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The solutions of classical and nonlocal nonlinear Schrödinger equations with nonzero backgrounds: Bilinearisation and reduction approach

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Abstract

In this paper we develop a bilinearisation-reduction approach to derive solutions to the classical and nonlocal nonlinear Schrödinger (NLS) equations with nonzero backgrounds. We start from the second order Ablowitz-Kaup-Newell-Segur coupled equations as an unreduced system. With a pair of solutions (q_0, r_0) we bilinearize the unreduced system and obtain solutions in terms of quasi double Wronskians. Then we implement reductions by introducing constraints on the column vectors of the Wronskians and finally obtain solutions to the reduced equations, including the classical NLS equation and the nonlocal NLS equations with reverse-space, reverse-time and reverse-space-time, respectively. With a set of plane wave solution (q_0, r_0) as a background solution, we present explicit formulae for these column vectors. As examples, we analyze and illustrate solutions to the focusing NLS equation and the reverse-space nonlocal NLS equation. In particular, we present formulae for the rouge waves of arbitrary order for the focusing NLS equation.

1 Introduction

It is common knowledge that the nonlinear Schrödinger-type equations admit carrier waves and solitons, and that breathers and other solutions (e.g. rogue waves) are the modulations of carrier waves. Meanwhile, many (1+1)-dimensional soliton equations admit solitons with either zero or nonzero asymptotic behaviours as $|x| \to \infty$. As for the one of the most popular nonlinear integrable models, the focusing nonlinear Schrödinger (NLS) equation,

$$iq_t = q_{xx} + 2|q|^2 q, \tag{1}$$

where i is the imaginary unit, $|q|^2 = qq^*$ and q^* stands for the complex conjugate of q, early investigations of the solutions of this equation with nonzero boundary conditions were due to Kuznetsov [40], Kawata and Inoue [38, 39] and Ma [47]. They solved the NLS equation (1) with nonzero boundary conditions, i.e., $|q(x,t)| \to$ const. as $x \to \pm\infty$, by means of the inverse scattering transform. Faddeev and Takhtajan have also done important work in this area (see for instance the monograph Ref.[25] and references therein). Besides, the NLS equation with different asymmetric nonzero boundary conditions has been studied in [20, 37, 14, 21, 59]. The defocusing NLS equation,

$$iq_t + q_{xx} - 2|q|^2 q = 0, \tag{2}$$

has dark solitons with nonzero boundary condition ($|q|$ goes to a positive constant as $|x| \to \infty$). Zakharov and Shabat are pioneers who studied the two NLS equations using tools of integrability [67, 68]. Hirota derived bright soliton solution for the equation (1) and dark soliton solution for (2) by using bilinear method, respectively in [34] and [36]. For more references about the integrability of NLS equations one can refer to [8] and the references therein.

From the point of view of the Darboux transformation, for any seed solution q_0 of the NLS equation (1) (i.e. q_0 satisfies (1)), the envelope of the solution q generated from the Darboux transformation has a form (see equation (4.3.10) in [48])

$$|q|^2 = |q_0|^2 + \partial_x^2 \ln f. \tag{3}$$

In this context, when $q_0 \neq 0$, we say the resulting solution q is a solution with a nonzero background q_0. Various methods for the systematic construction of solutions of equation (1) with a plane wave background $q_0 = \sqrt{\alpha} e^{-2i\alpha t}$ have been established, where α is a positive constant. One can replace q with $qe^{-2i\alpha t}$ in equation (1), which leads it to the form

$$iq_t = q_{xx} + 2(|q|^2 - \alpha)q. \tag{4}$$

Mathematically, this implies, compared with (1), that the envelope $|q|$ gains a positive lift $\sqrt{\alpha}$ such that $|q| \to \sqrt{\alpha}$ as $|x| \to \infty$. However, the plane wave background does bring interesting behavior of $|q|$ more than that. The simplest solution (corresponding to one-soliton) of the equation (4) is a breather [39, 40, 47], not the usual soliton. In a special limit the breather yields a localized rational solution [52], which is nowadays used to describe a rogue wave. The rational solution was also derived by Matveev and Salle via the Darboux transformation (see §4.3 in [48], where the rogue wave is called "exulton" solution). The second order rational solution of equation (4) was derived in 1985 in [10], using a similar way as in [52]. The rational solution of arbitrary order of the NLS equation was first constructed in 1986 in [24], where explicit formula of the solution was presented in an elegant way and nonsingular property of the solution was proved as well (see also [23] for an alternative proof). "Rogue waves" is the name given by oceanographers to isolated large amplitude waves, which occur more frequently than expected for normal, Gaussian distributed, statistical events (cf.[51]). After rogue waves was observed in optic experiment in 2007 [57], it started to draw new attention and the research on rogue waves has become a hot topic. One can refer to the review [51] for more references. Mathematically, higher order rational solutions of the NLS equation (4) can be obtained using the Darboux transformation via a special limit procedure [29, 32], from a bilinear approach using reduction of the Kadomtsev-Petviashvili τ functions [50], and from inverse scattering transform [13]. There are also some research of the NLS equation on the elliptic function background, e.g. [17, 26].

In this paper, we will derive solutions with a plane wave background for the NLS equation (1) and (2) and their nonlocal versions by using bilinear method but in a completely different way from [50].

Our idea is to solve the second order Ablowitz-Kaup-Newell-Segur (AKNS) coupled equations

$$iq_t = q_{xx} - 2q^2 r, \tag{5a}$$

$$ir_t = -r_{xx} + 2r^2 q \tag{5b}$$

as an unreduced system, which, for instance, yields the NLS equation (1) via reduction $r = -q^*$. We can bilinearize this unreduced system and present solutions of the bilinear equations in terms of double Wronskians. Then, we impose constraints on the column vectors of the double Wronskians so that the desired reduction holds and thus we get solutions to the reduced equation. Such an idea was first proposed in [18, 19] for obtaining solutions for the nonlocal integrable equations. Nonlocal integrable systems were first systematically proposed by Ablowitz and Musslimani in 2013 [5] and have drawn intensive attention (e.g.[72, 1, 63, 65, 7, 45, 15, 4, 53, 30, 46, 54, 55]). The bilinearisation-reduction approach has proved effective in deriving solutions not only to the nonlocal systems but also to the classical equations (e.g. [22, 42, 43, 44, 56, 60, 61]). In this paper, we introduce transformation

$$q = q_0 + \frac{g}{f}, \quad r = r_0 + \frac{h}{f} \tag{6}$$

for the unreduced system (5). Here (q_0, r_0) are an arbitrary set of solution of (5). It will be seen that for the NLS equation (1), $|q_0|$ does act as a background of the envelope $|q|$, see equation (4.3.10) in [48] and equation (73) in this paper). In this context we also call (q_0, r_0) a set of background solution of the system (5). We will employ (6) to bilinearize the unreduced system (5) and present (quasi) double Wronskian solutions to the bilinear equations. Then we will implement reduction technique to obtain solutions to the reduced equations listed in (9)-(12).

The paper is organized as follows. In Sec.2 we recall the classical and nonlocal reductions of the unreduced AKNS system (5). In Sec.3 we derive the bilinear form of (5) with a set of background solution (q_0, r_0) and derive (quasi) double Wronskian solutions to the bilinear equations. In Sec.4 the reduction technique is implemented and explicit form of solutions with plane wave background solutions are obtained for the reduced equations. Then in Sec.5 we investigate dynamics of some obtained solutions for the classical NLS equation and the nonlocal NLS equations with nonzero backgrounds. Finally, Sec.6 serves for presenting conclusions.

2 The second order AKNS system and its reductions

The second order coupled AKNS equations (5), where $q = q(x,t)$ and $r = r(x,t)$ are functions of $(x,t) \in \mathbb{R}^2$, has been studied as a classical coupled system in past decades. Recently it was found that this system is related to the cubic nonlinear Klein-Gordon equation, see [7]. Its Lax pairs consist of the well known AKNS (or Zakharov-Shabat (ZS)-AKNS) spectral problem [67, 2, 3],

$$\begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_x = M \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}, \quad M = \begin{pmatrix} \lambda & q \\ r & -\lambda \end{pmatrix}, \tag{7}$$

and the corresponding time evolution part

$$i \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}_t = N \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}, \quad N = \begin{pmatrix} 2\lambda^2 - qr & 2\lambda q + q_x \\ 2\lambda r - r_x & -2\lambda^2 + qr \end{pmatrix}, \tag{8}$$

in which λ is spectral parameter, $\lambda_t = 0$, Φ and Ψ are wave functions.

In the following we list possible one-component equations reduced from equation (5). These equations will be considered in this paper. Equation (5) admits the following reductions (see [6] and reference therein)

$$iq_t = q_{xx} - 2\delta q^2 q^*, \qquad r = \delta q^*, \tag{9}$$
$$iq_t = q_{xx} - 2\delta q^2 q^*(-x), \qquad r = \delta q^*(-x), \tag{10}$$
$$iq_t = q_{xx} - 2\delta q^2 q(-t), \qquad r = \delta q(-t), \tag{11}$$
$$iq_t = q_{xx} - 2\delta q^2 q(-x,-t), \qquad r = \delta q(-x,-t), \tag{12}$$

where $\delta = \pm 1$, $q(-x) = q(-x,t)$, $q(-t) = q(x,-t)$ and $q(-x,-t)$ indicate the reverse-space, reverse-time and reverse-space-time, respectively.

3 Bilinearisation and solutions of the AKNS system (5)

In this section, we develop the double Wronskian technique to construct exact solutions of the second order AKNS system (5) with nonzero background solution (q_0, r_0).

3.1 Bilinearisation

Suppose that (q_0, r_0) are a set of solution to the second order AKNS system (5). To introduce nonzero backgrounds, we consider the dependent variable transformation (i.e. (6))

$$q = q_0 + \frac{g}{f}, \quad r = r_0 + \frac{h}{f}, \tag{13}$$

with which the system (5) can be decoupled into the following bilinear form of f, g and h,

$$D_x^2 f \cdot f = -2gh - 2q_0 hf - 2r_0 gf, \tag{14a}$$
$$(D_x^2 - iD_t - 2q_0 r_0) g \cdot f + q_0 D_x^2 f \cdot f = 0, \tag{14b}$$
$$(D_x^2 + iD_t - 2q_0 r_0) h \cdot f + r_0 D_x^2 f \cdot f = 0, \tag{14c}$$

where D_x and D_t are the well known Hirota bilinear operators defined as [35]

$$D_x^m D_t^n f \cdot g = (\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^n f(x,t) g(x',t')|_{x'=x, t'=t}.$$

Note that when $q_0 = r_0 = 0$ the above bilinear form (14) degenerates to the case of zero background (cf. equations (1.5.1)-(1.5.3) in [16]).

To have solutions of (14), we expanding f, g and h as the series

$$f(x,t) = 1 + f^{(2)}\varepsilon^2 + f^{(4)}\varepsilon^4 + \cdots + f^{(2j)}\varepsilon^{2j} + \cdots, \tag{15a}$$

$$g(x,t) = g^{(1)}\varepsilon + g^{(3)}\varepsilon^3 + \cdots + g^{(2j+1)}\varepsilon^{2j+1} + \cdots, \tag{15b}$$

$$h(x,t) = h^{(1)}\varepsilon + h^{(3)}\varepsilon^3 + \cdots + h^{(2j+1)}\varepsilon^{2j+1} + \cdots, \tag{15c}$$

where ε is an arbitrary number, $\{f^{(j)}, g^{(k)}, h^{(l)}\}$ are functions to be determined. Consider a special case,

$$q_0 = A_0 e^{2iA_0^2 t}, \quad r_0 = A_0 e^{-2iA_0^2 t}, \tag{16}$$

where A_0 is an arbitrary constant. By calculation we can find out 1-, 2- and 3-soliton solutions, which agree with the following general expressions,

$$g = A_0 e^{2iA_0^2 t} \sum_{\mu=0,1} \exp\Bigg[\sum_{j=1}^{N} \mu_j(\xi_j + a_j) + \sum_{1\le i<j}^{N} \mu_i\mu_j B_{ij}\Bigg], \tag{17a}$$

$$h = A_0 e^{-2iA_0^2 t} \sum_{\mu=0,1} \exp\Bigg[\sum_{j=1}^{N} \mu_j(\xi_j + b_j) - \sum_{1\le i<j}^{N} \mu_i\mu_j B_{ij}\Bigg], \tag{17b}$$

$$f = \sum_{\mu=0,1} \exp\Bigg[\sum_{j=1}^{N} \mu_j\xi_j + \sum_{1\le i<j}^{N} \mu_i\mu_j A_{ij}\Bigg], \tag{17c}$$

where

$$e^{a_j} = \frac{-2k_j^4 - 2ik_j^2 w_j}{k_j^4 + w_j^2}, \quad e^{b_j} = \frac{-2k_j^4 + 2ik_j^2 w_j}{k_j^4 + w_j^2}, \tag{17d}$$

$$\xi_j = k_j x + \omega_j t + \xi_j^{(0)}, \quad \omega_j = \pm\sqrt{4A_0^2 k_j^2 - k_j^4}, \tag{17e}$$

$$e^{A_{ij}} = \frac{-k_i^2 k_j^2 + 2A_0^2\left(k_i^2 + k_j^2\right) - \sqrt{4A_0^2 k_i^2 - k_i^4}\sqrt{4A_0^2 k_j^2 - k_j^4}}{2A_0^2(k_i + k_j)^2}, \tag{17f}$$

$$e^{B_{ij}} = \frac{i(k_i - k_j)\left(-k_j^2\sqrt{4A_0^2 k_i^2 - k_i^4} + k_i^2\sqrt{4A_0^2 k_j^2 - k_j^4}\right)}{2k_i^2 k_j^2 (k_i + k_j)}, \tag{17g}$$

the summation of μ means to take all possible $\mu_j = \{0, 1\}$ for $j = 1, 2, \cdots, N$.

Note that in [41] the system (5) was also bilinearized via the following transformation

$$q = \frac{G}{F}, \quad r = \frac{H}{F},$$

and the bilinear form is

$$(D_x^2 - iD_t - \lambda)G \cdot F = 0,$$
$$(D_x^2 + iD_t - \lambda)H \cdot F = 0,$$
$$(D_x^2 - \lambda)F \cdot F = -2GH,$$

where $\lambda \in \mathbb{R}$. One-soliton and two-soliton solutions they derived (see equation (58) and (82) in [41]) are shown to be associated with a more general plane wave background (see (48), which degenerates to (16) when $a = 0$ and $A_0 = B_0$).

3.2 Quasi double Wronskian solutions of the AKNS system (5)

We now derive double Wronskian solutions of the second order AKNS system (5).

We extend the Lax pair (7) and (8) to the following matrix system

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix}_x = \mathcal{M} \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} A & q_0 I_{2m+2} \\ r_0 I_{2m+2} & -A \end{pmatrix}, \tag{18}$$

$$i \begin{pmatrix} \phi \\ \psi \end{pmatrix}_t = \mathcal{N} \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} 2A^2 - q_0 r_0 I_{2m+2} & 2Aq_0 + q_{0,x} I_{2m+2} \\ 2Ar_0 - r_{0,x} I_{2m+2} & -2A^2 + q_0 r_0 I_{2m+2} \end{pmatrix}, \tag{19}$$

where $A \in \mathbb{C}_{(2m+2)\times(2m+2)}$ is an arbitrary complex matrix, I_{2m+2} is the $(2m+2)$-th order identity matrix, ϕ and ψ are two $(2m+2)$-th column vectors. Introduce vectors ϕ_k and ψ_k by

$$\phi_k = A^k \phi, \quad \psi_k = (-A)^k \psi, \tag{20}$$

and define determinants[1]

$$f = |\widehat{\phi}_m; \widehat{\psi}_m|, \quad g = 2|\widehat{\phi}_{m+1}; \widehat{\psi}_{m-1}|, \quad h = -2|\widehat{\phi}_{m-1}; \widehat{\psi}_{m+1}|, \tag{21}$$

where $\widehat{\phi}_m$ stands for the consecutive columns $(\phi_0, \phi_1, \cdots, \phi_m)$. Then, solutions of the bilinear system (14) are described as the following.

Theorem 1. The bilinear system (14) has solutions (21), where ϕ and ψ in (20) satisfy (18) and (19), and (q_0, r_0) are given solutions of the system (5). Furthermore, matrix A and any matrix that is similar to it lead to the same solution of the AKNS system (5) through the transformation (13).

The proof will be sketched in Appendix A. Later we only need to consider the canonical forms of A, i.e. A being diagonal or a Jordan block.

