frac{1}{2}v\partial_z g + \frac{1}{2}f\partial_z u + \partial_z^2 v)\\
&\quad + \varepsilon_2(\frac{1}{4}vu^2 - \frac{1}{4}vf^2 + \frac{1}{2}u\partial_z f - \frac{1}{2}f\partial_z u)\\
-\frac{z}{3}f + K_1 &= \varepsilon_1(\frac{1}{4}fg^2 - \frac{1}{4}fu^2 - u\partial_z v - \frac{1}{2}v\partial_z u - \frac{1}{2}f\partial_z g - \partial_z^2 f)\\
&\quad + \varepsilon_2(-\frac{1}{4}fv^2 + \frac{1}{4}fg^2 - v\partial_z u - \frac{1}{2}u\partial_z v - \frac{1}{2}f\partial_z g - \partial_z^2 f)\\
-\frac{z}{3}g + C_2 &= \varepsilon_1(\frac{1}{4}gf^2 - \frac{1}{4}gv^2 + \frac{1}{2}f\partial_z f - \frac{1}{2}v\partial_z v)\\
&\quad + \varepsilon_2(\frac{1}{4}gf^2 - \frac{1}{4}gu^2 + \frac{1}{2}f\partial_z f - \frac{1}{2}u\partial_z u)\,,
\end{aligned} \tag{19}$$

where $C_i, K_i, i = 1, 2$ are integration constants.

Here we comment that it is enough to chose any direction in $\varepsilon_1 - \varepsilon_2$ plane because of a presence of previously noticed automorphisms that establish an equivalence (by substitution) between any of the one-parameter ε models in a self-similarity limit.

For example, we notice a symmetry between $\varepsilon_2 = 0$ limit of equation (19) and $\varepsilon_1 = 0$ limit of equation (19) via

$$\begin{aligned}
\varepsilon_2 \longleftrightarrow\ \varepsilon_1, \quad (v, K_2) \longleftrightarrow\ (u, C_1), \quad (u, C_1) \longleftrightarrow\ (v, K_2)\\
(f, K_1) \longleftrightarrow\ (f, K_1), \quad (g, C_2) \longleftrightarrow\ (g, C_2)
\end{aligned} \tag{20}$$

These substitutions follow from to symmetries F_1, F_4 from equation (16). Note that equations (19) with arbitrary $\varepsilon_1, \varepsilon_2$ remain invariant under transformations (20) that interchange u and v.

We further point out that symmetry extends to any direction in the $\varepsilon_1 - \varepsilon_2$ plane. For example we can transform the system of equations (19) with $\varepsilon_1 = 0$ into the system of equations (19) with $\varepsilon_1 + \varepsilon_2 = 0$ with only the parameter ε such that $\varepsilon = \varepsilon_1 - \varepsilon_2$ as follows

$$\begin{aligned} &\varepsilon_1 = 0 \longleftrightarrow \ \varepsilon_1 + \varepsilon_2 = 0, \quad \varepsilon_2 \longleftrightarrow \ \frac{\varepsilon}{2}, \\ &(v, K_2) \longleftrightarrow \ (f, K_2), \quad (f, K_1) \longleftrightarrow \ (u, K_1) \\ &(u, C_1) \longleftrightarrow \ (v, C_1), \quad (g, C_2) \longleftrightarrow \ (g, C_2)\,. \end{aligned} \tag{21}$$

Thus for simplicity we will from now on only consider the self-similarity limit for the case of $\varepsilon_2 = 0$ rewritten as:

$$\begin{aligned} -\frac{z}{3}v_+ + K_+ &= \varepsilon(\frac{1}{4}v_-(u^2 - g^2) - u\partial_z v_- + \frac{1}{2}v_-\partial_z g - \frac{1}{2}v_-\partial_z u + \partial_z^2 v_-) \\ -\frac{z}{3}v_- + K_- &= \varepsilon(\frac{1}{4}v_+(u^2 - g^2) + u\partial_z v_+ + \frac{1}{2}v_+\partial_z g + \frac{1}{2}v_+\partial_z u + \partial_z^2 v_+) \\ -\frac{z}{3}u + C_1 &= \frac{\varepsilon}{4}(uv_+v_- - (v_+\partial_z v_- - v_-\partial_z v_+)) \\ -\frac{z}{3}g + C_2 &= \frac{\varepsilon}{4}(-gv_+v_- - \partial_z(v_+v_-))\,, \end{aligned} \tag{22}$$

where

$$v_\pm = v \pm f, \ \ K_\pm = K_1 \pm K_2, \ \ \varepsilon = \varepsilon_1\,.$$

First, we note that equations (22) can be made independent of ε through the substitution $\mathcal{S}$:

$$\mathcal{S} : v_\pm \to (\varepsilon)^{-1/3} v_\pm, \ \ Ku \to (\varepsilon)^{-1/3} u, \ \ Kg \to (\varepsilon)^{-1/3} g, \ \ z \to (\varepsilon)^{1/3} z\,. \tag{23}$$

It is instructive to leave the equations (22) in the current form as the change of variables we will perform to arrive at the Hamiltonian formalism will lead anyway to canonical coordinates that are invariant under the above transformation $\mathcal{S}$.

The equations (22) are explicitly invariant under F_2:

$$F_2 : v_\pm \leftrightarrow \pm v_\pm, \ \varepsilon \to -\varepsilon, \ g \to g, u \to u$$

From the last two equations of (22) we derive

$$\begin{aligned} \frac{v_-'}{v_-} &= \frac{1}{2}(u - g) - \frac{2}{\varepsilon v_+ v_-}\big[(C_2 - \frac{zg}{3}) + (C_1 - \frac{zu}{3})\big] \\ \frac{v_+'}{v_+} &= -\frac{1}{2}(u + g) + \frac{2}{\varepsilon v_+ v_-}\big[(C_1 - \frac{zu}{3}) - (C_2 - \frac{zg}{3})\big]\,. \end{aligned} \tag{24}$$

The first order derivatives for u, g are:

$$\begin{aligned} (zg)_z &= -\frac{z}{4}(v_-^2 + v_+^2) + \frac{6}{\varepsilon\, v_+\, v_-}\big[(C_2 - \frac{z}{3}g)^2 - (C_1 - \frac{z}{3}u)^2\big] + \frac{3}{4}(K_+\, v_+ + K_-\, v_-) \\ (zu)_z &= \frac{z}{4}(v_-^2 - v_+^2) + \frac{3}{4}(K_+\, v_+ - K_-\, v_-)\,. \end{aligned} \tag{25}$$

