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An exercise in experimental mathematics: calculation of the algebraic entropy of a map

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Abstract

We illustrate the use of the notion of derived recurrences introduced earlier to evaluate the algebraic entropy of self-maps of projective spaces. We in particular give an example, where a complete proof is still awaited, but where different approaches are in such perfect agreement that we can trust we get to an exact result. This is an instructive example of experimental mathematics.

1 Introduction: algebraic entropy of maps

We deal with birational self-maps of N-dimensional projective space \mathcal{P}_N . Such maps are given as polynomial maps of degree d when written in terms of the $N+1$ homogeneous coordinates. Birationality means that the inverse maps are also polynomial (not necessarily of the same degree). The iterates can be evaluated polynomially, and the degree d_n of the nth iterate is uniquely defined once all common factors to the homogeneous coordinates are removed.

The algebraic entropy [\[1,](#page-8-1) [2\]](#page-8-2) is defined from the sequence of degrees $\{d_n\}$ by

$$
\epsilon = \lim_{n \to \infty} \frac{1}{n} \log(d_n) \tag{1}
$$

Some authors also use the terminology 'dynamic degree' [\[3\]](#page-8-3) or 'dynamical degree' (see [\[4\]](#page-8-4) and references therein) $\delta = \lim_{n \to \infty} d_n^{1/n}$, of which ϵ is the logarithm. Our definition is not without relation to the spectral radius of the map induced in homology, which already appeared in previous definitions of entropy [\[5,](#page-8-5) [6,](#page-8-6) [7\]](#page-9-0) and to the notion of complexity introduced in [\[8\]](#page-9-1).

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The limit [\(1\)](#page-0-1) always exists and is invariant by any birational change of coordinates. It is an excellent -if not the best- detector of integrability of birational discrete-time systems: $integrability = vanishing of the entropy.$

Beyond this use as an integrability criterion, an important question is to determine the set of non zero values the entropy can take. On aspect of the problem is then, given a map, to find explicitly the exact value of ϵ .

This can be achieved - especially in the two dimensional case which is of paramount importance since it encompasses the discrete Painlevé equations $[9, 10]$ $[9, 10]$ - by a singularity analysis of the maps: in that case one may eventually define a rational variety obtained by blow-ups of points, where the maps are diffeomorphisms, and the induced action on the Picard group gives the answer (see [\[11,](#page-9-4) [12,](#page-9-5) [13\]](#page-9-6)).

We are interested in situations where this approach cannot be used. In particular, going beyond the "easy" two-dimensional case, makes the singularity analysis much more intricate, see for example [\[14,](#page-9-7) [15,](#page-9-8) [16,](#page-9-9) [17\]](#page-9-10).

The most elementary thing one can do is to evaluate as many terms as possible of the sequence degrees, using one's favourite formal calculus software.

An approximate value of ϵ may then possibly be obtained by calculating the successive ratios d_{n+1}/d_n . Let us call this low brow analysis 'Method 0'. It gives an idea of the value of ϵ .

A better approach is to look for a generating function for the sequence of degrees. This function is by definition

$$
g(s) = \sum_{k=0}^{k=\infty} d_k s^k
$$
 (2)

This method ('Method 1') - which at first may look too heuristic - works extremely well, and this is due to the underlying algebraic structure of the problem.

A third method ('Method 2'), introduced in [\[18\]](#page-9-11) and expanded in [\[19\]](#page-9-12), is an analysis of the form of the iterates. If the factor structure of the iterates happens to stabilise, we may rewrite the maps in a different way, giving immediately the value of the entropy.

The main point is actually that the asymptotic behaviour which ϵ measures can be extracted from a finite piece of the sequence $\{d_n\}$. The fundamental reason is the fact that more than often, the sequence $\{d_n\}$ verifies a finite recurrence relation. In addition this recurrence relation has integer coefficients, yielding for the entropy the remarkable property that it is the logarithm of an algebraic integer. This last property was conjectured [\[1\]](#page-8-1) to be true for all birational self-maps of projective spaces, and its generality is now questioned [\[20\]](#page-9-13).