Strictly speaking, the above f, g, h in (21) are not double Wronskians that are defined by arranging columns by increasing the order of derivatives of ϕ and ψ. We may call them quasi double Wronskians. Note that when A is diagonal the results in Theorem 1 are the same as those derived from the Darboux transformation (cf. §4.2 in [48]). When A is a Jordan block, the corresponding solutions can be obtained using a limit procedure from those solutions which are derived from a diagonal matrix A (e.g. §4.3 in [48]). We also note that when the background solution (q_0, r_0) is independent of x, we may covert f, g, h given in (21) to double Wronskians.

Theorem 2. Suppose that the $(2m+2)\times(2m+2)$ double Wronskians[2]

$$f = (-1)^m |\widehat{m}; \widehat{m}|, \ g = -2|\widehat{m+1}; \widehat{m-1}|, \ h = 2|\widehat{m-1}; \widehat{m+1}|, \tag{22}$$

[1] When $m = 0$ we have $g = 2|\phi_0, \phi_1|$, $h = -2|\psi_0, \psi_1|$.

[2] When $m = 0$, g and h take the form $g = -2|\phi, \partial_x \phi|$, $h = 2|\psi, \partial_x \psi|$.

where $|\widehat{n};\widehat{m}|$ denotes a $(m+n+2)$ double Wronskian defined as (see [49])

$$|\widehat{m+j};\widehat{m-j}| = |0,1,\cdots,m+j;0,1,\cdots,m-j| = |\phi,\partial_x\phi,\ldots,\partial_x^{m+j}\phi;\psi,\partial_x\psi,\ldots,\partial_x^{m-j}\psi|,$$

ϕ and ψ are $(2m+2)$-th order column vectors. When ϕ and ψ meet the condition

$$\phi_x = A\phi + q_0\psi, \quad i\phi_t = 2\phi_{xx} - q_0r_0\phi - 2q_0\psi_x, \tag{23a}$$

$$\psi_x = r_0\phi - A\psi, \quad i\psi_t = -2\psi_{xx} + q_0r_0\psi + 2r_0\phi_x, \tag{23b}$$

where A is a $(2m+2)\times(2m+2)$ complex matrix, (q_0,r_0) satisfy (5) but are independent of x, then f,g,h defined in (22) are solutions to the bilinear equations (14). Furthermore, matrix A and any matrix that is similar to it lead to the same solution to the AKNS system (5) through the transformation (13).

The proof will be given in Appendix B.

4 Reduction and solutions

For convenience we call (5) the unreduced system and (9)-(12) the reduced equations. In the previous section we have already obtained solutions in terms of quasi double Wronskians (21) (see Theorem 1 for the unreduced system (5)). In this section we implement reductions by imposing constraints on A and ψ so that (21) can provides solutions to the reduced equations (9)-(12). Such a reduction technique was first introduced in [18, 19].

4.1 Reduction of the Wronskian solution

Let us directly present results and then prove them.

Theorem 3. Let A and T be matrices in $\mathbb{C}_{(2m+2)\times(2m+2)}$. Solutions of the reduced equations (9)-(12) are given in the following, respectively.
(1) The classical NLS equation (9) has solution

$$q = q_0 + \frac{g}{f}, \tag{24a}$$

$$f = |\widehat{\phi}_m; T\widehat{\phi}^*_m|, \quad g = 2|\widehat{\phi}_{m+1}; T\widehat{\phi}^*_{m-1}|, \tag{24b}$$

where q_0 is a solution of equation (9) such that $r_0^* = \delta q_0$, vector ϕ is a solution of matrix equations

$$\phi_x = A\phi + q_0T\phi^*, \tag{25a}$$

$$i\phi_t = (2A^2 - \delta q_0q_0^*I_{2m+2})\phi + (2Aq_0 + q_{0x}I_{2m+2})T\phi^*, \tag{25b}$$

and A and T obey the relation

$$AT + TA^* = 0, \quad TT^* = \delta I_{2m+2}, \quad \delta = \pm 1. \tag{26}$$

(2) For the reverse-space nonlocal NLS equation (10), its solution is given by

$$q = q_0 + \frac{g}{f}, \tag{27a}$$

$$f = (-1)^{\frac{m(m+1)}{2}}|\widehat{\phi}_m; T\widehat{\phi}^*_m(-x)|, \quad g = 2(-1)^{\frac{m(m-1)}{2}}|\widehat{\phi}_{m+1}; T\widehat{\phi}^*_{m-1}(-x)|, \tag{27b}$$

where q_0 is a solution of equation (10) such that $r_0^*(x) = \delta q_0(-x)$, vector ϕ is a solution of matrix equations

$$\phi_x(x) = A\phi(x) + q_0(x)T\phi^*(-x), \tag{28a}$$
$$i\phi_t(x) = (2A^2 - \delta q_0(x)q_0^*(-x)I_{2m+2})\phi(x) + (2Aq_0 + q_{0,x}(x)I_{2m+2})T\phi^*(-x), \tag{28b}$$

and A and T obey the relation

$$AT - TA^* = 0, \quad TT^* = -\delta I_{2m+2}, \quad \delta = \pm 1. \tag{29}$$

(3) For the reverse-time nonlocal NLS equation (11), its solution is given by

$$q = q_0 + \frac{g}{f}, \tag{30a}$$
$$f = |\widehat{\phi}_m; T\widehat{\phi}_m(-t)|, \quad g = 2|\widehat{\phi}_{m+1}; T\widehat{\phi}_{m-1}(-t)|, \tag{30b}$$

where q_0 is a solution of equation (11) such that $r_0(t) = \delta q_0(-t)$, ϕ is a solution of matrix equations

$$\phi_x(t) = A\phi(t) + q_0(t)T\phi(-t), \tag{31a}$$
$$i\phi_t(t) = (2A^2 - \delta q_0(t)q_0(-t)I_{2m+2})\phi(t) + (2Aq_0(t) + q_{0,x}(t)I_{2m+2})T\phi(-t), \tag{31b}$$

and A and T obey the relation

$$AT + TA = 0, \quad T^2 = \delta I_{2m+2}, \quad \delta = \pm 1. \tag{32}$$

(4) For the reverse-space-time nonlocal NLS equation (12), its solution is given by

$$q = q_0 + \frac{g}{f}, \tag{33a}$$
$$f = (-1)^{\frac{m(m+1)}{2}}|\widehat{\phi}_m; T\widehat{\phi}_m(-x,-t)|, \quad g = 2(-1)^{\frac{m(m-1)}{2}}|\widehat{\phi}_{m+1}; T\widehat{\phi}_{m-1}(-x,-t)|, \tag{33b}$$

where q_0 is a solution of equation (12) such that $r_0(x,t) = \delta q_0(-x,-t)$, vector ϕ is a solution of matrix equations

$$\phi_x(x,t) = A\phi(x,t) + q_0(x,t)T\phi(-x,-t), \tag{34a}$$
$$i\phi_t(x,t) = (2A^2 - \delta q_0(x,t)q_0(-x,-t))\phi(x,t) + (2Aq_0(x,t) + q_{0,x}(x,t))T\phi(-x,-t), \tag{34b}$$

and A and T obey the relation

$$AT - TA = 0, \quad T^2 = -\delta I_{2m+2}, \quad \delta = \pm 1. \tag{35}$$

Proof. We employ the classical NLS equation (9) as an illustrating example. Introduce constraint on ψ,

$$\psi = T\phi^*, \tag{36}$$

where T is a certain matrix in $\mathbb{C}_{(2m+2)\times(2m+2)}$. First, it can be verified that when $r_0 = \delta q_0^*$ and A and T satisfy (26), the constraint (36) reduces (18) and (19) to (25). In fact, taking (18) as an example, under (36) and $r_0 = \delta q_0^*$, we rewrite (18) as

$$\phi_x = A\phi + q_0 T\phi^*, \tag{37a}$$
$$T\phi_x^* = \delta q_0^*\phi - AT\phi^*, \tag{37b}$$

where (37a) is nothing but (25a). Making use of (26), equation (37b) multiplied by δT^* from the left gives rise to the complex conjugate of (37a). This indicates (18) reduces to (25a). In a similar way one can find (19) reduces to (25b) in this case.

Next, with the constraint (36), we can rewrite the quasi double Wronskians (21) as

$$f = |\widehat{\phi}_m; \widehat{\psi}_m| = |\widehat{\phi}_m; T\widehat{\phi}^*_m|, \tag{38a}$$
$$g = 2|\widehat{\phi}_{m+1}; \widehat{\psi}_{m-1}| = 2|\widehat{\phi}_{m+1}; T\widehat{\phi}^*_{m-1}|, \tag{38b}$$
$$h = -2|\widehat{\phi}_{m-1}; \widehat{\psi}_{m+1}| = -2|\widehat{\phi}_{m-1}; T\widehat{\phi}^*_{m+1}|. \tag{38c}$$

Making use of (26) we find that

$$f = |\widehat{\phi}_m; T\widehat{\phi}^*_m| = |T||\delta T^*\widehat{\phi}_m; \widehat{\phi}^*_m|.$$

Then, switching the first $(m+1)$ columns and the last $(m+1)$ columns and picking the parameter δ out yield

$$f = (-\delta)^{m+1}|T||\widehat{\phi}^*_m; T^*\widehat{\phi}_m| = (-\delta)^{m+1}|T|f^*.$$

In a similar way we can prove

$$h = -(-\delta)^m |T| g^*.$$

Thus we have

$$\frac{r}{q^*} = \frac{r_0 + h/f}{q_0^* + g^*/f^*} = \frac{\delta q_0^* + \delta g^*/f^*}{q_0^* + g^*/f^*} = \delta,$$

i.e. $r = \delta q^*$, which is the reduction by which we get (9) from (5).

The proof of nonlocal cases is similar to the classical one. For the reverse-space nonlocal NLS equation (10), the reduction is implemented by taking

$$\psi = T\phi^*(-x) \tag{39}$$

together with (29). Here and below we note that we do not write out independent variables unless the inverse of them are involved. Relations between Wronskians are

$$f = \delta^{m+1}|T|f^*(-x), \quad h = \delta^m |T| g^*(-x),$$

which yield

$$\frac{r}{q^*(-x)} = \frac{r_0 + h/f}{q_0^*(-x) + g^*(-x)/f^*(-x)} = \frac{\delta q_0^*(-x) + \delta g^*(-x)/f^*(-x)}{q_0^*(-x) + g^*(-x)/f^*(-x)} = \delta,$$

i.e. $r = \delta q^*(-x)$, which reduces the unreduced system (5) to equation (10).

For the reverse-time nonlocal NLS equation (11), the reduction is implemented by taking

$$\psi = T\phi(-t) \tag{40}$$

together with (32). Relations between Wronskians are

$$f = (-\delta)^{m+1}|T|f(-t), \quad h = -(-\delta)^m|T|g(-t),$$

which yield

$$\frac{r}{q(-t)} = \frac{r_0 + h/f}{q_0(-t) + g(-t)/f(-t)} = \frac{\delta q_0(-t) + \delta g(-t)/f(-t)}{q_0(-t) + g(-t)/f(-t)} = \delta,$$

i.e. $r = \delta q(-t)$, which reduces (5) to (11).

For the reverse-space-time nonlocal NLS equation (12), we start from

$$\psi = T\phi(-x,-t) \tag{41}$$

and (35). Relations between Wronskians are

$$f = \delta^{m+1}|T|f(-x,-t), \quad h = \delta^m|T|g(-x,-t),$$

which yield

$$\frac{r}{q(-x,-t)} = \frac{r_0 + h/f}{q_0(-x,-t) + g(-x,-t)/f(-x,-t)} = \frac{\delta q_0(-x,-t) + \delta g(-x,-t)/f(-x,-t)}{q_0(-x,-t) + g(-x,-t)/f(-x,-t)} = \delta,$$

i.e. $r = \delta q(-x,-t)$, which reduces (5) to (12).

■

4.2 Matrices A and T

We look for explicit forms of A and T in Theorem 3. Equations (26) and (29) can be unified to be

$$AT + \sigma TA^* = 0, \quad TT^* = \sigma\delta I, \quad \sigma, \delta = \pm 1, \tag{42}$$

and equations (32) and (35) can be unified to be

$$AT + \sigma TA = 0, \quad T^2 = \sigma\delta I, \quad \sigma, \delta = \pm 1. \tag{43}$$

Consider special solutions to these matrix equations, i.e. A and T are block matrices

$$A = \begin{pmatrix} K_1 & \mathbf{0} \\ \mathbf{0} & K_4 \end{pmatrix}, \quad T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}, \tag{44}$$

where T_i and K_i are $(m+1)\times(m+1)$ matrices. Then solutions to equations (42) and (43) can be listed out as in the Table 1 and 2, cf.[18].

In addition, equation (29) admits real solution in the form (44) for the case $\delta = -1$, where

$$K_1 = \mathbf{K}_{m+1}, K_4 = \mathbf{H}_{m+1}, \ \mathbf{K}_{m+1}, \mathbf{H}_{m+1} \in \mathbb{R}_{(m+1)\times(m+1)}, \tag{45a}$$

$$T_1 = -T_4 = \mathbf{I}_{m+1}, T_2 = T_3 = \mathbf{0}_{m+1}, \tag{45b}$$

$$\text{or } T_1 = T_4 = \mathbf{I}_{m+1}, T_2 = T_3 = \mathbf{0}_{m+1}. \tag{45c}$$

Table 1. T and A for equation (42)

	(σ,δ)	T	A
(26)	$(1,1)$	$T_1=T_4=\mathbf{0}_{m+1}, T_2=T_3=\mathbf{I}_{m+1}$	$K_1=\mathbf{K}_{m+1}, K_4=-\mathbf{K}^*_{m+1}$
	$(1,-1)$	$T_1=T_4=\mathbf{0}_{m+1}, T_2=-T_3=\mathbf{I}_{m+1}$	
(29)	$(-1,1)$	$T_1=T_4=\mathbf{0}_{m+1}, T_2=-T_3=\mathbf{I}_{m+1}$	$K_1=\mathbf{K}_{m+1}, K_4=\mathbf{K}^*_{m+1}$
	$(-1,-1)$	$T_1=T_4=\mathbf{0}_{m+1}, T_2=T_3=\mathbf{I}_{m+1}$	

Table 2. T and A for equation (43)

	(σ,δ)	T	A
(32)	$(1,1)$	$T_1=T_4=\mathbf{0}_{m+1}, T_2=T_3=\mathbf{I}_{m+1}$	$K_1=\mathbf{K}_{m+1}, K_4=-\mathbf{K}_{m+1}$
	$(1,-1)$	$T_1=T_4=\mathbf{0}_{m+1}, T_2=-T_3=\mathbf{I}_{m+1}$	
(35)	$(-1,1)$	$T_1=-T_4=i\mathbf{I}_{m+1}, T_2=T_3=\mathbf{0}_{m+1}$	$K_1=\mathbf{K}_{m+1}, K_4=\mathbf{H}_{m+1}$
	$(-1,-1)$	$T_1=-T_4=\mathbf{I}_{m+1}, T_2=T_3=\mathbf{0}_{m+1}$	

Besides, equation (26) with $\delta=1$ can have pure imaginary solution (44) and (45) where in (45a) $\mathbf{K}_{m+1},\mathbf{H}_{m+1}\in i\mathbb{R}_{(m+1)\times(m+1)}$.