Introducing for convenience

$$G = zg, \quad U = zu\,,$$

we can rewrite equations (24),(25) as

$$\begin{aligned}
v'_- &= \frac{v_-}{2z}(U-G) - \frac{2}{\varepsilon v_+}\big[(C_2 - \frac{G}{3}) + (C_1 - \frac{U}{3})\big]\,,\\
v'_+ &= -\frac{v_+}{2z}(U+G) - \frac{2}{\varepsilon v_-}\big[(C_2 - \frac{G}{3}) - (C_1 - \frac{U}{3})\big]\,,\\
(G)_z &= -\frac{z}{4}(v_-^2 + v_+^2) + \frac{6}{\varepsilon\, v_+\, v_-}\big[(C_2 - \frac{1}{3}G)^2 - (C_1 - \frac{1}{3}U)^2\big] + \frac{3}{4}(K_+\, v_+ + K_-\, v_-)\,,\\
(U)_z &= \frac{z}{4}(v_-^2 - v_+^2) + \frac{3}{4}(K_+\, v_+ - K_-\, v_-)\,.
\end{aligned} \tag{26}$$

There is one further change of variables needed to end up with equations that are manifestly Hamilton equations, namely:

$$\bar{G} = \frac{1}{3}G - C_2, \quad \bar{U} = \frac{1}{3}U - C_1\,.$$

Equations for $\bar{G}, \bar{U}$ variables are:

$$\begin{aligned}
\left(\bar{G} + \bar{U}\right)_z &= -\frac{z}{2\cdot 3}v_+^2 + \frac{2}{\varepsilon\, v_+\, v_-}\left(\bar{G} + \bar{U}\right)\left(\bar{G} - \bar{U}\right) + \frac{1}{2}K_+\, v_+\,,\\
\left(\bar{G} - \bar{U}\right)_z &= -\frac{z}{2\cdot 3}v_-^2 + \frac{2}{\varepsilon\, v_+\, v_-}\left(\bar{G} + \bar{U}\right)\left(\bar{G} - \bar{U}\right) + \frac{1}{2}K_-\, v_-\,.
\end{aligned} \tag{27}$$

To end up with the polynomial Hamilton equations we further introduce :

$$F_+ = \frac{\bar{G} + \bar{U}}{v_+}, \quad F_- = \frac{\bar{G} - \bar{U}}{v_-}.$$

Using this notation the first two of equations (26) can be rewritten as:

$$v'_\pm = -\frac{3}{2z}v_\pm^2 F_\pm - \frac{3}{2z}v_\pm(C_2 \pm C_1) + \frac{2}{\varepsilon}F_\mp\,. \tag{28}$$

From equations (27) and (28) we obtain

$$(F_\pm)' = -\frac{z}{3\cdot 2}v_\pm + \frac{1}{2}K_\pm + \frac{3}{2z}v_\pm F_\pm^2 + \frac{3}{2z}(C_2 \pm C_1)F_\pm\,. \tag{29}$$

Define now the Hamiltonian :

$$\begin{aligned}
H &= \left(\frac{3}{4z}v_+^2 F_+^2 + \frac{1}{2}K_+ v_+ - \frac{z}{3\cdot 4}v_+^2 + \frac{3}{2z}(C_1 + C_2)F_+ v_+\right) - \frac{2}{\varepsilon}F_+ F_-\\
&\quad + \left(\frac{3}{4z}v_-^2 F_-^2 + \frac{1}{2}K_- v_- - \frac{z}{3\cdot 4}v_-^2 + \frac{3}{2z}(C_1 - C_2)F_- v_-\right)\,.
\end{aligned} \tag{30}$$

which is polynomial in all variables such that it reproduces equations (28)-(29) through

$$(F_\pm)' = \frac{\delta}{\delta v_\pm}H, \quad (v_\pm)' = -\frac{\delta}{\delta F_\pm}H\,. \tag{31}$$

Note that the "plus" and "minus" parts of H in (30) are connected by only one term $-\frac{2}{\varepsilon}F_+F_-$.

The transformation

$$F_+ \to F_+ + \frac{a}{v_+}, \;\; v_+ \to v_+$$

or

$$v_+ \to v_+ + \frac{a}{F_+}, \;\; F_+ \to F_+$$

leaves only the first part of Hamiltonian (30) invariant (up to a constant).

We will now attempt to cast equations (28)-(29) in a form of equations that are manifestly $D_4^{(1)}$ invariant [1].

First, we apply the redefinition

$$F_\pm \to F_\pm + \frac{z}{3}, \quad v_\pm \to v_\pm \,,$$

to obtain

$$\begin{aligned}(F_+)' &= \frac{3}{2z}v_+F_+^2 + v_+F_+ + \frac{3}{2z}(C_1+C_2)F_+ + \left(\frac{1}{2}K_+ + \frac{1}{2}(C_1+C_2) - \frac{1}{3}\right), \\ v_+' &= -\frac{3}{2z}v_+^2F_+ - \frac{1}{2}v_+^2 - \frac{3}{2z}v_+(C_1+C_2) + \frac{2}{\varepsilon}(F_- + \frac{z}{3}),\end{aligned} \tag{32}$$

and

$$\begin{aligned}(F_-)' &= \frac{3}{2z}v_-F_-^2 + v_-F_- - \frac{3}{2z}(C_1-C_2)F_- + \left(\frac{1}{2}K_- - \frac{1}{2}(C_1-C_2) - \frac{1}{3}\right), \\ v_-' &= -\frac{3}{2z}v_-^2F_- - \frac{1}{2}v_-^2 - \frac{3}{2z}v_-(C_2-C_1) + \frac{2}{\varepsilon}(F_+ + \frac{z}{3}).\end{aligned} \tag{33}$$

We now further substitute

$$F_+ \to zF_p,\; F_- \to zF_m,\; v_+ \to \frac{1}{z}v_p,\; v_- \to \frac{1}{z}v_m, \tag{34}$$

to obtain for F_p, v_p equations :

$$\begin{aligned}(F_p)' &= \frac{1}{z}\Big[\frac{3}{2}v_pF_p^2 + v_pF_p + \frac{3}{2}(C_1+C_2-\frac{2}{3})F_p + \left(\frac{1}{2}K_+ + \frac{1}{2}(C_1+C_2) - \frac{1}{3}\right)\Big] \\ v_p' &= \frac{1}{z}\Big[-\frac{3}{2}v_p^2F_p - \frac{1}{2}v_p^2 - \frac{3}{2}v_p(C_1+C_2-\frac{2}{3})\Big] + \frac{2z^2}{\varepsilon}(F_m + \frac{1}{3}).\end{aligned} \tag{35}$$