The possible drop of degrees of the iterates (meaning that d_n is strictly lower than d_1^n), the nature of the generating function of the sequence of degrees, as well as the stabilisation phenomenon of the form of the iterates are all footprints of the singularity structure of the iterations. Our point is that - apart from the simplest two dimensional case, as is the warm-up example we have chosen, a full analysis of this structure is often difficult for the higher dimensional maps. We stick to an experimental approach, using our calculation tools to their limits, and do not embark into the singularity analysis, leaving it to further study..

2 Warm-up: a well studied two dimensional map

As a good example of what can be done for self-maps of \mathcal{P}_2 , we start from the prototype of algebraically stable (aka "confining") map with positive entropy given in [\[21\]](#page-9-14).

$$
\varphi : [x, y, z] \longrightarrow [x^3 + az^3 - yx^2, x^3, x^2z]
$$
\n(3)

which is the transcription as a map in \mathcal{P}_2 of the simple order 2 recurrence

$$
u_{n+1} + u_{n-1} = u_n + \frac{a}{u_n^2} \tag{4}
$$

This map has been shown to have positive entropy by various methods, among which the construction of a rational surface over \mathcal{P}_2 where the singularities are resolved [\[22,](#page-9-15) [23\]](#page-9-16). The lift of the map to the Picard group of this variety is a linear map whose maximal eigenvalue (spectral radius) gives the entropy.

A possible first step is to calculate the beginning of the sequence of degrees:

 ${d_n} = 1, 3, 9, 27, 73, 195, 513, 1347, 3529, 9243, 24201, 63363, ...$ (5)

and then extract as much as information from this limited amount of data.

Method 0: The most naive - but already useful - thing to do is to see how the ratio d_{n+1}/d_n evolves with n:

Clearly these numbers point to a value of the order of $log(2.618...)$ for the entropy. This is not enough if we want the exact value, but it already tells us that the entropy is not vanishing. In other words the recurrence does not fall into the integrable class [\[21\]](#page-9-14).

Method 1: It is possible to fit the sequence [\(5\)](#page-2-0) with the rational generating function

$$
g(s) = \frac{3s^3 + 1}{(1 - s)(1 + s)(s^2 - 3s + 1)}
$$
(6)

The entropy is the logarithm of the inverse of the smallest modulus of the poles of g , since this is what governs the growth of the Taylor expansion of $g(s)$. This gives $\epsilon =$ $\log((3+\sqrt{5})/2) \simeq \log(2.618033988...)$

To go beyond the approximation of Method 0 and the heuristic nature of Method 1, go to Method 2, keeping in mind that it may imply much heavier calculations.

For the map [\(3\)](#page-2-1) the form of the iterates does stabilise to the pattern

$$
p_k = [A_{k-3}^3 A_k, A_{k-4} A_{k-1}^3, z A_{k-3}^2 A_{k-2}^2 A_{k-1}^2]
$$
\n⁽⁷⁾

and the recurrence relation between the blocks A_k is just

$$
A_k^3 A_{k-3}^3 + a z^3 A_{k-1}^6 A_{k-2}^6 - A_{k-1}^3 A_{k-4} A_k^2 = A_{k-3}^2 A_{k-2}^3 A_{k+1}^3
$$
 (8)

We call [\(8\)](#page-3-0) the *derived recurrence* of the original one [\(4\)](#page-2-2). Notice that it is the same as equ (4.6) of $[24]$, where it was obtained by a different approach. Recurrence (8) extends over a string of length 6. This relation is not quadratic nor multi-linear, but it allows to prove [\(7\)](#page-3-1), providing the recurrence condition on the degrees of the iterates of φ , and the value of the entropy $\epsilon = \log((3 + \sqrt{5})/2)$ (same as above).

The properly conducted singularity analysis, blowing up enough points of \mathcal{P}_2 , and looking at the induced map on the Picard group of the rational variety constructed in this way, confirms the value $\epsilon = \log((3 + \sqrt{5})/2)[22, 23]$ $\epsilon = \log((3 + \sqrt{5})/2)[22, 23]$ $\epsilon = \log((3 + \sqrt{5})/2)[22, 23]$.