Due to the fact that A and any matrix that is similar to it generate same solutions to the system (5) (see Theorem 1), we only need to consider the canonical forms[3] of A. That is, $\mathbf{K}_{m+1}$ can either be

$$\mathbf{K}_{m+1}=\mathrm{Diag}(k_1,k_2,\cdots,k_{m+1}),\quad k_i\in\mathbb{C}, \tag{46}$$

or $\mathbf{K}_{m+1}=J_{m+1}(k),\ k\in\mathbb{C}$, where

$$J_{m+1}(k)=\begin{pmatrix} k & 0 & 0 & \dots & 0 & 0\\ 1 & k & 0 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 1 & k \end{pmatrix}_{(m+1)\times(m+1)}. \tag{47}$$

4.3 Plane wave background solution q_0

Considering the expression (13) (and also (73)) we can call q_0 and r_0 to be background solutions of q and r, respectively. The unreduced system (5) admits a set of plane wave solutions

$$q_0=A_0e^{i[ax+(a^2+2A_0B_0)t]},\quad r_0=B_0e^{-i[ax+(a^2+2A_0B_0)t]}, \tag{48}$$

[3] A general case for $\mathbf{K}_{m+1}$ is the block diagonal form

$$\mathbf{K}_{m+1}=\mathrm{Diag}\left(J_{h_1}(k_1),J_{h_2}(k_2),\cdots,J_{h_s}(k_s),\mathrm{Diag}(k_{s+1},\cdots,k_{s+n})\right),$$

where each $J_{h_j}(k_j)$ is an $h_j\times h_j$ Jordon block matrix defined as (47), $\mathrm{Diag}(k_{s+1},\cdots,k_{s+n})$ is an $n\times n$ diagonal matrix and $n+\sum_{j=1}^{s}h_j=m+1$. In this case, ϕ is just composed accordingly since (18) and (19) are linear system of ϕ and ψ. Thus, we will only consider two limiting cases, (46) and (47). Note that matrix A corresponds to the eigenvalues of the AKNS spectral problem (7), which means (46) is for the case of simple distinct eigenvalues and (47) is for the case of one eigenvalue of geometric multiplicity one and algebraic multiplicity $m+1$.

where A_0, B_0 and a are arbitrary complex constants. It is easy to find that the reduced classical and nonlocal NLS equations (9)-(12) admit the following solutions, respectively,

$$q_0 = A_0 e^{i[ax+(a^2+2\delta|A_0|^2)t]}, \quad (A_0 \in \mathbb{C},\ a \in \mathbb{R}), \tag{49a}$$

$$q_0 = A_0 e^{-ax+i(-a^2+2\delta|A_0|^2)t}, \quad (A_0 \in \mathbb{C},\ a \in \mathbb{R}), \tag{49b}$$

$$q_0 = A_0 e^{2i\delta A_0^2 t}, \quad (A_0 \in \mathbb{C}), \tag{49c}$$

$$q_0 = A_0 e^{i[ax+(a^2+2\delta A_0^2)t]}, \quad (A_0, a \in \mathbb{C}). \tag{49d}$$

Our purpose is to write out explicit Wronskian vectors ϕ that respectively satisfy the conditions (25), (28), (31) and (34) for given background solutions q_0. We are going to consider the simple case where q_0 are given in (49). If making use of some symmetries, we may start from a simpler background solution

$$q_0 = e^{2i\delta t} \tag{50}$$

instead of (49).

Proposition 1. The classical and nonlocal NLS equations (9)-(12) admit the following symmetries.
(1) Classical NLS equation (9):

- Galilean symmetry: if $q(x,t)$ solves the NLS equation (9) with the background solution q_0 given in (49a), then $Q(X,Y) = q(x,t)e^{-iax-ia^2t}$, where $X = x + 2at$ and $Y = t$, solves the NLS equation (9) with $Q(X,Y)$, i.e.

$$iQ_Y = Q_{XX} - 2\delta Q^2 Q^*, \tag{51}$$

  of which $Q_0(X,Y) = q_0 e^{-iax-ia^2t} = A_0 e^{2i\delta|A_0|^2 Y}$ is a solution.

- Scaling symmetry: if $q(x,t)$ solves the NLS equation (9) with a background solution $q_0(x,t) = A_0 e^{2i\delta|A_0|^2 t}$, then $Q(X,Y) = \frac{1}{A_0}q(x,t)$, where $X = |A_0|x$ and $Y = |A_0|^2 t$, also solves the NLS equation (51), of which $Q_0(X,Y) = \frac{1}{A_0}q_0 = e^{2i\delta Y}$ is a solution.

(2) Reverse-space nonlocal NLS equation (10):

- Galilean symmetry: if $q(x,t)$ solves the reverse-space NLS equation (10) with the background solution $q_0(x,t)$ given in (49b), then $Q(X,Y) = q(x,t)e^{ax+ia^2t}$, where $X = x + 2iat$ and $Y = t$, also solves the reverse-space NLS equation (10) with $Q(X,Y)$, i.e.

$$iQ_Y = Q_{XX} - 2\delta Q^2 Q^*(-X), \tag{52}$$

  of which $Q_0(X,Y) = q_0 e^{ax+ia^2t} = A_0 e^{2i\delta|A_0|^2 Y}$ is a solution.

- Scaling symmetry: if $q(x,t)$ solves the reverse-space NLS equation (10) with a background solution $q_0(x,t) = A_0 e^{2i\delta|A_0|^2 t}$, then $Q(X,Y) = \frac{1}{A_0}q(x,t)$, where $X = |A_0|x,\ Y = |A_0|^2 t$, also solves the reverse-space NLS equation (52) of which $Q_0(X,Y) = \frac{1}{A_0}q_0 = e^{2i\delta Y}$ is a solution.

(3) Reverse-time nonlocal NLS equation (11):

- Scaling symmetry: if $q(x,t)$ solves the reverse-time NLS equation (11) with the background solution $q_0(x,t)$ given in (49c), then $Q(X,Y)=\frac{1}{A_0}q(x,t)$, where $X=A_0x$ and $Y=A_0^2t$, also solves the reverse-time NLS equation (11) with $Q(X,Y)$, i.e.

$$iQ_Y=Q_{XX}-2\delta Q^2Q(-Y), \tag{53}$$

of which $Q_0(X,Y)=\frac{1}{A_0}q_0=e^{2i\delta Y}$ is a solution.

(4) Reverse-space-time nonlocal NLS equation (12):

- Galilean symmetry: if $q(x,t)$ solves the reverse-space-time NLS equation (12) with the background solution $q_0(x,t)$ given in (49d), then $Q(X,Y)=q(x,t)e^{-iax-ia^2t}$, where $X=x+2at$ and $Y=t$, also solves the reverse-space-time NLS equation (12) with $Q(X,Y)$, i.e.

$$iQ_Y=Q_{XX}-2\delta Q^2Q(-X,-Y), \tag{54}$$

of which $Q_0(X,Y)=q_0e^{-iax-ia^2t}=A_0e^{2i\delta A_0^2Y}$ is a solution.

- Scaling symmetry: if $q(x,t)$ solves the reverse-space-time NLS equation (12) with a background solution $q_0(x,t)=A_0e^{2i\delta A_0^2t}$, then $Q(X,Y)=\frac{1}{A_0}q(x,t)$, where $X=A_0x$ and $Y=A_0^2t$, also solves the reverse-space-time NLS equation (54), of which $Q_0(X,Y)=\frac{1}{A_0}q_0=e^{2i\delta Y}$ is a solution.

Based on these symmetries of equations (9)-(12), we only need to consider the unified background solution q_0 given in (50).

4.4 Wronskian column vectors ϕ and ψ

4.4.1 Vectors ϕ and ψ for the unreduced system (5)

We start with a pair of background solutions

$$q_0=e^{2i\delta t},\quad r_0=\delta e^{-2i\delta t},\quad \delta=\pm 1 \tag{55}$$

of the unreduced system (5). Note that the background solution (q_0,r_0) agrees with the reductions used in the equations (9)-(12). Substituting $(q,r)=(q_0,r_0)$ into the matrix equations (7) and (8), we find the following solutions of wave functions,

$$\Phi(\lambda,c,d)=\delta\left[c\left(\lambda-\sqrt{\lambda^2+\delta}\right)e^{-\sqrt{\lambda^2+\delta}(x-2i\lambda t)}+d\left(\lambda+\sqrt{\lambda^2+\delta}\right)e^{\sqrt{\lambda^2+\delta}(x-2i\lambda t)}\right]e^{i\delta t}, \tag{56a}$$

$$\Psi(\lambda,c,d)=\left(ce^{-\sqrt{\lambda^2+\delta}(x-2i\lambda t)}+de^{\sqrt{\lambda^2+\delta}(x-2i\lambda t)}\right)e^{-i\delta t}, \tag{56b}$$

in which c and d are constants (or functions of λ). Define

$$\phi=\left(\Phi(k_1,c_1,d_1),\Phi(k_2,c_2,d_2),\cdots,\Phi(k_{2m+2},c_{2m+2},d_{2m+2})\right)^T, \tag{57a}$$

$$\psi=\left(\Psi(k_1,c_1,d_1),\Psi(k_2,c_2,d_2),\cdots,\Psi(k_{2m+2},c_{2m+2},d_{2m+2})\right)^T. \tag{57b}$$

Then, the quasi double Wronskians (21) composed by the above ϕ and ψ provide solutions to the unreduced system (5) via the transformation (13) where the background solutions take (55). With regards to (57), the matrix $A = \mathrm{Diag}(k_1, k_2, \cdots, k_{2m+2})$ in (18) and (19). One can also take

$$\phi = \left(\Phi(k_1, c_1, d_1), \frac{\partial_{k_1}}{1!}\Phi(k_1, c_1, d_1), \cdots, \frac{\partial_{k_1}^{2m+1}}{(2m+1)!}\Phi(k_1, c_1, d_1) \right)^T, \tag{58a}$$

$$\psi = \left(\Psi(k_1, c_1, d_1), \frac{\partial_{k_1}}{1!}\Psi(k_1, c_1, d_1), \cdots, \frac{\partial_{k_1}^{2m+1}}{(2m+1)!}\Psi(k_1, c_1, d_1) \right)^T, \tag{58b}$$

to get multiple pole solutions corresponding to $A = J_{2m+2}(k_1)$, defined as in (47).

4.4.2 Vector ϕ for the reduced equations

To present vector ϕ for the reduced equations (9)-(12), let us define

$$\phi = (\phi^+, \phi^-)^T, \quad \phi^\pm = \left(\phi^\pm_{(1)}, \phi^\pm_{(2)}, \cdots, \phi^\pm_{(m+1)}\right)^T, \tag{59a}$$

$$\psi = (\psi^+, \psi^-)^T, \quad \psi^\pm = \left(\psi^\pm_{(1)}, \psi^\pm_{(2)}, \cdots, \psi^\pm_{(m+1)}\right)^T, \tag{59b}$$

where $\phi^\pm_{(j)}$ and $\psi^\pm_{(j)}$ are scalar functions. Note that a general form for the constraints (36), (39), (40) and (41) is

$$\psi = TC^\epsilon \phi(\alpha x, \beta t), \quad \alpha, \beta = \pm 1, \ \epsilon = 0, 1, \tag{60}$$

where C stands for an operator for complex conjugation: $C^\epsilon \phi = \phi^*$ when $\epsilon = 1$ and $C^\epsilon \phi = \phi$ when $\epsilon = 0$.

There are only two types of T in Sec.4.2, block skew-diagonal or block diagonal. We can present vector ϕ according to the type of T.

Case 1: T being block skew-diagonal: In this case,

$$T = \begin{pmatrix} 0 & I_{m+1} \\ \gamma I_{m+1} & 0 \end{pmatrix}, \quad \gamma = \pm 1, \tag{61}$$

which is for equation (9), (10) and (11), see Table 1 and Table 2. Vector ϕ takes the form

$$\phi = (\phi^+, C^\epsilon \psi^+(\alpha x, \beta t))^T, \tag{62}$$

where $(\epsilon, \alpha, \beta)$ takes $(1, 1, 1)$ for (9), $(1, -1, 1)$ for (10) and $(0, 1, -1)$ for (11). When $\mathbf{K}_{m+1}$ is diagonal as given in (46), we have

$$\phi^+_{(j)} = \Phi(k_j, c_j, d_j), \quad \psi^+_{(j)} = \Psi(k_j, c_j, d_j), \ j = 1, 2, \cdots, m+1, \tag{63}$$

where Φ and Ψ are defined in (56). When $\mathbf{K}_{m+1} = J_{m+1}(k_1)$ is the Jordan matrix as given in (47), we take

$$\phi^+_{(j)} = \frac{\partial_{k_1}^{j-1}}{(j-1)!}\Phi(k_1, c, d), \quad \psi^+_{(j)} = \frac{\partial_{k_1}^{j-1}}{(j-1)!}\Psi(k_1, c, d), \ j = 1, 2, \cdots, m+1. \tag{64}$$

Case 2: T being block diagonal: In this case,

$$T=\begin{pmatrix} I_{m+1} & 0 \\ 0 & \gamma I_{m+1} \end{pmatrix},\quad \gamma=\pm 1 \text{ or } \gamma=-i, \tag{65}$$

which is associated with equation (12), equation (10) with $\delta=-1$ and equation (9) with $\delta=1$, see Table 2 and (45).

To describe the relations between c_j and d_j in a more succinct and symmetric way, we rewrite (56a) and (56b) as

$$\Phi=\frac{(\hat{c}\lambda+\hat{c}\sqrt{\lambda^2+\delta}+\hat{d})e^{\sqrt{\lambda^2+\delta}(x-2i\lambda t)}+(-\hat{c}\lambda+\hat{c}\sqrt{\lambda^2+\delta}-\hat{d})e^{-\sqrt{\lambda^2+\delta}(x-2i\lambda t)}}{2\sqrt{\lambda^2+\delta}}e^{\delta it}, \tag{66a}$$

$$\Psi=\frac{(-\hat{d}\lambda+\hat{d}\sqrt{\lambda^2+\delta}+\delta\hat{c})e^{\sqrt{\lambda^2+\delta}(x-2i\lambda t)}+(\hat{d}\lambda+\hat{d}\sqrt{\lambda^2+\delta}-\delta\hat{c})e^{-\sqrt{\lambda^2+\delta}(x-2i\lambda t)}}{2\sqrt{\lambda^2+\delta}}e^{-\delta it}, \tag{66b}$$

where we have introduced $\hat{c},\hat{d}\in\mathbb{C}$ and taken in (56) that

$$c=\frac{\hat{d}\lambda+\hat{d}\sqrt{\lambda^2+\delta}-\delta\hat{c}}{2\sqrt{\lambda^2+\delta}},\quad d=\frac{-\hat{d}\lambda+\hat{d}\sqrt{\lambda^2+\delta}+\delta\hat{c}}{2\sqrt{\lambda^2+\delta}}.$$

Then, consider diagonal case where

$$\mathbf{K}_{m+1}=\mathrm{Diag}(k_1,k_2,\cdots,k_{m+1}),\quad \mathbf{H}_{m+1}=\mathrm{Diag}(h_1,h_2,\cdots,h_{m+1}). \tag{67}$$

For the reverse-space-time NLS equation (12), we have

$$\delta=1:\ \phi^+_{(j)}=\Phi(k_j,\hat{c}^+_j,i\hat{c}^+_j),\quad \phi^-_{(j)}=\Phi(h_j,\hat{c}^-_j,-i\hat{c}^-_j), \tag{68}$$

$$\delta=-1:\ \phi^+_{(j)}=\Phi(k_j,\hat{c}^+_j,\hat{c}^+_j),\quad \phi^-_{(j)}=\Phi(h_j,\hat{c}^-_j,-\hat{c}^-_j), \tag{69}$$

where Φ is defined in (66a), $k_j,h_j\in\mathbb{C}$, $\hat{c}^+_j$ and $\hat{c}^-_j$ can be arbitrary complex functions of k_j and h_j, respectively. For the classical NLS equation (9) with $\delta=1$ and the reverse-space NLS equation (10) with $\delta=-1$, we have

$$\phi^+_{(j)}=\Phi(k_j,\hat{c}^+_j,\hat{c}^{+*}_j),\quad \phi^-_{(j)}=\Phi(h_j,\hat{c}^-_j,-\hat{c}^{-*}_j),\quad \text{when } T \text{ is (45b)}, \tag{70}$$

$$\phi^+_{(j)}=\Phi(k_j,\hat{c}^+_j,\hat{c}^{+*}_j),\quad \phi^-_{(j)}=\Phi(h_j,\hat{c}^-_j,\hat{c}^{-*}_j),\quad \text{when } T \text{ is (45c)}, \tag{71}$$

where $k_j,h_j\in i\mathbb{R}$ for equation (9) with $\delta=1$ and $k_j,h_j\in\mathbb{R}$ for equation (10) with $\delta=-1$.

When

$$\mathbf{K}_{m+1}=J_{m+1}(k_1),\quad \mathbf{H}_{m+1}=J_{m+1}(h_1),$$

we have

$$\phi^+_{(s)}=\frac{\partial^{s-1}_{k_1}}{(s-1)!}\phi^+_{(1)},\quad \phi^-_{(s)}=\frac{\partial^{s-1}_{h_1}}{(s-1)!}\phi^-_{(1)},\ s=1,2,\cdots,m+1, \tag{72}$$

where $\phi^\pm_{(1)}$ are defined as in (68), (69), (70) or (71), varying with the equation considered.

5 Examples of dynamics of solutions

Nonzero background can bring new features for the classical and nonlocal NLS equations. In this section we analyze some solutions and illustrate their dynamics. The classical NLS equation (1) and reverse-space nonlocal NLS equation (10) with $\delta = -1$ will serve as main models.