Introducing $\alpha_1+\alpha_2 = (C_1+C_2-\frac{2}{3})$ and $\alpha_2 = \frac{1}{2}K_+ + \frac{1}{2}(C_1+C_2) - \frac{1}{3}$ we can rewrite the above equations as

$$\begin{aligned}(F_p)' &= \frac{1}{z}\Big[\frac{3}{2}v_pF_p^2 + v_pF_p + \frac{3}{2}(\alpha_1+\alpha_2)F_p + \alpha_2\Big] \\ v_p' &= \frac{1}{z}\Big[-\frac{3}{2}v_p^2F_p - \frac{1}{2}v_p^2 - \frac{3}{2}(\alpha_1+\alpha_2)v_p\Big] + \frac{2z^2}{\varepsilon}(F_m + \frac{1}{3}).\end{aligned} \tag{36}$$

For the "−" sector we obtain

$$\begin{aligned}(F_m)' &= \frac{1}{z}\Big[\frac{3}{2}v_m F_m^2 + v_m F_m - \frac{3}{2}(C_1 - C_2 + \frac{2}{3})F_m + \Big(\frac{1}{2}K_- - \frac{1}{2}(C_1 - C_2) - \frac{1}{3}\Big)\Big] \\ v_m' &= \frac{1}{z}\Big[-\frac{3}{2}v_m^2 F_- - \frac{1}{2}v_m^2 - \frac{3}{2}v_m(C_2 - C_1 - \frac{2}{3})\Big] + \frac{2z^2}{\varepsilon}(F_p + \frac{1}{3})\,.\end{aligned} \tag{37}$$

Introducing $\alpha_3 + \alpha_4 = -(C_1 - C_2 + \frac{2}{3})$ and $\alpha_4 = \frac{1}{2}K_- - \frac{1}{2}(C_1 - C_2) - \frac{1}{3}$ we can compactly rewrite the above equations as

$$\begin{aligned}(F_m)' &= \frac{1}{z}\Big[\frac{3}{2}v_m F_m^2 + v_m F_m + \frac{3}{2}(\alpha_3 + \alpha_4)F_m + \alpha_4\Big] \\ v_m' &= \frac{1}{z}\Big[-\frac{3}{2}v_m^2 F_m - \frac{1}{2}v_m^2 - \frac{3}{2}(\alpha_3 + \alpha_4)v_m\Big] + \frac{2z^2}{\varepsilon}(F_p + \frac{1}{3})\,.\end{aligned} \tag{38}$$

Equations (36) and (38) can be obtained from the Hamiltonian:

$$\begin{aligned}H &= \frac{1}{z}\left(\frac{3}{4}v_p^2 F_p^2 + \frac{1}{2}v_p^2 F_p + \frac{3}{2}(\alpha_1 + \alpha_2)F_p v_p + \alpha_2 v_p\right) - \frac{2z^2}{\varepsilon}(F_m + \frac{1}{3})(F_p + \frac{1}{3}) \\ &+ \frac{1}{z}\left(\frac{3}{4}v_m^2 F_m^2 + \frac{1}{2}v_m^2 F_m + \frac{3}{2}(\alpha_3 + \alpha_4)F_m v_m + \alpha_4 v_m\right)\,.\end{aligned} \tag{39}$$

through

$$(F_i)' = \frac{\delta}{\delta v_i}H, \quad (v_i)' = -\frac{\delta}{\delta F_i}H, \quad i = p, m\,. \tag{40}$$

The author of [1] has proposed such system as two coupled Painlevé III equations involving four variables and derived by symmetry consideration as a system that admits affine Weyl group symmetry of type $D_4^{(1)}$.

Comparing equations (36) and (38) we notice presence of π_0 automorphism :

$$\pi_0 : \quad v_p, F_p \leftrightarrow v_m, F_m, \quad \alpha_2 \leftrightarrow \alpha_4,\ \alpha_1 \leftrightarrow \alpha_3\,,$$

that transforms equation (36) into (38) and vice-versa.

In addition we introduce a variable α_0 defined by the condition $2\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 =$ const [1]. The constant used to define α_0 will be fixed below by a symmetry transformation s_0, that mixes the "+/−" sectors to be defined below. In [1] that constant is set to 1 consistently with Sasano's normalization (different from ours).

Furthermore we also find the following Bäcklund transformation s_2 :

$$s_2 :\ v_p \to v_p + \frac{2\alpha_2}{F_p},\ F_p \to F_p,\ \alpha_2 \to -\alpha_2\,, \tag{41}$$

that keeps equations (36) invariant. The consequence of $s_2(\alpha_2) = -\alpha_2$ is that $s_2(\alpha_0) = \alpha_0 + \alpha_2$ just to keep the condition $2\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 =$ const unchanged.

Similarly the following Bäcklund transformation :

$$s_4 :\ v_m \to v_m + \frac{2\alpha_4}{F_m},\ F_m \to F_m,\ \alpha_4 \to -\alpha_4\,, \tag{42}$$

will keep equations (38) invariant.

Note that $s_2^2 = s_4^2 = 1, \quad s_2 s_4 = s_4 s_2$ and $\pi_0 s_2 \pi_0 = s_4$.

Furthermore, inspired by the automorphism (17), we define the two automorphisms:

$$\begin{aligned} &\pi_1 : \; v_p \to -v_p, \; F_p \to -\frac{2}{3} - F_p, \; \varepsilon \to -\varepsilon, \; \alpha_1 \to \alpha_2 \to \alpha_1 \\ &\pi_3 : \; v_m \to -v_m, \; F_m \to -\frac{2}{3} - F_m, \; \varepsilon \to -\varepsilon, \; \alpha_3 \to \alpha_4 \to \alpha_3 \,, \end{aligned} \tag{43}$$

that both keep equations (36) - (38) invariant and satisfy

$$\pi_1^2 = \pi_3^2 = 1, \quad \pi_0 \pi_1 \pi_0 = \pi_3, \quad \pi_0 \pi_3 \pi_0 = \pi_1 \,.$$

Coincidently, all the canonical coordinates v_p, v_m, F_p, F_m have been defined in such a way that they are invariant under transformation $\mathcal{S}$ defined in relation (23), while the substitution $z \to (\varepsilon)^{1/3} z$ allows to eliminate ε completely from equations (36) - (38). With ε being replaced by 1, one can alternatively define the automorphisms π_1, π_3 involving a change of the sign of $z \to -z$ instead of $\varepsilon \to -\varepsilon$, as it was done in [1].