So all Methods 0,1,2, and the complete singularity analysis agree perfectly. Method 0 is approximate but useful, Method 1 is providing us with an educated guess, and suggests a candidate for the value of ϵ . This value turns out to be exact, as one can prove using Method 2, or with the full desingularisation of the map.

Remark: The sequence [\(5\)](#page-2-0) is registered in the On-Line Encyclopedia of Integer Sequences (https://oeis.org/) under number A084707.

At this point it is interesting to notice that the recurrence relation [\(8\)](#page-3-0) belongs the 'Somos-like' family (see [\[25,](#page-10-1) [26,](#page-10-2) [27\]](#page-10-3) and https://faculty.uml.edu//jpropp/somos/history.txt). Although A_{k+1} is given as a rational fraction in terms of the previous A's, the recurrence has the so called Laurent property [\[19,](#page-9-12) [28\]](#page-10-4). Moreover, if one launches the recurrence with appropriate polynomial initial conditions, the values one obtains are, by construction, multivariate polynomials.

3 A more challenging map

Monomial maps are known to behave in a particular way, as far as the sequence of degrees of their iterates is concerned. Birational monomial maps have an entropy which is the logarithm of an algebraic integer, but the generating function of the sequence is not necessarily rational [\[29\]](#page-10-5).

We will now examine a map acting in dimension larger than two. Consider the recurrence of order 4 [\[30\]](#page-10-6):

$$
x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(1 - x_{n-1})x_{n-3}}\tag{9}
$$

This recurrence defines a birational (almost monomial) map in \mathcal{P}_4 .

$$
\varphi : [x, y, z, u, t] \to [x z t^2, x u y (t - y), u y^2 (t - y), u y (t - y) z, u y (t - y) t]
$$
(10)

with inverse

$$
\psi : [x, y, z, u, t] \to [x y z (t - z), x z^{2} (t - z), x z u (t - z), y u t^{2}, x z (t - z) t]
$$
(11)

The direct calculation of the sequence of degrees yields

 ${d_n} = 1, 4, 5, 9, 11, 16, 21, 30, 43, 61, 86, 120, 168, 234, 329, 459, 645, 902, 1267, 1771,$, 3476, 4871, 6822, 9555, 13384, 18745, 26256, 36774, 51507, 72143, 101043, , 198223, 277633, 388864, 544644, 762846, 1068451, 1496494, 2096019, , 4111826, 5759091, 8066291, 11297797, 15823888, 22163239, 31042218, , 60896502, 85292724, 119462566, 1677321393, 234353404, 328239604, \cdots (12)

3.1 Method 0

We have the following sequence of ratios d_{n+1}/d_n :

., 1.250000000, 1.800000000, 1.222222222, 1.454545455, 1.312500000, 1.428571429, .433333333, 1.418604651, 1.409836066, 1.395348837, 1.400000000, 1.392857143, .405982906, 1.395136778, 1.405228758, 1.398449612, 1.404656319, 1.397790055, .402597403, 1.399355878, 1.401323360, 1.400533771, 1.400615655, 1.400732601, .400552899, 1.400693518, 1.400594150, 1.400636319, 1.400644573, 1.400593266, .400631414, 1.400631695, 1.400609415, 1.400640414, 1.400602781, 1.400632340, .400611657, 1.400620150, 1.400619715, 1.400615166, 1.400621177, 1.400616417, .400618778, 1.400618574, 1.400617129, 1.400619051, 1.400617392, 1.400618409, .400618221, 1.400617789, 1.400618487, 1.400617772, 1.400618294, 1.400618034, . . .

There is no doubt that the sequence converges, and that the numerical value of the entropy is around log(1.400618...), but we want to have its exact value.

3.2 Method 1

It is not possible to fit the sequence [\(12\)](#page-4-0) with a satisfactory rational generating function. This could come from two different reasons: either the generating function is not rational, either the information we have is not sufficient. Unfortunately, the explicit calculation of the degrees cannot go much further due to the practical limitations of the formal calculus software we have at hand...