5.1 The classical focusing NLS equation

It follows from the transformation (13) and the bilinear form (14) that the envelope $|q|$ of the solution to the focusing NLS equation (1) with a background solution q_0 can be expressed as (also see [12, 58])

$$|q|^2 = |q_0|^2 + \partial_x^2 \ln f, \tag{73}$$

where $f = |\widehat{\phi}_m; T\widehat{\phi}^*_m|$ is the quasi double Wronskian. For the focusing NLS equation (1), $A = \mathrm{Diag}(\mathbf{K}_{m+1}, -\mathbf{K}^*_{m+1})$, T is given by (61) with $\gamma = -1$, ϕ is given by (62) with $(\epsilon, \alpha, \beta) = (1, 1, 1)$. In principle, solutions to equation (1) can be determined by the eigenvalue structure of $\mathbf{K}_{m+1}$. One can investigate these solutions according to the canonical form of $\mathbf{K}_{m+1}$.

5.1.1 Breathers

Case 1: $\mathbf{K}_{m+1}$ being a complex diagonal matrix

When $\mathbf{K}_{m+1}$ is a diagonal matrix (46), following (62) we have $\phi = (\phi^+, \phi^-)^T$ where the entries in $\phi^\pm$ are

$$\phi^+_{(j)} = \Phi(k_j, c_j, d_j), \quad \phi^-_{(j)} = \psi^{+*}_{(j)}, \quad \psi^+_{(j)} = \Psi(k_j, c_j, d_j),$$

Φ and Ψ are given as (56). When the background solution takes $q_0 = e^{-2it}$, we have

$$\phi^+_{(j)} = \left(\left(-k_j + \sqrt{k_j^2 - 1}\right) e^{-\sqrt{k_j^2-1}(x-2ik_jt)-\xi_j^{(0)}} - \left(k_j + \sqrt{k_j^2 - 1}\right) e^{\sqrt{k_j^2-1}(x-2ik_jt)+\xi_j^{(0)}}\right) e^{-it}, \tag{74a}$$

$$\psi^+_{(j)} = \left(e^{-\sqrt{k_j^2-1}(x-2ik_jt)-\xi_j^{(0)}} + e^{\sqrt{k_j^2-1}(x-2ik_jt)+\xi_j^{(0)}}\right) e^{it} \tag{74b}$$

where we have taken $c_j = e^{-\xi_j^{(0)}}, d_j = e^{\xi_j^{(0)}}$ with $\xi_j^{(0)}$ being an arbitrary functions of k_j.

When $m = 0$ we have from (24b) that

$$f = |\phi, T\phi^*|, \tag{75}$$

where

$$\phi = \begin{pmatrix} \phi^+_{(1)} \\ \psi^{+*}_{(1)} \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{76}$$

Note that in this case we have

$$-f = \left|\phi^+_{(1)}\right|^2 + \left|\psi^+_{(1)}\right|^2,$$

which is positive definite when $\phi^{+}_{(1)}$ and $\psi^{+}_{(1)}$ are defined as in (74).

By calculation we find

$$f = -(A_1 \cosh 2X_1(x,t) + A_2 \sinh 2X_1(x,t) + A_3 \cos 2X_2(x,t) - A_4 \sin 2X_2(x,t)), \quad (77)$$

where

$$\begin{aligned}
&A_1 = 2(1 + a_1^2 + b_1^2 + u_{11}^2 + u_{12}^2),\ A_2 = 4(a_1 u_{11} + b_1 u_{12}),\\
&A_3 = 2(1 + a_1^2 + b_1^2 + u_{11}^2 + u_{12}^2),\ A_4 = 4(a_1 u_{12} - b_1 u_{11}),\\
&X_1(x,t) = u_{11}x + 2B_1 t + \xi_{1R}^{(0)},\ B_1 = a_1 u_{12} + b_1 u_{11},\\
&X_2(x,t) = u_{12}x + 2B_2 t + \xi_{1I}^{(0)},\ B_2 = b_1 u_{12} - a_1 u_{11},\\
&k_1 = a_1 + ib_1,\ \sqrt{k_1^2 - 1} = u_{11} + iu_{12},\ \xi_1^{(0)} = \xi_{1R}^{(0)} + i\xi_{1I}^{(0)},
\end{aligned}$$

and $a_1, b_1, u_{11}, u_{12}, \xi_{1R}^{(0)}, \xi_{1I}^{(0)} \in \mathbb{R}$. Since $\sqrt{k_1^2-1}$ is a double-valued function of k, here we consider the branch

$$\begin{aligned}
u_{11} &= \sqrt[4]{(a_1^2 - b_1^2 - 1)^2 + (2a_1b_1)^2}\cos\left(\frac{1}{2}(\arg(a_1 + 1 + ib_1) + \arg(a_1 - 1 + ib_1)\right),\\
u_{12} &= \sqrt[4]{(a_1^2 - b_1^2 - 1)^2 + (2a_1b_1)^2}\sin\left(\frac{1}{2}(\arg(a_1 + 1 + ib_1) + \arg(a_1 - 1 + ib_1)\right)
\end{aligned}$$

without loss of generality. Further we introduce

$$\tan\theta = \frac{A_4}{A_3},$$

such that (77) is rewritten as

$$f = -\left(A_1 \cosh 2X_1(x,t) + A_2 \sinh 2X_1(x,t) + \sqrt{A_3^2 + A_4^2}\cos(2X_2(x,t) + \theta)\right). \quad (78)$$

Noticing that $A_1 > |A_2| > 0$ for all $k_1 \neq 0$, from the above expression and (73), $|q|^2$ behaves like a wave traveling along the line $X_1 = 0$ and oscillating periodically with a period determined by $2X_2 + \theta = 2j\pi$, $j \in \mathbb{Z}$. Note that the case $a_1 = 0$ or $a_1 = \pm 1, b_1 = 0$ yields $|q|^2 = 1$, which is trivial and we do not consider.

To see more details, we rewrite (78) in terms of the following coordinates,

$$\left(x,\ z = t + \frac{u_{11}}{2B_1}x + \frac{\xi_{1R}^{(0)}}{2B_1}\right), \quad (79)$$

which gives rise to

$$\begin{aligned}
f = -\Bigg\{&A_1 \cosh(4B_1 z + 2\xi_{1R}^{(0)}) + A_2 \sinh(4B_1 z + 2\xi_{1R}^{(0)})\\
&+\sqrt{A_3^2 + A_4^2}\cos\left(4B_2\left(z + \frac{a_1(u_{11}^2 + u_{12}^2)}{2B_1B_2}x + \frac{\xi_{1I}^{(0)}}{2B_2} - \frac{\xi_{1R}^{(0)}}{2B_1}\right) + \theta\right)\Bigg\}.
\end{aligned} \quad (80)$$

In terms of (79) we can see that (73) with (80) provides a breather traveling along the straight line $z =$ constant and oscillating with a period with respect to x,

$$P = \left| \frac{2\pi B_1}{a_1(u_{11}^2 + u_{12}^2)} \right|. \tag{81}$$

An illustration is given in Fig.1(a), which describes a moving breather. Such a breather is also known as the Tajiri-Watanabe breather (see Fig.4 in [58]). In 1998 Tajiri and Watanabe derived and studied breathers of the focusing NLS equation using Hirota's bilinear method [58].

Back to the expression (78). Stationary breathers appear when $b_1 = 0$. More precisely, when $|a_1| > 1$ and $b_1 = 0$, which leads to $u_{11} = \sqrt{a_1^2 - 1}$ and $u_{12} = 0$, we have $B_1 = 0$ and then $X_1(x,t) = u_{11}x + \xi_{1R}^{(0)}$. In this case we can have a breather stationary with respect to x, where

$$\begin{aligned} f = -\Bigg(& 2a_1^2 \cosh 2\left(\sqrt{a_1^2 - 1}x + \xi_{1R}^{(0)}\right) + 4a_1\sqrt{a_1^2 - 1}\sinh 2\left(\sqrt{a_1^2 - 1}x + \xi_{1R}^{(0)}\right) \\ &+4\cos 2\left(-2a_1\sqrt{a_1^2 - 1}t + \xi_{1I}^{(0)}\right)\Bigg). \end{aligned} \tag{82}$$

It follows that a stationary breather oscillating in time with period $P_t = \frac{\pi}{|2a_1\sqrt{a_1^2-1}|}$, which is known as the Kuznetsov-Ma breather [40, 47]. It is described in Fig.1(b). In another case where $|a_1| < 1$ and $b_1 = 0$, which leads to $u_{11} = 0$ and $u_{12} = \sqrt{1 - a_1^2}$, from (78) we have

$$f = -\left(2\cosh 2\left(2a_1\sqrt{1 - a_1^2}t + \xi_{1R}^{(0)}\right) + 4a_1^2 \cos\left(2(\sqrt{1 - a_1^2}x + \xi_{1I}^{(0)}) + \theta\right)\right) \tag{83}$$

with $\tan\theta = \frac{\sqrt{1-a_1^2}}{2a_1}$. This will gives rise to a breather traveling along the line $t = -\frac{\xi_{1R}^{(0)}}{2a_1\sqrt{1-a_1^2}}$ and being periodic with respect to x with the period $P_x = \frac{\pi}{\sqrt{1-a_1^2}}$. Such a breather is known as the Akhmediev breather [11], which was first studied by Akhmediev in [11] and then bear his name. Stability of the Akhmediev and Kuznetsov-Ma breathers was studied recently [28, 31]. The Akhmediev breather is perpendicular to the Kuznetsov-Ma breather, as depicted in Fig.1 (b) and (c).

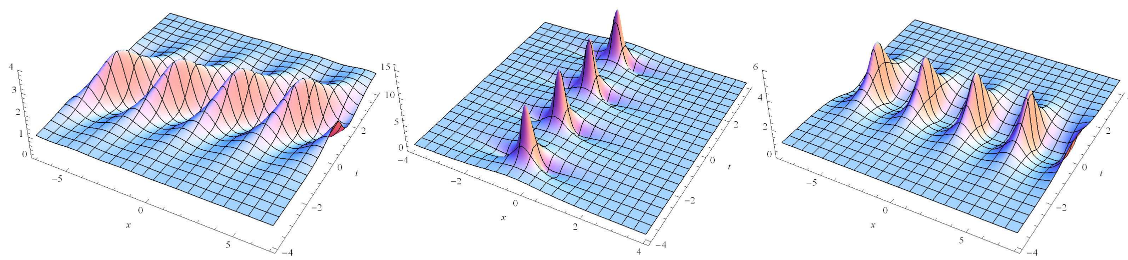
The envelope of two-breather solution can be obtained via (73) by taking $m = 1$ in quasi double Wronskians (24b), i.e.

$$f = |\phi, \phi_1; \psi, \psi_1| \tag{84a}$$

with

$$\phi = \left(\phi_{(1)}^+, \phi_{(2)}^+; \psi_{(1)}^{+*}, \psi_{(2)}^{+*}\right)^T, \quad \psi = \left(\psi_{(1)}^+, \psi_{(2)}^+; -\phi_{(1)}^{+*}, -\phi_{(2)}^{+*}\right)^T, \tag{84b}$$

in which $\phi_{(j)}^+$ and $\psi_{(j)}^+$ are defined as in (74). There are various types of two-breather interactions. As examples Fig.2 illustrates interactions between two Tajiri-Watanabe breathers, interaction of the Akhmediev breather and Kuznetsov-Ma breather and interaction of two Akhmediev breathers, in Fig.2 (a), (b) and (c), respectively.



(a) (b) (c)

Fig. 1. Shape and motion of one breather solution of the focusing NLS equation (1) with a background solution $q_0 = e^{-2it}$. The envelope is given by (73) where f is (77). (a) 3D-plot for a moving breather associated with (80) for $a_1 = 0.3, b_1 = -0.3, \xi_{1R}^{(0)} = \xi_{1I}^{(0)} = 0$. (b) 3D-plot for the Kuznetsov-Ma breather associated with (82) for $a_1 = 1.2, \xi_{1R}^{(0)} = \xi_{1I}^{(0)} = 0$. (c) 3D-plot for the Akhmediev breather associated with (83) for $a_1 = 0.5, \xi_{1R}^{(0)} = \xi_{1I}^{(0)} = 0$.

Case 2: $\mathbf{K}_{m+1}$ being a Jordan matrix

Let $\phi_{(1)}^{+}$ and $\psi_{(1)}^{+}$ be defined as in (74), and we define

$$\begin{aligned}
&\phi_{(j)}^{+} = \frac{\partial_{k_1}^{j-1}}{(j-1)!}\phi_{(1)}^{+}, \quad \psi_{(j)}^{+} = \frac{\partial_{k_1}^{j-1}}{(j-1)!}\psi_{(1)}^{+},\\
&\phi_{(j)}^{-} = \psi_{(j)}^{+*}, \quad \psi_{(j)}^{-} = -\phi_{(j)}^{+*}, \qquad j = 1, 2, \cdots, m+1.
\end{aligned}$$

The corresponding f composed by the above elements yields breathers when $\mathbf{K}_{m+1}$ is the Jordan matrix $J_{m+1}(k_1)$ as given in (47). For the simplest Jordan block solution of the focusing NLS equation (1) with the background solution $q_0 = e^{-2it}$, we have $m = 1$ and f composed by

$$\phi = \left(\phi_{(1)}^{+}, \partial_{k_1}\phi_{(1)}^{+}; \psi_{(1)}^{+*}, \left(\partial_{k_1}\psi_{(1)}^{+}\right)^{*}\right)^{T}, \quad \psi = \left(\psi_{(1)}^{+}, \partial_{k_1}\psi_{(1)}^{+}; -\phi_{(1)}^{+*}, -\left(\partial_{k_1}\phi_{(1)}^{+}\right)^{*}\right)^{T}. \tag{85}$$

Such a breather is described in Fig.3.

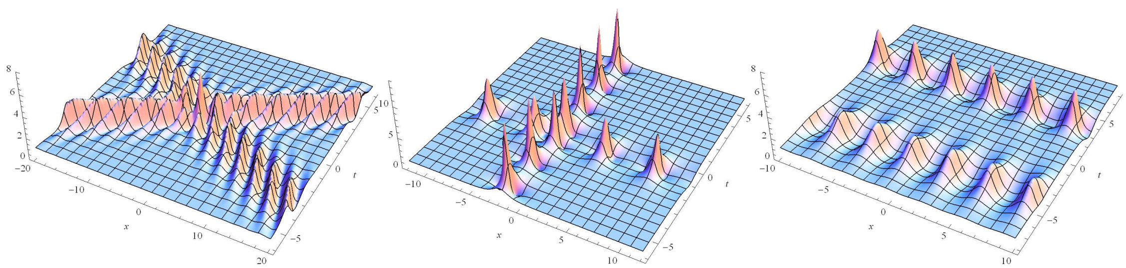
5.1.2 Rational solutions and rogue waves

Rational solutions can be obtained as a limit case of breathers when taking $k_j \to 1$. This can be seen from the expression (74). Since the Akhmediev breathers and Kuznetsov-Ma breathers are generated when $b_j = 0$, rational solutions can also be understood as a limit of these two types of breathers. In principle, for getting rational solutions, in A we should take $\mathbf{K}_{m+1} = J_{m+1}(1)$, but the limit procedure needs to be elaborated.