The other two Bäcklund transformations are (s_0, s_3 in notation of [1]) but here relabeled as :

$$\begin{aligned} &s_1 : \; v_p \to v_p + \frac{2\alpha_1}{F_p + \frac{2}{3}}, \; F_p \to F_p, \;\; \alpha_1 \to -\alpha_1, \; \alpha_2 \to \alpha_2, \; \alpha_0 \to \alpha_0 + \alpha_1 \\ &s_3 : \; v_m \to v_m + \frac{2\alpha_3}{F_m + \frac{2}{3}}, \; F_m \to F_m, \;\; \alpha_3 \to -\alpha_3, \; \alpha_4 \to \alpha_4, \; \alpha_0 \to \alpha_0 + \alpha_3 \,. \end{aligned} \tag{44}$$

They both square to one : $s_1^2 = s_3^2 = 1$. Also the Bäcklund transformations satisfy :

$$\begin{aligned} \pi_i s_i \pi_i &= s_{i+1}, \quad \pi_i s_{i+1} \pi_i = s_i, \quad i = 1, 3, \\ \pi_i s_{i\pm 2} \pi_i &= s_{i\pm 2}, \quad \pi_i s_{i\pm 3} \pi_i = s_{i\pm 3}, \quad +/- \quad \text{for} \quad i = 1/3 \\ \pi_0 s_i \pi_0 &= s_{i+2}, \quad i = 1, 2 \end{aligned}$$

Finally we need to prove invariance under s_0 that mixes the $+/-$ sectors. When this Bäcklund transformation is defined as

$$\begin{aligned} &s_0(F_p) = F_p - \frac{2\alpha_0 \, v_m}{v_p v_m - \frac{4}{3\varepsilon} z^3}, \;\; s_0(v_p) = v_p, \; s_0(\alpha_1) = \alpha_1 + \alpha_0, \; s_0(\alpha_2) = \alpha_2 + \alpha_0 \\ &s_0(F_m) = F_m - \frac{2\alpha_0 \, v_p}{v_p v_m - \frac{4}{3\varepsilon} z^3}, \;\; s_0(v_m) = v_m, \; s_0(\alpha_3) = \alpha_3 + \alpha_0, \; s_0(\alpha_4) = \alpha_4 + \alpha_0 \\ &s_0(\alpha_0) = -\alpha_0 \,, \end{aligned} \tag{45}$$

the equations (36) and (38) are invariant if the condition,

$$2\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -2 \,,$$

holds. As remarked before our normalization is different from the one used by Sasano [1] and the differences also include different powers of z in equations (45) and in the Hamiltonian (39).

Note that $s_0^2 = 1$ because $\bar{\alpha}_0 = -\alpha_0$ and $\pi_0 s_0 \pi_0 = s_0$, $\pi_i s_0 \pi_i$, $i = 1, 3$. It is easy to verify that the $D_4^{(1)}$ s_i, $1, 2, 3, 4$ Bäcklund transformations satisfy :

$$
\begin{aligned}
s_i^2 &= 1, \quad i = 1, 2, 3, 4\,, \\
s_i s_j &= s_j s_i, \quad i, j = 1, 2, 3, 4\,, \\
s_i s_0 s_i &= s_0 s_i s_0, \quad i = 1, 2, 3, 4\,,
\end{aligned}
$$

where the last two identities are equivalent to the standard $D_4^{(1)}$ relations $(s_i s_j)^2 = 1, (s_0 s_i)^3 = 1$. This is in contrast to the $A_l^{(1)}$ affine Weyl symmetry group multiplications for which it holds that $(s_i s_{i\pm 1})^3 = 1$ and $(s_i s_{i\pm 2})^2 = 1$. These examples clearly illustrate a difference from the $D_4^{(1)}$ structure encountered above.

The steps shown in this section complete the systematic derivation of the $D_4^{(1)}$ Hamiltonian system starting from the integrable hierarchy of D_4 symmetry. We will return to the model with two independent parameters $\varepsilon_i, i = 1, 2$ in a separate publication.

This work illustrates the power of algebraic methods to derive systems invariant under affine Weyl groups that should lend itself well to generalizations to other group structures.

Recently, the Sasano systems of four-dimensional Painlevé type equations with affine Weyl group symmetry of type $D_6^{(1)}$ [5] were derived as isomonodromic deformation equations in [6, 7], which suggests that a similar analysis will apply to coupled Painlevé III models with four canonical variables of reference [1] obtained in this paper from the self-similarity limit.

A Algebraic background on $so(2n)$

Here we discuss the Lie algebra so(2n) and its loop algebra that underlies the zero-curvature considerations. The algebra $so(2n) = \{X \in gl(2n, C) | X + X^T = 0\}$ is generated by $2n \times 2n$ anti-symmetric matrices $L_{i,j} = -L_{j,i}$ with components

$$
(L_{i,j})_{k,l} = \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}, \quad i, j = 1, \ldots, 2n\,. \tag{46}
$$

These $\frac{1}{2}(2n)(2n-1)$ matrices form a basis for the $so(2n)$ Lie algebra with the commutation relations :

$$
[L_{i,j}, L_{m,n}] = \delta_{i,m} L_{j,n} + \delta_{j,n} L_{i,m} - \delta_{i,n} L_{j,m} - \delta_{j,m} L_{i,m}\,, \tag{47}
$$

where we followed Olive's convention [4]. The Cartan sub-algebra generators are:

$$
H_i = i L_{2i-1,2i}, \quad 1, 2, \ldots, n\,.
$$

The relevant commutation relations in accordance to (47) are:

$$
\begin{aligned}
[H_i, L_{2j-1,2k-1}] &= i\delta_{i,j} L_{2j,2k-1} - i\delta_{i,k} L_{2i,2j-1} \\
[H_i, L_{2j,2k-1}] &= -i\delta_{i,j} L_{2j-1,2k-1} + i\delta_{i,k} L_{2j,2k} \\
[H_i, L_{2j-1,2k}] &= i\delta_{i,j} L_{2j,2k} - i\delta_{i,k} L_{2j-1,2k-1} \\
[H_i, L_{2j,2k}] &= -i\delta_{i,j} L_{2j-1,2k} - i\delta_{i,k} L_{2j,2k-1}
\end{aligned} \tag{48}
$$

The roots are:

$$\boldsymbol{\alpha} = \epsilon\, \mathbf{e}_j + \eta\, \mathbf{e}_k, \quad j \neq k$$

with independent $\epsilon, \eta = \pm 1$ and $e_i, i = 1, \ldots, n$ with $(e_i, e_j) = \delta_{i,j}$ being a basis for R^n.