3.3 Method 2

The images p_n of the generic starting point $p_0 = [x, y, z, u, t]$ are made of products of factors B_k with $B_1 = t - y$ and $B_2 = t - x$, and the further B_k 's are the proper transforms[\[31\]](#page-10-7) of B_{k-1} .

The first coordinate of p_n is a product of some B_j 's with various powers, B_n not appearing, and some adventive monomials in x, y, z, u, t , which remain of low degree.

The other four coordinates are of the same form, but all contain B_n with power 1.

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The outcome¹ is that the form indeed stabilises after order 27, and remains unchanged up to the maximum order we were able to reach. To give an idea the 44th iterate looks like

 $p_{44} := [y u^3 t^3 B_{18} B_{19} B_{20} B_{23} B_{24}^2 B_{25}^2 B_{26}^3 B_{27}^3 B_{28} B_{31}^2 B_{33} B_{33}^3 B_{34}^4 B_{35}^2 B_{38} B_{39} B_{40}^2 B_{41}^3 B_{42}^2,$ $x^2 z u^4 t^2 B_{18} B_{19} B_{22}^2 B_{23}^2 B_{24} B_{25}^2 B_{26}^3 B_{27} B_{29}^2 B_{30}^2 B_{31} B_{32} B_{33}^4 B_{34}^2 B_{36} B_{37}^2 B_{38} B_{39} B_{40}^3 B_{41}^2 B_{44}$ $x^2 y u^2 t^3 B_{18} B_{21}^2 B_{22}^3 B_{23} B_{24} B_{25}^2 B_{26} B_{28}^2 B_{29}^4 B_{30} B_{31} B_{32}^2 B_{33}^2 B_{35}^2 B_{36}^3 B_{27}^2 B_{38} B_{39}^2 B_{40}^2 B_{43} B_{44}$ $x^2y^2t^3B_{20}^2B_{21}^3B_{22}^2B_{23}B_{24}B_{27}^2B_{28}^4B_{29}^3B_{30}B_{31}^2B_{34}^3B_{35}^3B_{36}^3B_{37}^2B_{38}^2B_{39}B_{42}B_{43}B_{44}$ $x y u^2 B_{18} B_{19} B_{20} B_{21} B_{22} B_{25} B_{26}^2 B_{27}^2 B_{28}^2 B_{29}^2 B_{33}^2 B_{34}^2 B_{35}^2 B_{36}^2 B_{37} B_{40} B_{41} B_{42} B_{43} B_{44}$

and the equation giving B_{44} is

$$
equ44: \{ B_{17} \mathbf{B}_{44} = B_{19} B_{20} B_{26} B_{27}^2 B_{34}^2 B_{35} B_{41} B_{42} - B_{21} B_{22}^2 B_{23} B_{24} B_{25} B_{29}^2 B_{30} B_{31} B_{32}^2 B_{36} B_{37} B_{38} B_{39}^2 B_{40} t^3 x \}
$$
(13)

This is a recurrence of order 27, and it gives for the rate of growth of the degrees the largest root of

$$
r^{44} - r^{42} - r^{41} - r^{35} - 2r^{34} - 2r^{27} - r^{26} - r^{20} - r^{19} + r^{17}
$$
\n
$$
\tag{14}
$$

which happens to be 1.400618098... to be compared with the approximate value $\epsilon \simeq$ $log(1.400618...)$ we had from the original sequence of degrees.

Remark: In (14) we neglected the factors in x, y, z, u, t appearing in (13). These factors are present and will be taken into account in the next section. The main point is that they do not affect the value of the entropy.

Happily enough we have a candidate for the exact value of the entropy. It is the logarithm of an algebraic integer.

Notice that the derived recurrence if of much larger order than the initial one.