Let us consider (56) and rewrite them in the form

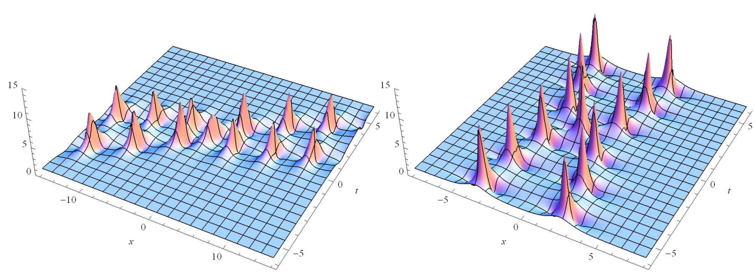
$$\Phi(\kappa, c, d) = \left(c(\kappa)\left(-\sqrt{\kappa^2+1}+\kappa\right)e^{-\kappa(x-2i\sqrt{\kappa^2+1}t)} - d(\kappa)\left(\sqrt{\kappa^2+1}+\kappa\right)e^{\kappa(x-2i\sqrt{\kappa^2+1}t)}\right)e^{-it}, \tag{86a}$$

$$\Psi(\kappa, c, d) = \left(c(\kappa)e^{-\kappa(x-2i\sqrt{\kappa^2+1}t)-\xi_j^{(0)}} + d(\kappa)e^{\kappa(x-2i\sqrt{\kappa^2+1}t)+\xi_j^{(0)}}\right)e^{it}, \tag{86b}$$



(a) (b) (c)

Fig. 2. Shape and motion of two-breather interactions of the focusing NLS equation (1) with a background solution $q_0 = e^{-2it}$. The envelope is given by (73) where f is (84). (a) 3D-plot for $a_1 = 0.3, b_1 = 0.5, a_2 = 0.3, b_2 = -0.5, \xi_{1R}^{(0)} = \xi_{1I}^{(0)} = \xi_{2R}^{(0)} = \xi_{2I}^{(0)} = 0$. (b) 3D-plot for $a_1 = 1.2, b_1 = 0, a_2 = 0.8, b_2 = 0, \xi_{1R}^{(0)} = 1, \xi_{1I}^{(0)} = 0, \xi_{2R}^{(0)} = 1, \xi_{2I}^{(0)} = 0$. (c) 3D-plot for $a_1 = 0.3, b_1 = 0, a_2 = 0.5, b_2 = 0, \xi_{1R}^{(0)} = 2, \xi_{1I}^{(0)} = 0, \xi_{2R}^{(0)} = -2, \xi_{2I}^{(0)} = 0$.



(a) (b)

Fig. 3. Shape and motion of Jordan block solution of the focusing NLS equation (1) with a background solution $q_0 = e^{-2it}$. The envelope is given by (73) where f is composed by (85). (a) 3D-plot for $a_1 = 0.8, b_1 = -0.15, \xi_{1R}^{(0)} = \xi_{1I}^{(0)} = 0$. (b) 3D-plot for $a_1 = 1.2, b_1 = 0, \xi_{1R}^{(0)} = \xi_{1I}^{(0)} = 0$.

where we have taken $\delta = -1$ and introduce $\kappa = \sqrt{k^2 - 1}$, and we take $c(\kappa)$ and $d(\kappa)$ to be functions of κ. Impose constraint $c(\kappa) = -d(-\kappa)$ and take formal expressions

$$c(\kappa) = \sum_{j=0}^{\infty} s_j \kappa^j, \ d(\kappa) = \sum_{j=0}^{\infty} (-1)^{j+1} s_j \kappa^j, \tag{87}$$

where s_j are arbitrary complex parameters. We denote the above $\Phi(\kappa, c, d)$ and $\Psi(\kappa, c, d)$ with (87) by Φ_{odd} and Ψ_{odd} respectively. Expand them in terms of κ as

$$\Phi_{odd} = \sum_{j=0}^{\infty} R_{2j+1} \kappa^{2j+1}, \quad \Psi_{odd} = \sum_{j=0}^{\infty} S_{2j+1} \kappa^{2j+1}, \tag{88}$$

in which

$$R_{2j+1}=\frac{1}{(2j+1)!}\frac{\partial^{2j+1}}{\partial\kappa^{2j+1}}\Phi_{odd}|_{\kappa=0},\quad S_{2j+1}=\frac{1}{(2j+1)!}\frac{\partial^{2j+1}}{\partial\kappa^{2j+1}}\Psi_{odd}|_{\kappa=0},\quad j=0,1,2,\cdots. \tag{89}$$

Define

$$\phi^{odd}=(R_1,R_3,\cdots,R_{2m+1},S_1^*,S_3^*,\cdots,S_{2m+1}^*)^T, \tag{90a}$$

$$\psi^{odd}=(S_1,S_3,\cdots,S_{2m+1},-R_1^*,-R_3^*,\cdots,-R_{2m+1}^*)^T=T(\phi^{odd})^*, \tag{90b}$$

where T takes the form (61) with $\gamma=-1$. It can be verified that ϕ^{odd} satisfies equation (25) where $q_0=e^{-2it}$, $\delta=-1$, T is given by (61) with $\gamma=-1$, and $A=\mathrm{Diag}(\mathbf{K}_{m+1},-\mathbf{K}_{m+1}^*)$ with

$$\mathbf{K}_{m+1}=\begin{pmatrix}\alpha_0 & 0 & 0 & \cdots & 0\\ \alpha_2 & \alpha_0 & 0 & \cdots & 0\\ \alpha_4 & \alpha_2 & \alpha_0 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ \alpha_{2m} & \alpha_{2m-2} & \cdots & \alpha_2 & \alpha_0\end{pmatrix}, \tag{91}$$

in which $\alpha_{2j}=\frac{1}{(2j)!}\partial_\kappa^{2j}\sqrt{\kappa^2+1}|_{\kappa=0}$, $(j=0,1,2,\cdots)$. Note also that A and T satisfy (26) with $\delta=-1$. Thus, the quasi double Wronskians

$$f=|\widehat{\phi}_m^{odd};\widehat{\psi}_m^{odd}|,\quad g=2|\widehat{\phi}_{m+1}^{odd};\widehat{\psi}_{m-1}^{odd}| \tag{92}$$

provide rational solutions to the focusing NLS equation (1) via (13) and the envelope via (73).

The first order rational solution (for $m=0$) is

$$q=-\left(1+\frac{4it-1}{\widetilde{x}^2+4t^2+\frac{1}{4}}\right)e^{-2it}, \tag{93}$$

where $\widetilde{x}=x+\frac{s_0-2s_1}{2s_0}$ with s_0,s_1 being coefficients of $c(k)$. Here we take $s_0,s_1\in\mathbb{R}$ for simplicity. We refer to it as the Peregrine soliton since it was first derived by Peregrine in [52]. Its envelope $|q|$ is localized in both space and time. It is also known as a rogue wave of the focusing NLS equation. The maximum value of $|q|$ is 3, occurring at $(x,t)=(-\frac{s_0-2s_1}{2s_0},0)$, three times hight of the background $|q_0|=1$. The envelope is depicted in Fig.4(a).

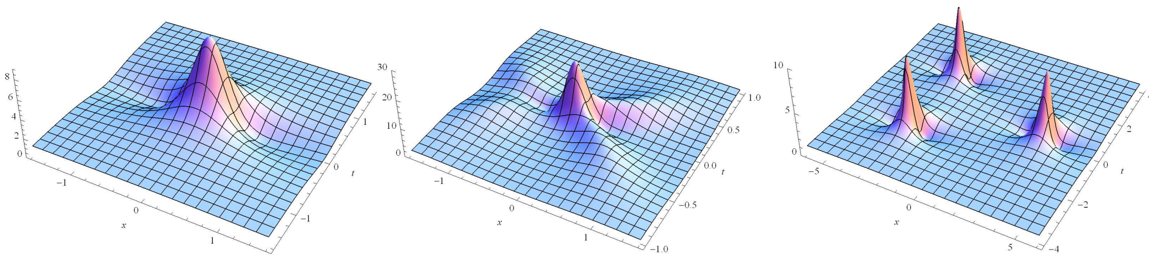
The general second order rational solution can be obtained from

$$q=q_0+\frac{g}{f},\quad g=2|\phi^{odd},\phi_1^{odd},\phi_2^{odd};\psi_1^{odd}|,\quad f=|\phi^{odd},\phi_1^{odd};\psi^{odd},\psi_1^{odd}|, \tag{94a}$$

where

$$\phi^{odd}=(R_1,R_2;S_1^*,S_2^*)^T,\quad \psi^{odd}=(S_1,S_2;-R_1^*,-R_2^*)^T. \tag{94b}$$

We skip explicit expression of q. The envelope of a typical second order rational solution is shown in Fig.4(b) with a symmetric shape and having a single maximum 5. In general,



(a) (b) (c)

Fig. 4. Shape and motion of rational solutions of the focusing NLS equation (1). (a) Envelope of the first order rational solution given by (93) with $s_0 = 1, s_1 = 0.5$. (b) Envelope of the second order rational solution given by (94) with $s_0 = 2, s_1 = 0.5$. (c) Envelope of the second order rational solution given by (94) with $s_0 = 1, s_1 = 0, s_2 = 10, s_3 = -20$.

the maximum amplitude of a nth-order rogue wave with one central main peak is $2n+1$ times of the height of the amplitude of the background plane wave [9, 62], (also see [62] where rogue wave with such pattern is called a "fundamental rogue wave"). The envelope of another typical second order rational solution has three peaks, as shown in Fig.4(c).

The third order rational solution is obtained by taking $m = 2$ in (92). Without presenting formulae, we depict some different patterns of the envelope of these solutions in Fig.5. Fig.5 (a) shows the pattern where there is only one central main peak, Fig.5 (d) and Fig.5 (e) show the pattern consisting basically of 6 well-separated fundamental part on a unit background, which are located on a triangle and a pentagon, respectively. Another two interesting patterns are shown in Fig.5 (b) and Fig.5 (c). Thus, it indicates that higher-order rogue waves contain richer structures. Note that recently it was found the pattern of rogue waves is related to the roots of Yablonskii-Vorob'ev polynomials [64, 66].

Apart from (90), one can also introduce Wronskian entries by imposing $c(\kappa) = d(-\kappa)$, i.e.,

$$c(\kappa) = \sum_{j=0}^{\infty} s_j \kappa^j, \ d(\kappa) = \sum_{j=0}^{\infty} (-1)^j s_j \kappa^j, \tag{95}$$

such that $\Phi(\kappa, c, d)$ and $\Psi(\kappa, c, d)$ given in (86) (denoted by Φ_{evev} and Ψ_{even}, respectively) can be expanded as

$$\Phi_{even} = \sum_{j=0}^{\infty} R_{2j} \kappa^{2j}, \quad \Psi_{even} = \sum_{j=0}^{\infty} S_{2j} \kappa^{2j}, \tag{96}$$

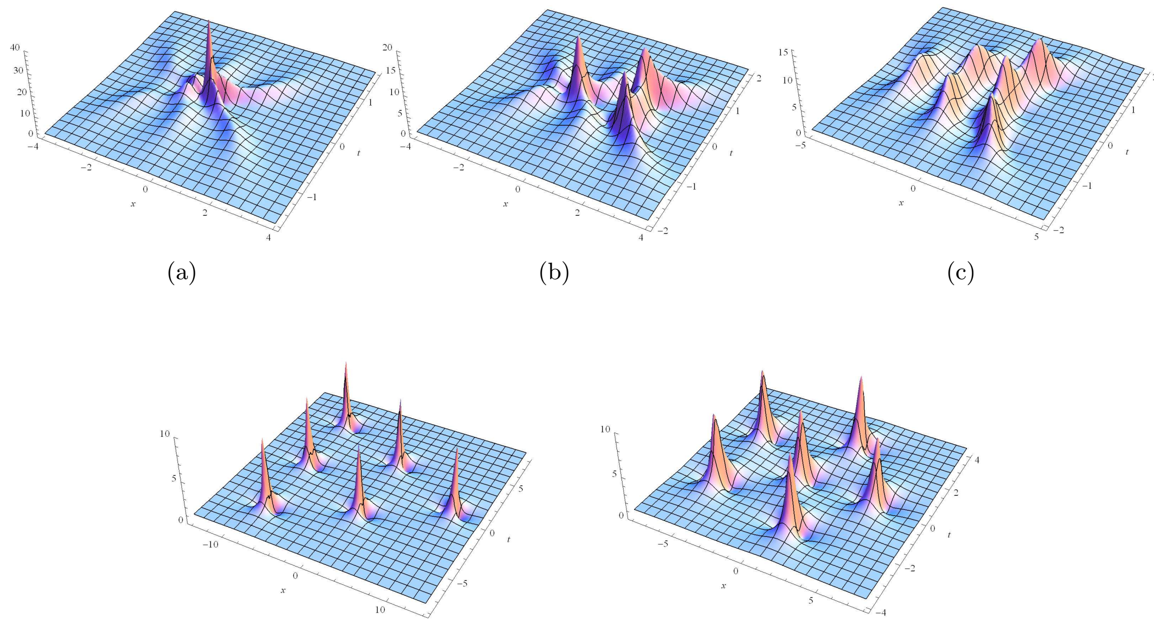
where

$$R_{2j} = \frac{1}{(2j)!} \frac{\partial^{2j}}{\partial \kappa^{2j}} \Phi_{even}|_{\kappa=0}, \ S_{2j} = \frac{1}{(2j)!} \frac{\partial^{2j}}{\partial \kappa^{2j}} \Psi_{even}|_{\kappa=0}, \quad j = 0, 1, 2, \cdots. \tag{97}$$

Then the vectors for the quasi double Wronskian are taken as

$$\phi^{even} = (R_0, R_2, \cdots, R_{2m}, S_0^*, S_2^*, \cdots, S_{2m}^*)^T, \tag{98a}$$

$$\psi^{even} = (S_0, S_2, \cdots, S_{2m}, -R_0^*, -R_2^*, \cdots, -R_{2m}^*)^T = T(\phi^{even})^*, \tag{98b}$$



(a) (b) (c)

(d) (e)

Fig. 5. Shape and motion of the envelope of the third order rational solution of the focusing NLS equation (1). (a) 3D plot for $s_0 = 1, s_1 = 0, s_2 = 0, s_3 = 0, s_4 = 0, s_5 = 0$. (b) 3D plot for $s_0 = 1, s_1 = 1, s_2 = 0, s_3 = 0, s_4 = 0, s_5 = 0$. (c) 3D plot for $s_0 = 1, s_1 = 0, s_2 = 0, s_3 = 1, s_4 = 0, s_5 = 0$. (d) 3D plot for $s_0 = -1, s_1 = 0, s_2 = 0, s_3 = 100, s_4 = -200, s_5 = 0$. (e) 3D plot for $s_0 = 1, s_1 = 0, s_2 = 100, s_3 = 1, s_4 = 0, s_5 = 200$.

where T is given by (61) with $\gamma = -1$. In this case $m = 0$ does not lead to a nontrivial solution but the solutions obtained by taking $m = 1$ and $m = 2$ correspond to (93) and (94a), which are the first order and second order rational solutions derived using ϕ^{odd} and ψ^{odd}.

One may conjecture that the m-th order rational solution derived using ϕ^{odd} and ψ^{odd} corresponds to the $(m+1)$-th order rational solution derived using ϕ^{even} and ψ^{even}. Similar connection is proved in the rational solutions of the discrete KdV-type equations, see [69]. We also note that the parameters $\{s_j\}$ (or $c(\kappa)$) play the same roles as the lower triangular Toeplitz matrices, cf.[70, 71]. An $(m+1)$-th order lower triangular Toeplitz matrix $\mathbf{P}_{m+1}$ is defined as

$$\mathbf{P}_{m+1} = \mathcal{T}_{m+1}[t_j]_0^m = \begin{pmatrix} t_0 & 0 & 0 & \cdots & 0 \\ t_1 & t_0 & 0 & \cdots & 0 \\ t_2 & t_1 & t_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ t_m & t_{m-1} & \cdots & t_1 & t_0 \end{pmatrix}, \quad t_j \in \mathbb{C}, \tag{99}$$

which commutes with $\mathbf{K}_{m+1}$ defied in (91). For the block diagonal matrix $Q = \mathrm{Diag}(\mathbf{P}_{m+1}, \mathbf{P}^*_{m+1})$, when T is given by (61) with $\gamma = -1$, and $A = \mathrm{Diag}(\mathbf{K}_{m+1}, -\mathbf{K}^*_{m+1})$ with (91), we have

$$AQ = QA,\ QT = TQ^*.$$

This indicates that for any ϕ that satisfies (25) with the above A and T, $\widetilde{\phi} = Q\phi$ is also a solution of (25). Moreover, if ϕ^{odd} in (90a) is derived with $c(\kappa) = 1$, then in $\widetilde{\phi} = Q\phi$, the parameters $\{t_j\}$ play the exactly same roles as $\{s_j\}$. In [70], for the KdV equation, the relation between $\mathbf{P}_{m+1}$ and $c(\kappa)$ is described, see Sec.2 of [70].

5.2 The defocusing reverse-space nonlocal NLS equation

In this section we investigate solutions of the defocusing reverse-space nonlocal NLS equation

$$iq_t = q_{xx} - 2q^2q^*(-x) \tag{100}$$

with the background solution $q_0 = e^{2it}$. This is the equation (10) with $\delta = 1$. Note that the reverse-space nonlocal NLS equation (10) is considered as a model with parity-time symmetry (see [5]). Efforts of finding physical applications of NLS type nonlocal integrable systems can also be found in [7, 46, 65], etc.