The associated step operators are

$$E_{\boldsymbol{\alpha}} = -\frac{1}{2}\left(L_{2j-1,2k-1} + i\epsilon L_{2j,2k-1} + i\eta L_{2j-1,2k} - \epsilon\eta L_{2j,2k}\right)$$

Number of roots is $\frac{1}{2}n(n-1) \times 2 \times 2 = 2n(n-1)$,which is the dimension of $so(2n)$, rank of Cartan sub-algebra is n.

It holds that

$$[H_i, E_{\boldsymbol{\alpha}}] = (\epsilon\delta_{i,j} + \eta\delta_{i,k})E_{\boldsymbol{\alpha}}\,,$$

as long as $\eta^2 = 1, \epsilon^2 = 1$.

All roots have equal length and satisfy $(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = 2$. The basis of simple roots is given by:

$$\begin{aligned} \boldsymbol{\alpha}_i &= \mathbf{e}_i - \mathbf{e}_{i+1},\ i = 1, \ldots, n-1 \\ \boldsymbol{\alpha}_n &= \mathbf{e}_{n-1} + \mathbf{e}_n, \end{aligned} \tag{49}$$

The inner product of simple roots

$$(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j) = \begin{cases} 2 & i = j & 1 \leq i, j \leq n \\ -1 & |i-j| = 1 & 1 \leq i, j \leq n-1 \\ 0 & |i-j| \geq 2 & 1 \leq i, j \leq n \\ 0 & i = n-1, j = n & \end{cases}$$

defines the corresponding Cartan matrix. For $so(2n)$ the roots and co-roots are identical, the highest root is

$$\psi = e_1 + e_2 = \boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2 + \ldots + 2\boldsymbol{\alpha}_{n-2} + \boldsymbol{\alpha}_{n-1} + \boldsymbol{\alpha}_n\,,$$

the Coxeter number h and the dual Coxeter number $h^\vee$ coincide and

$$h = h^\vee = 2n - 2\,.$$

For case of $so(8)$ these become

$$\psi = e_1 + e_2 = \boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2 + + \boldsymbol{\alpha}_3 + \boldsymbol{\alpha}_4, \qquad h = h^\vee = 6 \tag{50}$$

The fundamental weights Λ_i such that $2(\boldsymbol{\alpha}_i, \Lambda_j)/(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i) = \delta_{ij}$ are:

$$\Lambda_i = \sum_{j=1}^{i} e_j = \boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2 + \ldots + (i-1)\boldsymbol{\alpha}_{i-1} + i(\boldsymbol{\alpha}_i + \ldots + \boldsymbol{\alpha}_{n-2}) + \frac{i}{2}(\boldsymbol{\alpha}_{n-1} + \boldsymbol{\alpha}_n),\ i = 1, \ldots, n-2$$

$$\Lambda_{n-1} = \frac{1}{2}(e_1 + \ldots + e_{n-1} - e_n) = \frac{1}{2}(\boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2 + \ldots + (n-2)\boldsymbol{\alpha}_{n-2}) + \frac{n}{2}\boldsymbol{\alpha}_{n-1} + \frac{n-2}{2}\boldsymbol{\alpha}_n$$

$$\Lambda_n = \frac{1}{2}(e_1 + \ldots + e_{n-1} + e_n) = \frac{1}{2}(\boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2 + \ldots + (n-2)\boldsymbol{\alpha}_{n-2}) + \frac{n-2}{2}\boldsymbol{\alpha}_{n-1} + \frac{n}{2}\boldsymbol{\alpha}_n$$

Especially for $so(8)$ with $n = 4$ we find for weights and simple roots

$$\begin{aligned}
\Lambda_1 &= e_1, & \boldsymbol{\alpha}_1 &= e_1 - e_2,\\
\Lambda_2 &= e_1 + e_2, & \boldsymbol{\alpha}_2 &= e_2 - e_3,\\
\Lambda_3 &= \frac{1}{2}(e_1 + e_2 + e_3 - e_4), & \boldsymbol{\alpha}_3 &= e_3 - e_4,\\
\Lambda_4 &= \frac{1}{2}(e_1 + e_2 + e_3 + e_4), & \boldsymbol{\alpha}_4 &= e_3 + e_4,
\end{aligned}$$

we obtain for a sum of weights:

$$\Lambda = \sum_{i=1}^{4} \Lambda_i = 3e_1 + 2e_2 + e_3 \tag{51}$$

The product of Λ and a general root $\boldsymbol{\alpha} = \epsilon e_i + \eta e_j$

$$(\Lambda, \boldsymbol{\alpha}) \neq 0$$

for all $\boldsymbol{\alpha} = \epsilon e_i + \eta e_j$.

We will use (51) to define the principal gradation operator for $so(8)$:

$$Q = 6d + \sum_{i=1}^{n} \Lambda_i \cdot H = 6d + \Lambda \cdot H = 6d + (3e_1 + 2e_2 + e_3) \cdot H\,. \tag{52}$$