Why did Method 1 not work? $\overline{4}$

The monomial factors appearing in p_n have some structure:

Setting

$$
f(k) = \frac{B_{k-21}B_{k-20}^2B_{k-19}B_{k-18}B_{k-17}B_{k-13}^2B_{k-12}B_{k-11}B_{k-10}^2B_{k-6}B_{k-5}B_{k-4}B_{k-3}^2B_{k-2}}{B_{k-23}B_{k-22}B_{k-16}B_{k-15}^2B_{k-8}^2B_{k-7}B_{k-1}B_k}
$$

the k-th iterate p_k of the generic point $p_0 = [x, y, z, u, t]$ then reads

$$
p_k \simeq [\alpha_k \cdot f(k), \beta_k \cdot f(k-1), \gamma_k \cdot f(k-2), \delta_k \cdot f(k-3), \eta_k]
$$
\n(15)

where \simeq means equality up to a common factor.

The adventive factors $\alpha_k, \beta_k, \gamma_k, \delta_k, \eta_k$ are simple monomials in x, y, z, u, t. Two periods appear for these factors, namely 7 and 32. Defining

$$
\rho_k = x^{X_{\rho}(k)} y^{Y_{\rho}(k)} z^{Z_{\rho}(k)} u^{U_{\rho}(k)} t^{T_{\rho}(k)}, \quad \rho = \alpha, \beta, \gamma, \delta, \eta. \tag{16}
$$

¹Maple calculation up to order 17, insufficient, and then calculation using the V. Shoup's NTL C++ library $[32]$

The powers X_{ρ} , U_{ρ} and T_{ρ} are periodic with period 32, and Y_{ρ} and Z_{ρ} are periodic with period 7. There exist in addition simple relations between Y_{ρ} and Z_{ρ} as well as between U_{ρ} and $X_{\rho}, \rho = \alpha, \beta, \gamma, \delta, \eta$.

$$
Y_{\rho}(k) = Z_{\rho}(k+2) \quad \text{and} \quad U_{\rho}(k) = X_{\rho}(k+4), \quad \rho = \alpha, \beta, \gamma, \delta, \eta. \tag{17}
$$

We give here the values of X, Y, and T on one period starting at $k = 1$

 $X_{\alpha} = [1, 0, 0, 0, 0, 0, 2, 3, 2, 1, 1, 0, 0, 2, 4, 3, 1, 2, 0, 0, 1, 3, 3, 2, 2, 1, 0, 0, 1, 1, 1, 1],$ $X_{\beta} = [1, 1, 0, 0, 1, 0, 0, 2, 3, 1, 1, 2, 1, 0, 2, 4, 1, 1, 2, 2, 0, 1, 3, 2, 1, 2, 2, 0, 0, 1, 1, 0],$ $X_{\gamma} = [0, 1, 1, 0, 1, 1, 0, 0, 2, 2, 1, 2, 3, 1, 0, 2, 2, 1, 1, 4, 2, 0, 1, 2, 1, 1, 3, 2, 0, 0, 1, 0],$ $X_{\delta} = [0, 0, 1, 1, 1, 1, 1, 0, 0, 1, 2, 2, 3, 3, 1, 0, 0, 2, 1, 3, 4, 2, 0, 0, 1, 1, 2, 3, 2, 0, 0, 0],$ $X_n = [0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 1, 2, 2, 2, 2, 0, 0, 0, 2, 2, 2, 2, 1, 0, 0, 1, 1, 1, 1, 1, 0],$ Y_{α} = [0, 1, 2, 1, 0, 1, 0], $Y_{\beta} = [1, 0, 1, 2, 0, 0, 1],$ $Y_{\gamma} = [2, 1, 0, 1, 1, 0, 0],$ $Y_{\delta} = [1, 2, 1, 0, 0, 1, 0], \qquad Y_{\eta} = [1, 1, 1, 1, 0, 0, 0],$ $T_{\alpha} = [2, 4, 3, 1, 1, 0, 0, 2, 6, 6, 3, 3, 1, 0, 0, 4, 6, 5, 3, 3, 0, 0, 2, 4, 4, 3, 3, 1, 0, 0, 0, 0]$ $T_{\beta} = [0, 2, 4, 2, 1, 2, 2, 0, 2, 6, 4, 2, 3, 4, 0, 0, 4, 6, 2, 3, 4, 2, 0, 2, 4, 2, 2, 3, 2, 0, 0, 0],$ $T_{\gamma} = [0, 0, 2, 3, 2, 2, 4, 2, 0, 2, 4, 3, 2, 6, 4, 0, 0, 4, 3, 2, 4, 6, 2, 0, 2, 2, 1, 2, 4, 2, 0, 0],$ $T_{\delta} = [0, 0, 0, 1, 3, 3, 4, 4, 2, 0, 0, 3, 3, 5, 6, 4, 0, 0, 1, 3, 3, 6, 6, 2, 0, 0, 1, 1, 3, 4, 2, 0],$ $T_n = [1, 1, 1, 0, 0, 1, 3, 3, 3, 3, 1, 0, 0, 3, 3, 3, 3, 3, 3, 0, 0, 1, 3, 3, 3, 3, 1, 0, 0, 1, 1, 1, 1].$