5.2.1 Solitons and doubly periodic solutions

Solution to equation (100) with a background solution q_0 is written as

$$q = q_0 + \frac{g}{f},\quad f = |\widehat{\phi}_m; \widehat{\psi}_m|,\quad g = 2|\widehat{\phi}_{m+1}; \widehat{\psi}_{m-1}|, \tag{101}$$

where we take $q_0 = e^{2it}$. Consider the simplest case, $m = 0$. From the results in Table 1 and in Sec.4.4.2, we have

$$f = \begin{vmatrix} \phi^+_{(1)} & \psi^+_{(1)} \\ \psi^{+*}_{(1)}(-x) & -\phi^{+*}_{(1)}(-x) \end{vmatrix},\quad g = 2\begin{vmatrix} \phi^+_{(1)} & k_1\phi^+_{(1)} \\ \psi^{+*}_{(1)}(-x) & k_1^*\psi^{+*}_{(1)}(-x) \end{vmatrix}, \tag{102a}$$

where

$$\phi^+_{(1)} = \left(\alpha_1 e^{\sqrt{k_1^2+1}(x-2ik_1t)} + \beta_1 e^{-\sqrt{k_1^2+1}(x-2ik_1t)}\right)e^{it}, \tag{102b}$$

$$\psi^+_{(1)} = \left(\alpha_1\left(\sqrt{k_1^2+1} - k_1\right)e^{\sqrt{k_1^2+1}(x-2ik_1t)} - \beta_1\left(\sqrt{k_1^2+1} + k_1\right)e^{-\sqrt{k_1^2+1}(x-2ik_1t)}\right)e^{-it}, \tag{102c}$$

and in $\Phi(k, c, d)$ and $\Psi(k, c, d)$ defined in (56) we have taken $\delta = 1$,

$$c(k) = -\beta_1\left(\sqrt{k_1^2+1} + k_1\right),\quad d(k) = \alpha_1\left(\sqrt{k_1^2+1} - k_1\right)$$

with α_1 and β_1 as arbitrary functions of k.

The envelope $|q|$ of some solutions resulting from (102) is depicted in Fig.6, which exhibits features of two-soliton interactions, although the solution is from the simplest case, $m = 0$. In the following we implement asymptotic analysis so as to understand such features. To avoid singular and trivial solutions, we consider the special case where $k_1 = ib$, $b \in \mathbb{R}$. It turns out that the solution can be classified according to the sign of $1-b^2$.

Case 1: $|b| < 1$

We write solution q in terms of the following coordinates,

$$(X_1 = x + 2bt,\ t)\,.$$

This gives rise to

$$q = \frac{G}{F},$$

where

$$\begin{aligned}
G =& e^{2it}\Big\{1 + 2ib\Big[(\sqrt{1-b^2}+ib)e^{4b\sqrt{1-b^2}t} - \beta\beta^*(\sqrt{1-b^2}-ib)e^{-4b\sqrt{1-b^2}t}\\
&+\beta(\sqrt{1-b^2}+ib)e^{-2\sqrt{1-b^2}X_1+4b\sqrt{1-b^2}t} - \beta^*(\sqrt{1-b^2}-ib)e^{2\sqrt{1-b^2}X_1-4b\sqrt{1-b^2}t}\Big]\Big\},\\
F =& e^{4b\sqrt{1-b^2}t} + \beta\beta^* e^{-4b\sqrt{1-b^2}t} + \beta b(b - i\sqrt{1-b^2})e^{-2\sqrt{1-b^2}X_1+4b\sqrt{1-b^2}t}\\
&+ \beta^* b(b + i\sqrt{1-b^2})e^{2\sqrt{1-b^2}X_1-4b\sqrt{1-b^2}t},
\end{aligned}$$

and we have taken $\beta = \frac{\beta_1}{\alpha_1}$. When keeping X_1 to be constant, we find

$$|q|^2 \to 1 - \frac{2\sqrt{1-b^2}\beta_I}{\mathrm{sgn}[b]\left(|\beta|\cosh(2\sqrt{1-b^2}X_1 - \ln(|\beta\, b^{\pm 1}|)) + \mathrm{sgn}[b](\sqrt{1-b^2}\beta_I + b\beta_R)\right)},\ bt \to \pm\infty.$$

Similarly, in terms of the coordinate

$$(X_2 = x - 2bt,\ t)\,,$$

we get

$$|q|^2 \to 1 + \frac{2((2b^2-1)\beta_I - 2b\sqrt{1-b^2}\beta_R)\sqrt{1-b^2}}{\mathrm{sgn}[b]\left(|\beta|\cosh(2\sqrt{1-b^2}X_2 + \ln(|\beta\, b^{\pm 1}|)) + \mathrm{sgn}[b](\sqrt{1-b^2}\beta_I + b\beta_R)\right)},\ bt \to \pm\infty.$$

Here we have taken $\beta = \beta_R + i\beta_I, \beta_R, \beta_I \in \mathbb{R}$. Note that here we do not have formula (73), which indicates the background $|q_0|$ for the envelope $|q|$ in the classical case, however, since the background solution is $q_0 = e^{2it}$ which yields $|q_0| = 1$, the above asymptotic results indicate that $|q|$ has a background plane $|q| = 1$, which is equal to $|q_0|$.

For convenience let us call the above two solitons X_1-soliton and X_2-soliton, respectively. We further impose restriction $\mathrm{sgn}[b](\sqrt{1-b^2}\beta_I + b\beta_R) > 0$ so that the solution has no singularity. Then we have the following results on the asymptotic behaviors of X_j-solitons.

Theorem 4. Assume that $\mathrm{sgn}[b](\sqrt{1-b^2}\beta_I + b\beta_R) > 0$. In case $|b| < 1$ and when $bt \to \pm\infty$, the envelope of X_1-soliton asymptotically travels on a background $|q|^2 = 1$ and with characteristics

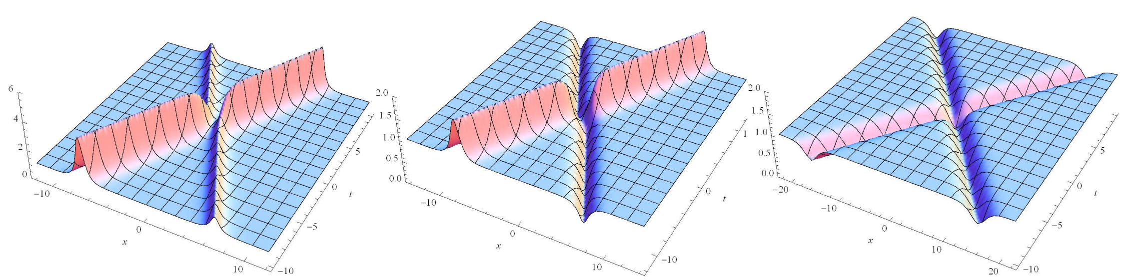
$$\begin{aligned}
&\text{trajectory}: \ x(t) = \frac{1}{2\sqrt{1-b^2}} \ln|\beta\, b^{\pm 1}| - 2bt,\\
&\text{velocity}: x'(t) = -2b,\\
&\text{amplitude}: 1 - \frac{2\sqrt{1-b^2}\beta_I}{\mathrm{sgn}[b]\left(|\beta| + \mathrm{sgn}[b](\sqrt{1-b^2}\beta_I + b\beta_R)\right)};
\end{aligned}$$

when $bt \to \pm\infty$, the $|q|^2$ of X_2-soliton asymptotically travels on a background $|q|^2 = 1$ and with characteristics

$$\begin{aligned}
&\text{top trace}: x(t) = -\frac{1}{2\sqrt{1-b^2}} \ln|\beta\, b^{\pm 1}| + 2bt,\\
&\text{velocity}: x'(t) = 2b,\\
&\text{amplitude}: 1 + \frac{2((2b^2-1)\beta_I - 2b\sqrt{1-b^2}\beta_R)\sqrt{1-b^2}}{\mathrm{sgn}[b]\left(|\beta| + \mathrm{sgn}[b](\sqrt{1-b^2}\beta_I + b\beta_R)\right)}.
\end{aligned}$$

Each soliton gets a phase shift $-2\ln|b|$ due to interaction.

The value of amplitude of each soliton can be either larger or less than the background $|q_0| = 1$. This indicates various types of interactions. Fig.6 exhibits three types of interactions. It is also notable that the value of amplitude of each soliton can be even equal to the background $|q_0| = 1$, which means the soliton can vanish on the background. This happens when $\beta_I = 0$ for the X_1-soliton and when $(2b^2-1)\beta_I - 2b\sqrt{1-b^2}\beta_R = 0$ for the X_2-soliton. Illustrations are given in Fig.7. Note that such a behavior usually appear in some coupled system and known as "ghost soliton", cf.[33].



(a) (b) (c)

Fig. 6. Interactions of the X_1-X_2 solitons for the defocusing reverse-space NLS equation (100): shape and motion of the envelope $|q|^2$ resulting from (102). (a) 3D plot for $b = 0.3, \beta = -1.6 - 0.4i$. (b) 3D plot for $b = 0.3, \beta = -0.5 + 0.1i$. (c) 3D plot for $b = -0.8, \beta = 1.4 - i$. In (a) and (b), X_2-soliton is the branch in up-right direction, and the other is X_1-soliton. In (c) X_1-soliton is the branch in up-right direction, and the other is X_2-soliton.

Case 2: $|b| > 1$

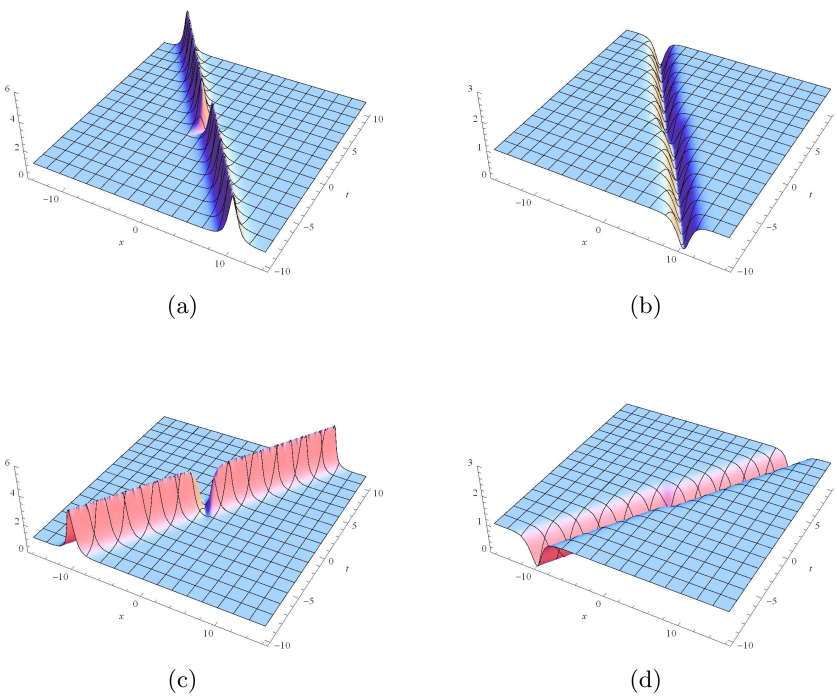


Fig. 7. Interactions of X_1-X_2 solitons with degeneration: shape and motion of the envelope $|q|^2$ resulting from (102). (a) 3D plot for $b = 0.5, \beta = 0.5 - \frac{\sqrt{3}}{2}i$. (b) 3D plot for $b = 0.5, \beta = -0.5 + \frac{\sqrt{3}}{2}i$. (c) 3D plot for $b = 0.5, \beta = -0.5$. (d) 3D plot for $b = 0.5, \beta = 0.5$. In (a) and (b), X_2-soliton vanishes in the background $|q|^2 = 1$. In (c) and (d), X_1-soliton vanishes in the background $|q|^2 = 1$.

In this case the one-soliton solution of equation (100) resulting from (102) can be written as

$$q = \frac{G}{F}, \tag{103a}$$

where

$$\begin{aligned}
G =& e^{2it}\Big\{1 + 2b\Big[((\sqrt{b^2-1} - b) - \beta\beta^*(\sqrt{b^2-1} + b))\cos(2\sqrt{b^2-1}x) \\
&+ i((\sqrt{b^2-1} - b) + \beta\beta^*(\sqrt{b^2-1} + b))\sin(2\sqrt{b^2-1}x) \\
&+ (\beta(\sqrt{b^2-1} - b) - \beta^*(\sqrt{b^2-1} + b))\cos(4b\sqrt{b^2-1}t) \\
&- i(\beta(\sqrt{b^2-1} - b) + \beta^*(\sqrt{b^2-1} + b))\sin(4b\sqrt{b^2-1}t)\Big]\Big\},
\end{aligned} \tag{103b}$$

$$\begin{aligned}
F =& - b((\sqrt{b^2-1} - b) - \beta\beta^*(\sqrt{b^2-1} + b))\cos(2\sqrt{b^2-1}x) \\
&- ib((\sqrt{b^2-1} - b) + \beta\beta^*(\sqrt{b^2-1} + b))\sin(2\sqrt{b^2-1}x) \\
&+ (\beta + \beta^*)\cos(4b\sqrt{b^2-1}t) + i(\beta^* - \beta)\sin(4b\sqrt{b^2-1}t).
\end{aligned} \tag{103c}$$

Solution (103) is doubly periodic with respect to both x and t and the periods are

$$P_x = \frac{\pi}{\sqrt{b^2-1}}, \quad P_t = \frac{\pi}{2b\sqrt{b^2-1}}. \tag{104}$$

The solution is plotted in Fig.8. Although there are some results on doubly periodic solutions, which are constructed by Jacobi elliptic functions in general, to our knowledge, the doubly periodic solution (103) to the defocusing reverse-space NLS equation (100) is not reported before.

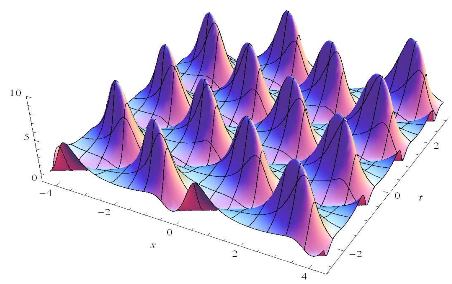


Fig. 8. Envelope of doubly periodic solution (103) for the defocusing reverse-space NLS equation (100), for $b = -1.25, \beta = 1$.

5.2.2 Rational solutions

According to Table 1, for equation (100), we have $A = \mathrm{Diag}(\mathbf{K}_{m+1}, \mathbf{K}^*_{m+1})$ and T given by (61) with $\gamma = -1$. In the following we consider solutions resulting from $\mathbf{K}_{m+1} = J_{m+1}(i)$.

Consider $\Phi(k, \hat{c}, \hat{d})$ and $\Psi(k, \hat{c}, \hat{d})$ defined in (66) where we take $\lambda = k$ and $\delta = 1$. Expanding them in terms of $(k-i)$ yields

$$\Phi(k, \hat{c}, \hat{d}) = \sum_{j=0}^{\infty} R_{j+1}(k-i)^j, \quad \Psi(k, \hat{c}, \hat{d}) = \sum_{j=0}^{\infty} S_{j+1}(k-i)^j, \tag{105}$$

where

$$R_{j+1} = \frac{1}{j!}\frac{\partial^j}{\partial k^j}\Phi(k, \hat{c}, \hat{d})\Big|_{k=i}, S_{j+1} = \frac{1}{j!}\frac{\partial^j}{\partial k^j}\Psi(k, \hat{c}, \hat{d})\Big|_{k=i}, \quad j = 0, 1, 2, \cdots. \tag{106}$$

Define

$$\phi = (R_1, R_2, \cdots, R_{m+1}, S_1^*(-x), S_2^*(-x), \cdots, S_{m+1}^*(-x))^T, \tag{107a}$$

$$\psi = (S_1, S_2, \cdots, S_{m+1}, -R_1^*(-x), -R_2^*(-x), \cdots, -R_{m+1}^*(-x))^T = T\phi^*(-x). \tag{107b}$$

It can be verified that ϕ satisfies (28) with the above mentioned A, T, $\delta = 1$ and $q_0 = e^{2it}$. This means, with such ϕ and ψ as basic column vectors, by the formula

$$q = q_0 + \frac{g}{f} = e^{2it} + \frac{g}{f}, \tag{108}$$

the quasi double Wronskians

$$f = |\widehat{\phi}_m; \widehat{\psi}_m|, \quad g = 2|\widehat{\phi}_{m+1}; \widehat{\psi}_{m-1}|$$

provide rational solutions to defocusing reverse-space NLS equation (100).