Note that

$$(\Lambda, \psi) = (3e_1 + 2e_2 + e_3, e_1 + e_2) = 5\,.$$

A.1 so(8) charge sectors and their bases

The underlying charge sectors are (with $m \in \mathbb{Z}$):

$$\begin{aligned}
\mathcal{G}^{(6m)} &= \{H_1^{(m)}, H_2^{(m)}, H_3^{(m)}, H_4^{(m)}\}\\
\mathcal{G}^{(6m+1)} &= \{E^{(m)}_{e_1-e_2}, E^{(m)}_{e_2-e_3}, E^{(m)}_{e_3-e_4}, E^{(m)}_{e_3+e_4}, E^{(m+1)}_{-\psi} = E^{(m+1)}_{-e_1-e_2}\}\\
\mathcal{G}^{(6m+2)} &= \{E^{(m)}_{e_1-e_3}, E^{(m)}_{e_2+e_4}, E^{(m)}_{e_2-e_4}, E^{(m+1)}_{-e_1-e_3}\}\\
\mathcal{G}^{(6m+3)} &= \{E^{(m)}_{e_1-e_4}, E^{(m)}_{e_1+e_4}, E^{(m)}_{e_2+e_3}, E^{(m+1)}_{-e_1-e_4}, E^{(m+1)}_{-e_1+e_4}, E^{(m+1)}_{-e_2-e_3}\}\\
\mathcal{G}^{(6m+4)} &= \{E^{(m)}_{e_1+e_3}, E^{(m+1)}_{-e_2+e_4}, E^{(m+1)}_{-e_2-e_4}, E^{(m+1)}_{-e_1+e_3}\}\\
\mathcal{G}^{(6m+5)} &= \{E^{(m)}_{e_1+e_2}, E^{(m+1)}_{-e_3+e_4}, E^{(m+1)}_{-e_3-e_4}, E^{(m+1)}_{-e_2+e_3}, E^{(m+1)}_{-e_1+e_2}\}\,.
\end{aligned} \tag{53}$$

The unique grade one semi-simple element in $\mathcal{G}^{(1)}$ is

$$\begin{aligned}
E^{(1)} &= \sum_{i=1}^{4} E^{(0)}_{\alpha_i} + E^{(1)}_{-\psi}\\
&= E^{(0)}_{e_1-e_2} + E^{(0)}_{e_2-e_3} + E^{(0)}_{e_3-e_4} + E^{(0)}_{e_3+e_4} + E^{(1)}_{-e_1-e_2}
\end{aligned} \tag{54}$$

where the sum was over all simple roots of $so(8)$ from (49).

Define the kernels $\mathcal{K}^{(i)} \in \mathcal{G}^{(i)}$ to be such that

$$[E^{(1)}, \mathcal{K}^{(i)}] = 0\,,$$

for $i = 2, 3, 4, 5$ and $\mathcal{G}^{(i)}$ as given in (53).

Given the grade 2 sector $\mathcal{G}^{(2)}$ in (53) we consider

$$\begin{aligned}
[E^{(1)}, E^{(0)}_{e_1-e_3}] &= -E^{(0)}_{e_1-e_4} - E^{(0)}_{e_1+e_4} - E^{(1)}_{-e_2-e_3} \\
[E^{(1)}, E^{(0)}_{e_2+e_4}] &= +E^{(0)}_{e_1+e_4} + E^{(0)}_{e_2+e_3} + E^{(1)}_{-e_1+e_4} \\
[E^{(1)}, E^{(0)}_{e_2-e_4}] &= +E^{(0)}_{e_1-e_4} + E^{(0)}_{e_2+e_3} + E^{(1)}_{-e_1-e_4} \\
[E^{(1)}, E^{(1)}_{-e_1-e_3}] &= -E^{(1)}_{-e_2-e_3} - E^{(1)}_{-e_1-e_4} - E^{(1)}_{-e_1+e_4}
\end{aligned} \tag{55}$$

Accordingly we find for

$$K^{(2)} = aE^{(0)}_{e_1-e_3} + bE^{(0)}_{e_2+e_4} + cE^{(0)}_{e_2-e_4} + dE^{(1)}_{-e_1-e_3}$$

that

$$[E^{(1)}, K^{(2)}] = 0$$

only for $a = b = c = d = 0$ and $\mathcal{K}^{(2)}$ is empty.

For

$$K^{(3)} = \varepsilon_1 E^{(0)}_{e_1-e_4} + \varepsilon_2 E^{(0)}_{e_1+e_4} + \varepsilon_3 E^{(0)}_{e_2+e_3} + \varepsilon_4 E^{(1)}_{-e_1-e_4} + \varepsilon_5 E^{(1)}_{-e_1+e_4} + \varepsilon_6 E^{(1)}_{-e_2-e_3}$$

we find that

$$[E^{(1)}, K^{(3)}] = 0 \quad \text{for} \quad \varepsilon_3 = -\varepsilon_1 - \varepsilon_2, \varepsilon_6 = -\varepsilon_1 - \varepsilon_2, \varepsilon_4 = \varepsilon_2, \varepsilon_5 = \varepsilon_1 \tag{56}$$

with arbitrary two parameters $\varepsilon_1, \varepsilon_2$ that parameterize $\mathcal{K}^{(3)}$. If we define elements in $\mathcal{K}^{(3)}$ that both satisfy (56):

$$\begin{aligned}
K^{(3)}_\varepsilon &= \varepsilon_1 E^{(0)}_{e_1-e_4} + \varepsilon_2 E^{(0)}_{e_1+e_4} - (\varepsilon_1 + \varepsilon_2) E^{(0)}_{e_2+e_3} + \varepsilon_2 E^{(1)}_{-e_1-e_4} + \varepsilon_1 E^{(1)}_{-e_1+e_4} - (\varepsilon_1 + \varepsilon_2) E^{(1)}_{-e_2-e_3} \\
K^{(3)}_\eta &= \eta_1 E^{(0)}_{e_1-e_4} + \eta_2 E^{(0)}_{e_1+e_4} - (\eta_1 + \eta_2) E^{(0)}_{e_2+e_3} + \eta_2 E^{(1)}_{-e_1-e_4} + \eta_1 E^{(1)}_{-e_1+e_4} - (\eta_1 + \eta_2) E^{(1)}_{-e_2-e_3}\,,
\end{aligned} \tag{57}$$

then

$$[K^{(3)}_\varepsilon, K^{(3)}_\eta] = 0\,,$$

for any two arbitrary sets $(\varepsilon_1, \varepsilon_2), (\eta_1, \eta_2)$. Thus $\mathcal{K}^{(3)}$ is, as expected, abelian.

For

$$K^{(4)} = \varepsilon_1 E^{(0)}_{e_1+e_3} + \varepsilon_2 E^{(1)}_{-e_2+e_4} + \varepsilon_3 E^{(1)}_{-e_2-e_4} + \varepsilon_4 E^{(1)}_{-e_1+e_3}\,,$$

we find that

$$[E^{(1)}, K^{(4)}] = 0 \quad \text{for} \quad \varepsilon_1 = 0, \varepsilon_2 = 0, \varepsilon_3 = 0, \varepsilon_4 = 0\,, \tag{58}$$

and $\mathcal{K}^{(4)}$ is empty.