All this means that there is a global period of $224 = 7 \times 32$ for these factors. The B_k verify the recurrence relation of order 27

$$
\mu_k \cdot B_{k-23} B_{k-22}^2 B_{k-21} B_{k-20} B_{k-19} B_{k-15}^2 B_{k-14} B_{k-13} B_{k-12}^2 B_{k-8} B_{k-7} B_{k-6} B_{k-5}^2 B_{k-4}
$$

+ $\nu_k \cdot B_{k-25} B_{k-24} B_{k-18} B_{k-17}^2 B_{k-10}^2 B_{k-9} B_{k-3} B_{k-2} - B_{k-27} B_k = 0$ (18)

where μ_k and ν_k are monomials in x, y, z, u, t with period 224. They can be evaluated from the explicit expression of p_k .

The generating function of both sequences (g_B for the degrees of the B's, and g_p for the p_k 's) are then readily obtained.

$$
g_B = \frac{P}{(s-1)(s+1)(s^6+s^5+s^4+s^3+s^2+s+1)(s^4+1)(s^8+1)(s^{16}+1)Q},
$$

with

$$
\begin{array}{rcl} P&=&s^{58}+s^{57}-s^{56}-2s^{55}-2s^{52}-2s^{51}+s^{50}+s^{49}-2s^{48}-2s^{47}+s^{46}-3s^{44}-2s^{43}\\ & &+2s^{42}+s^{41}-3s^{40}-2s^{39}+2s^{38}-4s^{36}-2s^{35}+3s^{34}+s^{33}-3s^{32}-s^{31}+2s^{30}\\ & &-3s^{29}-7s^{28}-3s^{27}+s^{26}-3s^{25}-5s^{24}-s^{23}+s^{22}-4s^{21}-6s^{20}-2s^{19}-4s^{17}\\ & &-4s^{16}-s^{15}-4s^{13}-5s^{12}-2s^{11}-s^{10}-4s^{9}-3s^{8}-3s^{5}-3s^{4}-s^{3}-s^{2}-2s-1, \end{array}
$$

and

$$
Q = s^{22} - s^{20} - s^{19} - s^{17} + s^{15} + s^{14} - s^{13} - s^{12} - s^{10} - s^9 + s^8 + s^7 - s^5 - s^3 - s^2 + 1.
$$

We also have

$$
g_p = \frac{R}{(1-s)(s+1)(s^2+1)(s^4+1)(s^6+s^5+s^4+s^3+s^2+s+1)(s^8+1)(s^{16}+1)Q}
$$

 \overline{D}

with

$$
R = 4 s^{59} + 9 s^{58} + 14 s^{57} + 16 s^{56} + 18 s^{55} + 15 s^{54} + 13 s^{53} + 10 s^{52} + 13 s^{51} + 15 s^{50}
$$

\n
$$
+ 19 s^{49} + 21 s^{48} + 22 s^{47} + 19 s^{46} + 15 s^{45} + 11 s^{44} + 11 s^{43} + 15 s^{42} + 18 s^{41}
$$

\n
$$
+ 23 s^{40} + 23 s^{39} + 22 s^{38} + 15 s^{37} + 11 s^{36} + 6 s^{35} + 6 s^{34} + 6 s^{33} + 12 s^{32} + 15 s^{31}
$$