The first order rational solution (for $m = 0$) is provided by

$$f = \begin{vmatrix} R_1 & S_1 \\ S_1^*(-x) & -R_1^*(-x) \end{vmatrix}, \quad g = 2\begin{vmatrix} R_1 & iR_1 \\ S_1^*(-x) & -iS_1^*(-x) \end{vmatrix} \tag{109a}$$

with

$$R_1 = (1 + (i + \hat{d})(x + 2t))e^{it}, \ S_1 = (\hat{d} + (1 - i\hat{d})(x + 2t))e^{-it}, \tag{109b}$$

where we have taken $\hat{c} = 1$ and $\hat{d}$ being a constant. Explicit form of the first order rational solution is given by

$$q = \frac{G}{F}, \tag{110a}$$

where

$$\begin{aligned} G =& e^{2it}\Big\{1 + 4i\Big[\hat{d}^* + (i + \hat{d})\hat{d}^*(x + 2t) + i(-i + \hat{d}^*)(-x + 2t) \\ & + i|i + \hat{d}|^2(x + 2t)(-x + 2t)\Big]\Big\}, \end{aligned} \tag{110b}$$

$$\begin{aligned} F =& 1 + |\hat{d}|^2 + (i + \hat{d})(1 - i\hat{d}^*)(x + 2t) + (-i + \hat{d}^*)(1 + i\hat{d})(-x + 2t) \\ & + 2|i + \hat{d}|^2(x + 2t)(-x + 2t). \end{aligned} \tag{110c}$$

Some illustrations are given in Fig.9 which exhibit soliton interactions. To understand the dynamics we investigate asymptotic behaviors of the above rational solution. We introduce a new coordinate

$$(X_1 = x + 2t, \ t)\,,$$

then rewrite (110) in this coordinate, keep X_1 to be constant and let $t \to \pm\infty$. It follows that

$$|q(X_1, t)|^2 \to 1 + \frac{8(1 + \hat{d}_I)}{4|i + \hat{d}|^2(X_1 + \frac{\hat{d}_R}{|i+\hat{d}|^2})^2 + \frac{(|\hat{d}|^2-1)^2}{|i+\hat{d}|^2}}, \quad t \to \pm\infty, \tag{111}$$

where $\hat{d} = \hat{d}_R + i\hat{d}_I$. For convenience, we call it X_1-soliton. It indicates that, asymptotically, this is a wave traveling on the background plane $|q|^2 = 1$, along the line $x = -2t - \frac{\hat{d}_R}{|i+\hat{d}|^2}$, with amplitude $1 + \frac{8|i+\hat{d}|^2(1+\hat{d}_I)}{(|\hat{d}|^2-1)^2}$ and without phase shift due to interaction. The wave can be above the background plane when $1 + \hat{d}_I > 0$, or below the background plane when $1 + \hat{d}_I < 0$, or vanishes in the background plane when $1 + \hat{d}_I = 0$. We further introduce a second coordinate frame

$$(X_2 = x - 2t, \ t)\,,$$

in terms of which we rewrite (110). Then keeping X_2 to be constant and letting $t \to \pm\infty$, we find

$$|q(X_2,t)|^2 \to 1 + \frac{8(\hat{d}_I + |\hat{d}|^2)}{4|i+\hat{d}|^2(X_2 - \frac{\hat{d}_R}{|i+\hat{d}|^2})^2 + \frac{(|\hat{d}|^2-1)^2}{|i+\hat{d}|^2}}, \quad t \to \pm\infty. \tag{112}$$

This implies that, when $t \to \pm\infty$, there is a wave (X_2-soliton for short) traveling on the background plane $|q|^2 = 1$, along the line $x = 2t + \frac{\hat{d}_R}{|i+\hat{d}|^2}$ with amplitude $1 + \frac{8|i+\hat{d}|^2(\hat{d}_I+|\hat{d}|^2)}{(|\hat{d}|^2-1)^2}$ and without phase shift due to interaction. The wave can be either above or below or vanishes in the background plane, depending on the sign of $\hat{d}_I + |\hat{d}|^2$. We summarize these asymptotic behaviors in the follow theorem.

Theorem 5. When $t \to \pm\infty$, the envelope $|q|^2$ of X_1-soliton asymptotically travels on a background $|q|^2 = 1$ and with characteristics

$$\begin{aligned}
&\text{trajectory}: \ x(t) = -2t - \frac{\hat{d}_R}{|i+\hat{d}|^2}, \\
&\text{velocity}: x'(t) = -2, \\
&\text{amplitude}: 1 + \frac{8|i+\hat{d}|^2(1+\hat{d}_I)}{(|\hat{d}|^2-1)^2};
\end{aligned}$$

when $t \to \pm\infty$, the $|q|^2$ of X_2-soliton asymptotically travels on a background $|q|^2 = 1$ and with characteristics

$$\begin{aligned}
&\text{top trace}: x(t) = 2t + \frac{\hat{d}_R}{|i+\hat{d}|^2}, \\
&\text{velocity}: x'(t) = 2, \\
&\text{amplitude}: 1 + \frac{8|i+\hat{d}|^2(\hat{d}_I+|\hat{d}|^2)}{(|\hat{d}|^2-1)^2}.
\end{aligned}$$

Asymptotically, no phase shift occurs for each soliton.

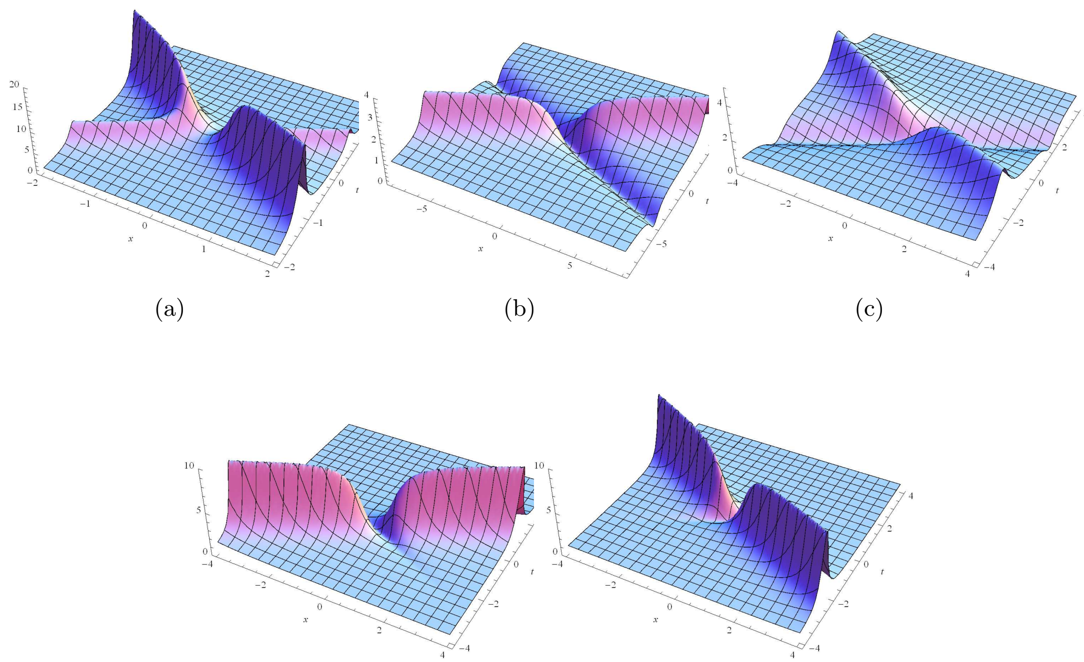
Various types of interactions are illustrated in Fig.9, which coincide with the above results of asymptotic analysis. Note that, considering the signs of $1 + \hat{d}_I$ and $\hat{d}_I + |\hat{d}|^2$, it is impossible to have both waves below the background plane, neither one wave below the background plane and another vanishing.

5.3 The defocusing reverse-time nonlocal NLS equation

For the defocusing reverse-time NLS equation

$$iq_t = q_{xx} - 2q^2 q(-t) \tag{113}$$

with nonzero background $q_0 = e^{2it}$, we can analyze solutions resulting from $k_1 = ib$, $b \in \mathbb{R}$, as we have done in Sec.5.2 for the reverse-space nonlocal NLS equation (100). However, it turns out that the analysis procedure of these solutions and their dynamics are all similar to those in Sec.5.2 for equation (100). Let us explain the statement below.



(a) (b) (c)

(d) (e)

Fig. 9. Interactions of the X_1-X_2 solitons for the defocusing reverse-space NLS equation (100): shape and motion of the envelope $|q|^2$ resulting from (110). (a) 3D plot for $\hat{d}=0.5$. (b) 3D plot for $\hat{d}=-2i$. (c) 3D plot for $\hat{d}=0.2-0.5i$. (d) 3D plot for $\hat{d}=0.5-i$. (e) 3D plot for $\hat{d}=0.5-0.5i$. In (a), (b) and (c), the branch in up-right direction stands for the X_2-soliton, the other is for the X_1-soliton. In (d), X_1-soliton vanishes in the background $|q|^2=1$. In (e), X_2-soliton vanishes in the background $|q|^2=1$.

Rewrite (102b) and (102c) as

$$\phi^{+}_{(1)}(x,t,\alpha_1,\beta_1)=\left(\alpha_1 e^{\eta_1(x,t)}+\beta_1 e^{-\eta_1(x,t)}\right)e^{it}, \tag{114a}$$

$$\psi^{+}_{(1)}(x,t,\alpha_1,\beta_1)=\left(\alpha_1 \tilde{d}(k_1) e^{\eta_1(x,t)}-\beta_1 \tilde{c}(k_1) e^{-\eta_1(x,t)}\right)e^{-it}, \tag{114b}$$

where

$$\eta_1(x,t)=\sqrt{k_1^2+1}(x-2ik_1t),\quad \tilde{c}(k_1)=\sqrt{k_1^2+1}+k_1,\quad \tilde{d}(k_1)=\sqrt{k_1^2+1}-k_1. \tag{114c}$$

According to Sec.4.4.2, for the reverse-space NLS equation (100), the vectors ϕ and ψ in the 2×2 double Wronskians f and g are

$$\phi=\phi_{[x]}(x,t,\alpha_1,\beta_1)=\begin{pmatrix}\phi^{+}_{(1)}(x,t,\alpha_1,\beta_1)\\ \psi^{+*}_{(1)}(-x,t,\alpha_1,\beta_1)\end{pmatrix},\quad \psi=\psi_{[x]}(x,t,\alpha_1,\beta_1)=T\phi^{*}_{[x]}(-x,t,\alpha_1,\beta_1), \tag{115}$$

and for the reverse-time NLS equation (113), we have

$$\phi = \phi_{[t]}(x,t,\alpha_1,\beta_1) = \left(\begin{array}{c} \phi^{+}_{(1)}(x,t,\alpha_1,\beta_1) \\ \psi^{+}_{(1)}(x,-t,\alpha_1,\beta_1) \end{array} \right), \quad \psi = \psi_{[t]}(x,t,\alpha_1,\beta_1) = T\phi_{[t]}(x,-t,\alpha_1,\beta_1), \tag{116}$$

where $T = \left(\begin{smallmatrix} 0 & 1 \\ \gamma & 0 \end{smallmatrix}\right)$ with $\gamma = \pm 1$, the subscripts $[x]$ and $[t]$ stand for the reverse-space and reverse-time, respectively. It can be verified that, when $k_1 = ib$, $b \in \mathbb{R}$ (as we have taken in Sec.5.2), we have

$$\eta_1(x,t) = \eta_1^*(x,t), \quad \eta_1(-x,t) = -\eta(x,-t), \quad \tilde{c}^*(k_1) = \tilde{d}(k_1). \tag{117}$$

It then follows that

$$\phi_{[x]}(x,t,\alpha_1,\beta_1) = B\phi_{[t]}(x,t,\beta_1^*,\alpha_1^*) \tag{118}$$

and

$$\psi_{[x]}(x,t,\alpha_1,\beta_1) = T\phi^*_{[x]}(-x,t,\alpha_1,\beta_1) = -BT\phi_{[t]}(x,-t,\beta_1^*,\alpha_1^*) = -B\psi_{[t]}(x,t,\beta_1^*,\alpha_1^*), \tag{119}$$

where $B = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. These relations indicate that $|q|^2$ resulting from the above ϕ and ψ for the reverse-time NLS equation (113) are similar to those of the reverse-space NLS equation (100). We skip presenting illustrations.

5.4 The defocusing reverse-space-time nonlocal NLS equation

For the solutions of the nonlocal defocusing reverse-space-time NLS equation (i.e. (12) with $\delta = 1$),

$$iq_t = q_{xx} - 2q^2 q(-x,-t) \tag{120}$$

with the plane wave background $q_0 = e^{2it}$, the matrices A and T take the form (see Table 2)

$$A = \left(\begin{array}{cc} \mathbf{K}_{m+1} & \mathbf{0}_{m+1} \\ \mathbf{0}_{m+1} & \mathbf{H}_{m+1} \end{array} \right), \quad T = \left(\begin{array}{cc} i\mathbf{I}_{m+1} & \mathbf{0}_{m+1} \\ \mathbf{0}_{m+1} & -i\mathbf{I}_{m+1} \end{array} \right). \tag{121}$$

Solution to equation (120) with the background solution $q_0 = e^{2it}$ is written as

$$q = q_0 + \frac{g}{f}.$$

For the case of $m = 0$, (i.e. one-soliton case), there are

$$f = \left| \begin{smallmatrix} \phi^{+}_{(1)} & i\phi^{+}_{(1)}(-x,-t) \\ \phi^{-}_{(1)} & -i\phi^{-}_{(1)}(-x,-t) \end{smallmatrix} \right|, \quad g = 2\left| \begin{smallmatrix} \phi^{+}_{(1)} & k_1\phi^{+}_{(1)} \\ \phi^{-}_{(1)} & h_1\phi^{-}_{(1)} \end{smallmatrix} \right|, \tag{122}$$

where in light of (68),

$$\phi^{+}_{(1)} = \Phi(k_1, \hat{c}^{+}_1, i\hat{c}^{+}_1),\ \phi^{-}_{(1)} = \Phi(h_1, \hat{c}^{-}_1, -i\hat{c}^{-}_1),$$

Φ is defined in (66a), $k_j, h_j \in \mathbb{C}$, $\hat{c}^{+}_j$ and $\hat{c}^{-}_j$ can be arbitrary complex functions of k_j and h_j, respectively. For the case of $m = 1$, (i.e., two-soliton case), we have

$$f = \begin{vmatrix} \phi^{+}_{(1)} & k_1\phi^{+}_{(1)} & i\phi^{+}_{(1)}(-x,-t) & -ik_1\phi^{+}_{(1)}(-x,-t) \\ \phi^{+}_{(2)} & k_2\phi^{+}_{(2)} & i\phi^{+}_{(2)}(-x,-t) & -ik_2\phi^{+}_{(2)}(-x,-t) \\ \phi^{-}_{(1)} & h_1\phi^{-}_{(1)} & -i\phi^{-}_{(1)}(-x,-t) & ih_1\phi^{-}_{(1)}(-x,-t) \\ \phi^{-}_{(2)} & h_2\phi^{-}_{(2)} & -i\phi^{-}_{(2)}(-x,-t) & ih_2\phi^{-}_{(2)}(-x,-t) \end{vmatrix},\ g = \begin{vmatrix} \phi^{+}_{(1)} & k_1\phi^{+}_{(1)} & k_1^2\phi^{+}_{(1)}(-x,-t) & i\phi^{+}_{(1)}(-x,-t) \\ \phi^{+}_{(2)} & k_2\phi^{+}_{(2)} & k_2^2\phi^{+}_{(2)}(-x,-t) & i\phi^{+}_{(2)}(-x,-t) \\ \phi^{-}_{(1)} & h_1\phi^{-}_{(1)} & h_1^2\phi^{-}_{(1)}(-x,-t) & -i\phi^{-}_{(1)}(-x,-t) \\ \phi^{-}_{(2)} & h_2\phi^{-}_{(2)} & h_2^2\phi^{-}_{(2)}(-x,-t) & -i\phi^{-}_{(2)}(-x,-t) \end{vmatrix}, \tag{123}$$

where $k_1, k_2, h_1, h_2 \in \mathbb{C}$ and

$$\phi^{+}_{(j)} = \Phi(k_j, \hat{c}^{+}_j, i\hat{c}^{+}_j),\quad \phi^{-}_{(j)} = \Phi(h_j, \hat{c}^{-}_j, -i\hat{c}^{-}_j),\ j = 1, 2,$$

Φ is defined in (66a), $k_j, h_j, \hat{c}^{\pm}_j \in \mathbb{C}$.