For

$$K^{(5)} = aE^{(0)}_{e_1+e_2} + bE^{(1)}_{-e_3+e_4} + cE^{(1)}_{-e_3-e_4} + dE^{(1)}_{-e_2+e_3} + eE^{(1)}_{-e_1+e_2}\,,$$

we find that

$$[E^{(1)}, K^{(5)}] = 0 \quad \text{for} \quad b = a, c = a, e = a, d = 2a\,, \tag{59}$$

with an arbitrary one parameter a that parametrizes $\mathcal{K}^{(5)}$.

For the (sub-algebras) $\mathcal{G}^{(3)}$ and $\mathcal{G}^{(1)}$ that have non-trivial two- and one-dimensional kernels, $\mathcal{K}^{(3)}$ and $\mathcal{K}^{(1)}$, respectively, it is useful to describe their bases.

For $\mathcal{G}^{(3)}$ from the relation (53) we will use the basis:

$$\begin{aligned}
V_1 &= E^{(0)}_{e_1-e_4} - E^{(0)}_{e_2+e_3} + E^{(1)}_{-e_1+e_4} - E^{(1)}_{-e_2-e_3}\,,\\
V_2 &= E^{(0)}_{e_1+e_4} - E^{(0)}_{e_2+e_3} + E^{(1)}_{-e_1-e_4} - E^{(1)}_{-e_2-e_3}\,,\\
V_3 &= E^{(0)}_{e_1-e_4} + E^{(0)}_{e_1+e_4} + E^{(1)}_{-e_2-e_3}\,,\\
V_4 &= -E^{(0)}_{e_1+e_4} - E^{(0)}_{e_2+e_3} - E^{(1)}_{-e_1+e_4}\,,\\
V_5 &= -E^{(0)}_{e_1-e_4} - E^{(0)}_{e_2+e_3} - E^{(1)}_{-e_1-e_4}\,,\\
V_6 &= E^{(1)}_{-e_1-e_4} + E^{(1)}_{-e_1+e_4} + E^{(1)}_{-e_2-e_3}\,,
\end{aligned} \tag{60}$$

with V_1, V_2 being the two matrices from (56) that span a basis for the kernel $\mathcal{K}^{(3)}$ of $E^{(1)}$ in $\mathcal{G}^{(3)}$, while V_3, V_4, V_5, V_6 span a basis for the image of $E^{(1)}$ in $\mathcal{G}^{(3)}$.

To analyze zero-curvature equations involving $\mathcal{G}^{(1)}$ from the relation (53) we will use the basis $E_1, \ldots, E_5$:

$$\begin{aligned}
&E_1 = E^{(1)},\;\; E_2 = -E^{(0)}_{e_1-e_2} + E^{(1)}_{-e_1-e_2}\\
&E_3 = E^{(0)}_{e_1-e_2} - E^{(0)}_{e_2-e_3} + E^{(1)}_{-e_1-e_2},\;\; E_4 = E^{(0)}_{e_2-e_3} - E^{(0)}_{e_3-e_4} - E^{(0)}_{e_3+e_4}\\
&E_5 = E^{(0)}_{e_3-e_4} - E^{(0)}_{e_3+e_4}\,,
\end{aligned} \tag{61}$$

for $\mathcal{G}^{(1)}$. The first element E_1 is obviously in kernel of $E^{(1)}$, while E_2, E_3, E_4, E_5 span the image of $E^{(1)}$. One can check that

$$\varepsilon_1 E_1 + \varepsilon_2 E_2 + \varepsilon_3 E_3 + \varepsilon_4 E_4 + \varepsilon_5 E_5 = 0 \;\;\to\;\; \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 0\,,$$

and the same basic relation for the V-basis.

B Main expressions of of the zero-curvature calculation

The coefficients $M_i, i = 1, \ldots, 4$ of the matrix $D^{(2)}$ defined in expressions (3), are explicitly given by solving the grade 3 equation (4):

$$\begin{aligned}
M_1 &= \frac{(\varepsilon_1+\varepsilon_2)}{2}(\phi_1+\phi_2+\phi_3) - \frac{(\varepsilon_1-\varepsilon_2)}{2}\phi_4\,,\\
M_2 &= \frac{(\varepsilon_1+\varepsilon_2)}{2}(-\phi_4+\phi_2+\phi_3) + \frac{(\varepsilon_1-\varepsilon_2)}{2}\phi_1\,,\\
M_3 &= \frac{(\varepsilon_1+\varepsilon_2)}{2}(\phi_2+\phi_3+\phi_4) - \frac{(\varepsilon_1-\varepsilon_2)}{2}\phi_1\,,\\
M_4 &= \frac{(\varepsilon_1+\varepsilon_2)}{2}(-\phi_1+\phi_2+\phi_3) + \frac{(\varepsilon_1-\varepsilon_2)}{2}\phi_4\,.
\end{aligned} \tag{62}$$

The coefficients $d_i, i = 2, \ldots, 5$ of the grade one element $D^{(1)}$ along the basis elements $E_i, i = 2, \ldots E_5$ given in expressions (61) are obtained from the grade 2 component of the zero curvature equations (1) to be

$$\begin{aligned}
d_2 &= -\frac{(\varepsilon_1+\varepsilon_2)}{2}\left(\phi_1\phi_2+\partial_x\phi_1\right) - \frac{(\varepsilon_1-\varepsilon_2)}{2}\left(\phi_3\phi_4-\partial_x\phi_4\right)\,,\\
d_3 &= \frac{(\varepsilon_1+\varepsilon_2)}{6}\left(-\phi_2\phi_3+\phi_2^2-2\phi_3^2-\phi_4^2+2\phi_1^2+3\partial_x(\phi_2+\phi_3)\right) - \frac{(\varepsilon_1-\varepsilon_2)}{6}\phi_1\phi_4\,,\\
d_4 &= \frac{(\varepsilon_1+\varepsilon_2)}{6}\left(\phi_2\phi_3-\phi_3^2+\phi_1^2-2\phi_4^2+2\phi_2^2+3\partial_x(\phi_2+\phi_3)\right) + \frac{(\varepsilon_1-\varepsilon_2)}{6}\phi_1\phi_4\,,\\
d_5 &= \frac{(\varepsilon_1+\varepsilon_2)}{2}\left(\phi_3\phi_4-\partial_x\phi_4\right) + \frac{(\varepsilon_1-\varepsilon_2)}{2}\left(\phi_1\phi_2+\partial_x\phi_1\right)
\end{aligned} \tag{63}$$