\n
$$
18 s^{30} + 15 s^{29} + 13 s^{28} + 7 s^{27} + 6 s^{26} + 5 s^{25} + 10 s^{24} + 13 s^{23} + 20 s^{22} + 21 s^{21}
$$

\n
$$
+ 22 s^{20} + 17 s^{19} + 14 s^{18} + 10 s^{17} + 10 s^{16} + 13 s^{15} + 17 s^{14} + 20 s^{13} + 20 s^{12} + 18 s^{11}
$$

\n
$$
+ 14 s^{10} + 12 s^9 + 9 s^8 + 11 s^7 + 13 s^6 + 16 s^5 + 15 s^4 + 13 s^3 + 9 s^2 + 5 s + 1.
$$

The entropy is the log of the inverse of the smallest root of the polynomial Q (which also happens to be its largest root) approximately $log(1.400618098)$ in perfect agreement with the numerical evaluation obtained from the explicit calculation of the 55 first terms of the sequence of degrees of the iterates of the map.

The degrees of the numerator and the denominator (respectively 59 and 60) of the generating function g_p indicate that we would have needed to evaluate the degree of the first 119 iterates to use Method 1. This is beyond the capabilities of the presently available formal calculus software.

Notice that the denominator of g_p is just $Q(s^7-1)(s^{32}-1)/(s-1)$, and the two periods 7 and 32 appear there naturally (they are present as well in q_B).

All this is not quite a proof, but we can be we have the exact value of the entropy, and this value is once more the logarithm of an algebraic integer!.

Conclusion $\bf{5}$

The derived recurrence (18) has two remarkable properties, having to do with Laurent/'Somos like' characteristics.

The first one is obtained by construction: if we take as initial conditions the factors B_k obtained from the iterates of the generic point $[x, y, z, u, t]$ of \mathcal{P}_4 , all further B's are multivariate polynomials.

Moreover -and this is another *experimental* fact- if we start from $B_i = 1, i = 1..27$ then again all further B's are polynomials in (x, y, z, u, t) , and of course integers if (x, y, z, u, t) are themselves. This comes from the fact that recurrence (18) has the Laurent property for arbitrary $[x, y, z, u, t]$, as is easy to check explicitly on the first iterates. The proof will come later.

In summary, the 'derivation' process of recurrences provides us with a *factory of Somos like recurrences*, keeping in mind that it is not a mere change of coordinates. It is a complete change of description.

This raises a number of questions:

• The process defines sequences of multivariate polynomials. What are the properties of these polynomials?

- Is it possible to predict the order of the derived recurrence? We gave two examples. In the first one the original recurrence was of order 2 and the derived one of order 5, and for the latter the original order was 4 and the new one is 27.
- Iterating the derivation process could produce more recurrences but may as well reach a fixed point. This should be investigated. In fact, when applied to the Somos-4 recurrence, it just reproduces Somos-4 with a periodic decoration similar to the factors μ_k and ν_k seen in $(18)^2$ $(18)^2$.
- The stabilisation of the form of the iterates is necessary for the existence of a derived recurrence. How can we characterise the systems having this property? This question is crucial since it ensures that the entropy is the log of an algebraic integer.
- In the specific case of discrete integrable systems (vanishing entropy) the derived recurrence may take the form of the Hirota quadratic relation between τ functions. The link has to be clarified.

There would be much more to say about last item of the previous list: Discrete Integrable Systems. Among the contributors to the subject Decio Levi was there from the first day. He was at the origin (together with Pavel Winternitz, and Luc Vinet) of the series of SIDE conferences [\[33\]](#page-10-9), a very important series of meeting in the field and he took an active part, as early as 1994, organising more than one of the meetings. I have had a long interaction with Decio, not only at the occasion of these conferences, but also of visits to Roma Tre (where I made the acquaintance of his then Ph.D. student Giorgio Gubbiotti, now a collaborator). Decio was always very supportive, and discussions with him very constructive. He will be sorely missed.

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