The components of $[A_0, D^{(1)}] = \sum_{i=2}^{5} C_i E_i$ can be calculated as

$$\begin{aligned}
C_2 &= -\phi_1 d_1 - \phi_2 d_2 - \phi_1 d_3\,,\\
C_3 &= -\phi_2 d_1 - \frac{2}{3}\phi_1 d_2 - \frac{1}{3}\phi_2 d_3 - \frac{1}{3}\phi_3 d_3 - \frac{1}{3}\phi_2 d_4 + \frac{2}{3}\phi_3 d_4 + \frac{1}{3}\phi_4 d_5\,,\\
C_4 &= -\phi_3 d_1 - \frac{1}{3}\phi_1 d_2 - \frac{2}{3}\phi_2 d_3 + \frac{1}{3}\phi_3 d_3 + \frac{1}{3}\phi_2 d_4 + \frac{1}{3}\phi_3 d_4 + \frac{2}{3}\phi_4 d_5\,,\\
C_5 &= -\phi_4 d_1 + \phi_4 d_4 + \phi_3 d_5\,.
\end{aligned} \tag{64}$$

Inserting these values of d_i and C_i into equation (11) we obtain

$$
\begin{aligned}
v_1 &= \frac{(\varepsilon_1+\varepsilon_2)}{4}(-\phi_1(\phi_2^2+\phi_3^2+\phi_4^2-\phi_1^2)+2\phi_1\partial_x(\phi_3+2\phi_2)+2\partial_x^2\phi_1) \\
&\quad + \frac{(\varepsilon_1-\varepsilon_2)}{2}(-\phi_2\phi_3\phi_4+\phi_2\partial_x\phi_4+\partial_x(\phi_3\phi_4)-\partial_x^2\phi_4)\,, \\
v_2 &= \frac{(\varepsilon_1+\varepsilon_2)}{4}(-\phi_2(\phi_1^2+\phi_3^2+\phi_4^2-\phi_2^2)+2\phi_4\partial_x\phi_4+2(\phi_2+\phi_3)\partial_x\phi_3-4\phi_1\partial_x\phi_1-2\partial_x^2(\phi_2+\phi_3)) \\
&\quad - \frac{(\varepsilon_1-\varepsilon_2)}{2}(\phi_1\phi_3\phi_4-\phi_1\partial_x\phi_4)\,, \\
v_3 &= \frac{(\varepsilon_1+\varepsilon_2)}{4}(-\phi_3(\phi_1^2+\phi_2^2+\phi_4^2-\phi_3^2)+4\phi_4\partial_x\phi_4-2(\phi_2+\phi_3)\partial_x\phi_2-2\phi_1\partial_x\phi_1-2\partial_x^2(\phi_2+\phi_3)) \\
&\quad - \frac{(\varepsilon_1-\varepsilon_2)}{2}(\phi_1\phi_2\phi_4+\phi_4\partial_x\phi_1)\,, \\
v_4 &= \frac{(\varepsilon_1+\varepsilon_2)}{4}(-\phi_4(\phi_1^2+\phi_2^2+\phi_3^2-\phi_4^2)-2\phi_4\partial_x(\phi_2+2\phi_3)+2\partial_x^2\phi_4) \\
&\quad - \frac{(\varepsilon_1-\varepsilon_2)}{2}(\phi_1\phi_2\phi_3+\phi_3\partial_x\phi_1+\partial_x(\phi_1\phi_2)+\partial_x^2\phi_1)\,.
\end{aligned} \tag{65}
$$

We can now insert the above values v_i into the t_3 flow expression (13) to obtain

$$
\begin{aligned}
\partial_{t_3}\phi_1 &= \partial_x\Big[\frac{(\varepsilon_1+\varepsilon_2)}{4}(-\phi_1(\phi_2^2+\phi_3^2+\phi_4^2-\phi_1^2)+2\phi_1\partial_x(\phi_3+2\phi_2)+2\partial_x^2\phi_1)\Big] \\
&\quad + \partial_x\Big[\frac{(\varepsilon_1-\varepsilon_2)}{2}(-\phi_2\phi_3\phi_4+\phi_2\partial_x\phi_4+\partial_x(\phi_3\phi_4)-\partial_x^2\phi_4)\Big]\,, \\
\partial_{t_3}\phi_2 &= \partial_x\Big[\frac{(\varepsilon_1+\varepsilon_2)}{4}(-\phi_2(\phi_1^2+\phi_3^2+\phi_4^2-\phi_2^2)+2\phi_4\partial_x\phi_4+2(\phi_2+\phi_3)\partial_x\phi_3-4\phi_1\partial_x\phi_1-2\partial_x^2(\phi_2+\phi_3)\Big] \\
&\quad - \partial_x\Big[\frac{(\varepsilon_1-\varepsilon_2)}{2}(\phi_3\phi_1\phi_4-\phi_1\partial_x\phi_4))\Big]\,, \\
\partial_{t_3}\phi_3 &= \partial_x\Big[\frac{(\varepsilon_1+\varepsilon_2)}{4}(-\phi_3(\phi_1^2+\phi_2^2+\phi_4^2-\phi_3^2)+4\phi_4\partial_x\phi_4-2(\phi_2+\phi_3)\partial_x\phi_2-2\phi_1\partial_x\phi_1-2\partial_x^2(\phi_2+\phi_3))\Big] \\
&\quad - \partial_x\Big[\frac{(\varepsilon_1-\varepsilon_2)}{2}(\phi_2\phi_1\phi_4+\phi_4\partial_x\phi_1)\Big]\,, \\
\partial_{t_3}\phi_4 &= \partial_x\Big[\frac{(\varepsilon_1+\varepsilon_2)}{4}(-\phi_4(\phi_2^2+\phi_3^2+\phi_1^2-\phi_4^2)-2\phi_4\partial_x(\phi_2+2\phi_3)+2\partial_x^2\phi_4)\Big] \\
&\quad + \partial_x\Big[\frac{(\varepsilon_1-\varepsilon_2)}{2}(-\phi_1\phi_3\phi_2-\phi_3\partial_x\phi_1-\partial_x(\phi_1\phi_2)-\partial_x^2\phi_1)\Big]\,.
\end{aligned} \tag{66}
$$

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