In the following, we list some solutions of reverse-space-time NLS equation (120) for the special form of $\mathbf{H}_{m+1}$.

When $\mathbf{H}_{m+1} = -\mathbf{K}_{m+1}$, one-soliton solution of equation (120) reads

$$q = \frac{G}{F}, \tag{124}$$

where

$$\begin{aligned} G =& e^{2it}\left\{1 - 2k_1\left[\cosh(2\sqrt{k_1^2+1}x) + \sqrt{k_1^2+1}\sin(4k_1\sqrt{k_1^2+1}t) + ik_1\cos(4k_1\sqrt{k_1^2+1}t)\right]\right\}, \\ F =& k_1\cosh(2\sqrt{k_1^2+1}x) - i\cos(4k_1\sqrt{k_1^2+1}t), \end{aligned}$$

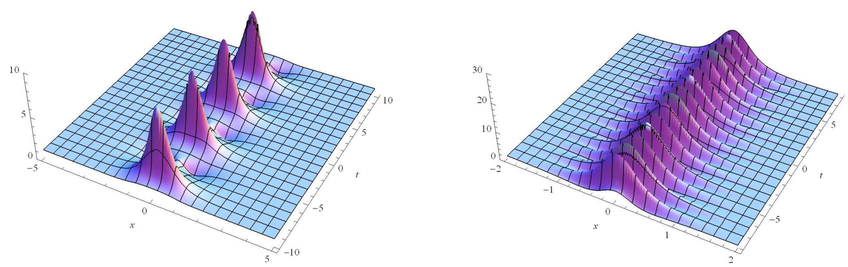
and we have taken $\hat{c}^{+}_1 = \hat{c}^{-}_1 = 1$. Nonsingular solution occurs when $k_1 \in \mathbb{R}$. In this case, the envelope of (124) describes a wave moving along the straight line $x = 0$ and oscillating in time with a period $P_t = \frac{\pi}{2|k_1|\sqrt{k_1^2+1}}$. Such a solution is depicted in Fig.10(a). Note that the period tends to 0 when $|k_1|$ is large enough, and such a change is illustrated in Fig.10(b).

When $\mathbf{H}_{m+1} = -\mathbf{K}^{*}_{m+1}$, by calculation, we get one-soliton solution which reads

$$q = e^{2it} + \frac{G}{F}, \tag{125}$$

where

$$\begin{aligned} G =& -2ia_1[(1 - a_1^2 - b_1^2 + v_{11}^2 + v_{12}^2 - 2ia_1)\cosh(2B_1) + 2(i(b_1v_{11} - a_1v_{12}) + v_{12})\sinh(2B_1) \\ & + (-1 + a_1^2 + b_1^2 + v_{11}^2 + v_{12}^2 + 2ia_1)\cos(2B_2) + 2i((a_1v_{11} + b_1v_{12}) + iv_{11})\sin(2B_2)]e^{2it}, \\ F =& (-1 + a_1^2 + b_1^2 + v_{11}^2 + v_{12}^2 + 2ia_1)\cosh(2B_1) + (1 - a_1^2 - b_1^2 + v_{11}^2 + v_{12}^2 - 2ia_1)\cos(2B_2), \end{aligned}$$



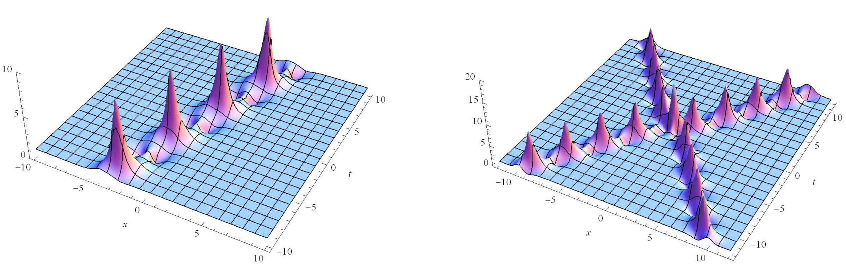
(a) (b)

Fig. 10. Shape and motion of stationary breather of $|q|^2$ resulting from (124) for the nonlocal defocusing reverse-space-time NLS equation (120). (a) 3D-plot for $k_1 = 0.3$. (b) 3D-plot for $k_1 = 1$.

we take $\hat{c}_1^+ = \hat{c}_1^- = 1$ and denote

$$\begin{aligned} k_1 =& a_1 + ib_1, \ \sqrt{k_1^2+1} = v_{11} + iv_{12}, \quad a_1, b_1, v_{11}, v_{12} \in \mathbb{R}, \\ B_1 =& v_{11}x + 2(a_1v_{12} + b_1v_{11})t, \ B_2 = v_{12}x + 2(b_1v_{12} - a_1v_{11})t. \end{aligned}$$

The envelope of (125) behaves like breather which travels along the line $x = -\frac{2(a_1v_{12}+b_1v_{11})t}{v_{11}}$. An illustration is given in Fig.11(a). Note that $a_1 = 0$ yields trivial solution and $b_1 = 0$ leads to the solution (124). We can also calculate two-soliton solution from (123) of this case, where we take $h_j = -k_j^*, j = 1, 2$. Its envelope describes a head-on collision of two breathers, as shown in Fig.11(b).



(a) (b)

Fig. 11. Shape and motion of the envelope $|q|^2$ of the nonlocal defocusing reverse-space-time NLS equation (120). (a) 3D-plot for the one breather resulting (125) with $k_1 = 0.3 - 0.15i$. (b) 3D-plot for two breathers interaction resulting from (123) with $k_1 = 0.5 + 0.3i, k_2 = 0.5 - 0.3i, \hat{c}_1^{\pm} = \hat{c}_2^{\pm} = 1$.

Finally, $\mathbf{H}_{m+1} = \mathbf{K}_{m+1}^*$ with $k_i \in \mathbb{C}$, we claim that dynamics of solutions are similar to the reverse-space case. The explanation is similar to what we have done in Sec.5.3. In fact, write the vectors ϕ and ψ with $h_1 = k_1^*$ in the double Wronskians (122) as

$$\phi = \phi_{[x,t]} = \begin{pmatrix} \phi_{(1)}^+(x,t,k_1,\hat{c}_1,i\hat{c}_1) \\ \phi_{(1)}^-(x,t,k_1^*,\hat{c}_1,-i\hat{c}_1) \end{pmatrix}, \psi = \psi_{[x,t]} = \begin{pmatrix} i\phi_{(1)}^+(-x,-t,k_1,\hat{c}_1,i\hat{c}_1) \\ -i\phi_{(1)}^-(-x,-t,k_1^*,\hat{c}_1,-i\hat{c}_1) \end{pmatrix}, \tag{126}$$

where the subscript $[x,t]$ stands for the reverse-space-time. We rewrite the vectors in (115) (for the reverse-space NLS equation (100)) as

$$\phi=\phi_{[x]}=\begin{pmatrix}\phi^{+}_{(1)}(x,t,k_1,c,d)\\ \psi^{+*}_{(1)}(-x,t,k_1,c,d)\end{pmatrix},\ \psi=\psi_{[x]}=\begin{pmatrix}\psi^{+}_{(1)}(x,t,k_1,c,d)\\ -\phi^{+*}_{(1)}(-x,t,k_1,c,d)\end{pmatrix}. \tag{127}$$

In the special case $c_1=1, d_1=i, \hat{c}_1=1$, the following relations hold:

$$\phi_{[x]}=C\phi_{[x,t]},\quad \psi_{[x]}=C\psi_{[x,t]}, \tag{128}$$

where $C=\left(\begin{smallmatrix}1&0\\0&-i\end{smallmatrix}\right)$. Besides, the construction of f and g, the same $A=\left(\begin{smallmatrix}k_1&0\\0&k_1^*\end{smallmatrix}\right)$ is used. This indicates that in the case $\mathbf{H}_{m+1}=\mathbf{K}^*_{m+1}$, the analysis of dynamics of solutions for the revere-space-time NLS equation will be similar to those of the revere-space NLS equation that has been investigated in Sec.5.2. We skip it.

6 Conclusions

In this paper, by means of the bilinearisation-reduction approach, solutions for classical and nonlocal NLS equations with nonzero backgrounds were constructed in a systematical way. Solutions are presented in terms of quasi double Wronskians. Asymptotic analysis and illustrations were provided to understand dynamics of solutions, in particular breathers and rogue waves of the classical focusing NLS equation (9) and solitons and rational solutions of the reverse-space nonlocal NLS equation (10). One can see that the nonzero backgrounds bring more interesting behaviors in the dynamics of solutions. In addition, although the solutions are given in terms of quasi double Wronskians (not standard double Wronskians), the reduction technique is still effective. In light of Theorem 2 one can also use the double Wronskians given in Theorem 2 if q_0 is independent of x. This bilinearisation-reduction technique can also be extended to the other integrable equations with nonzero backgrounds, which will be investigated elsewhere.

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A Proof of Theorem 1

To prove Theorem 1, we first recall the following lemmas.

Lemma 1. [27] Suppose that $\mathbf{D}$ is an arbitrary $s\times(s-2)$ matrix, and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are column vectors of order s, then

$$|\mathbf{D},\mathbf{a},\mathbf{b}||\mathbf{D},\mathbf{c},\mathbf{d}|-|\mathbf{D},\mathbf{a},\mathbf{c}||\mathbf{D},\mathbf{b},\mathbf{d}|+|\mathbf{D},\mathbf{a},\mathbf{d}||\mathbf{D},\mathbf{b},\mathbf{c}|=0. \tag{129}$$

Lemma 2. [70, 71] Suppose that $\Xi=(a_{js})_{M\times M}$ is an $M\times M$ matrix with column vector set $\{\alpha_j\}$ and row vector set $\{\beta_j\}$. $\mathcal{P}=(P_{js})_{M\times M}$ is an $M\times M$ operator matrix where each P_{js} is an operator. Then we have

$$\sum_{s=1}^{M}|\alpha_1,\cdots,\alpha_{s-1},\ C_s\circ\alpha_s,\ \alpha_{s+1},\cdots,\alpha_M|=\sum_{j=1}^{M}\begin{vmatrix}\beta_1\\ \vdots\\ \beta_{j-1}\\ R_j\circ\beta_j\\ \beta_{j+1}\\ \vdots\\ \beta_M\end{vmatrix}, \tag{130}$$

where $\{C_s\}$ and $\{R_j\}$ are respectively the column and row vector sets of $\mathcal{P}$, and "$\circ$" denotes $C_s\circ\alpha_s=(P_{1s}a_{1s},\ P_{2s}a_{2s},\cdots,P_{Ms}a_{Ms})^T$ and $R_j\circ\beta_j=(P_{j1}a_{j1},\ P_{j2}a_{j2},\cdots,P_{jM}a_{jM})$.

Proof of Theorem 1

Direct calculation yields

$$\begin{aligned}
f_x=&|\widehat{\phi}_{m-1},\phi_{m+1};\widehat{\psi}_m|+|\widehat{\phi}_m;\widehat{\psi}_{m-1},\psi_{m+1}|,\\
f_{xx}=&|\widehat{\phi}_{m-2},\phi_m,\phi_{m+1};\widehat{\psi}_m|+|\widehat{\phi}_{m-1},\phi_{m+2};\widehat{\psi}_m|+2|\widehat{\phi}_{m-1},\phi_{m+1};\widehat{\psi}_{m-1},\psi_{m+1}|\\
&+|\widehat{\phi}_m;\widehat{\psi}_{m-2},\psi_m,\psi_{m+1}|+|\widehat{\phi}_m;\widehat{\psi}_{m-1},\psi_{m+2}|\\
&+2q_0|\widehat{\phi}_{m-1};\widehat{\psi}_{m+1}|-2r_0|\widehat{\phi}_{m+1};\widehat{\psi}_{m-1}|,\\
if_t=&-2|\widehat{\phi}_{m-2},\phi_m,\phi_{m+1};\widehat{\psi}_m|+2|\widehat{\phi}_{m-1},\phi_{m+2};\widehat{\psi}_m|\\
&+2|\widehat{\phi}_m;\widehat{\psi}_{m-2},\psi_m,\psi_{m+1}|-2|\widehat{\phi}_m;\widehat{\psi}_{m-1},\psi_{m+2}|\\
&+2q_0|\widehat{\phi}_{m-1};\widehat{\psi}_{m+1}|+2r_0|\widehat{\phi}_{m+1};\widehat{\psi}_{m-1}|,\\
ig_t/2=&-2|\widehat{\phi}_{m-1},\phi_{m+1},\phi_{m+2};\widehat{\psi}_{m-1}|+2|\widehat{\phi}_m,\phi_{m+3};\widehat{\psi}_{m-1}|\\
&+2|\widehat{\phi}_{m+1};\widehat{\psi}_{m-3},\psi_{m-1},\psi_m|-2|\widehat{\phi}_{m+1};\widehat{\psi}_{m-2},\psi_{m+1}|\\
&+2q_0(|\widehat{\phi}_m;\widehat{\psi}_{m-1},\psi_{m+2}|+|\widehat{\phi}_{m-1},\phi_{m+1};\widehat{\psi}_{m-1},\psi_{m+1}|+|\widehat{\phi}_{m-2},\phi_m,\phi_{m+1};\widehat{\psi}_m|)\\
&-q_0r_0g-q_{0x}f_x,\\
g_x/2=&|\widehat{\phi}_m,\phi_{m+2};\widehat{\psi}_{m-1}|+|\widehat{\phi}_{m+1};\widehat{\psi}_{m-2},\psi_m|-q_0f_x,\\
g_{xx}/2=&|\widehat{\phi}_{m-1},\phi_{m+1},\phi_{m+2};\widehat{\psi}_{m-1}|+|\widehat{\phi}_m,\phi_{m+3};\widehat{\psi}_{m-1}|+2|\widehat{\phi}_m,\phi_{m+2};\widehat{\psi}_{m-2},\psi_m|\\
&+|\widehat{\phi}_{m+1};\widehat{\psi}_{m-3},\psi_{m-1},\psi_m|+|\widehat{\phi}_{m+1};\widehat{\psi}_{m-2},\psi_{m+1}|\\
&+q_0(|\widehat{\phi}_m;\widehat{\psi}_{m-1},\psi_{m+2}|-|\widehat{\phi}_{m-1},\phi_{m+2};\widehat{\psi}_m|\\
&-|\widehat{\phi}_m;\widehat{\psi}_{m-2},\psi_m,\psi_{m+1}|+|\widehat{\phi}_{m-2},\phi_m,\phi_{m+1};\widehat{\psi}_m|)-q_0f_{xx}-q_{0x}f_x.
\end{aligned}$$

Next, using Lemma 2 we derive some relations of quasi double Wronskians. Taking $\Xi=|\widehat{\phi}_m;\widehat{\psi}_m|$ and (for $1\le j\le 2m+2$)

$$P_{js}=\begin{cases}\partial_x-(-1)^m q_0\psi_m\circ\phi_m^{-1}\circ, & 0\le s\le m,\\ -\partial_x+(-1)^m r_0\phi_m\circ\psi_m^{-1}\circ, & m+1\le s\le 2m+2,\end{cases}$$

where the "$\circ$" is defined as in Lemma 2. One can find from the relation (130) that

$$(\mathrm{Tr}A)f = |\widehat{\phi}_{m-1}, \phi_{m+1}; \widehat{\psi}_m| + |\widehat{\phi}_m; \widehat{\psi}_{m-1}, \psi_{m+1}|,$$

where TrA stands for the trace of matrix A. Similarly, we can get

$$\begin{aligned}(\mathrm{Tr}A)^2 f =& |\widehat{\phi}_{m-2}, \phi_m, \phi_{m+1}; \widehat{\psi}_m| + |\widehat{\phi}_{m-1}, \phi_{m+2}; \widehat{\psi}_m| - 2|\widehat{\phi}_{m-1}, \phi_{m+1}; \widehat{\psi}_{m-1}, \psi_{m+1}| \\ &+ |\widehat{\phi}_m; \widehat{\psi}_{m-2}, \psi_m, \psi_{m+1}| + |\widehat{\phi}_m; \widehat{\psi}_{m-1}, \psi_{m+2}|.\end{aligned}$$

Substituting them into the left hand side of (14a) one obtains

$$\begin{aligned} ff_{xx} - f_x f_x &= 4|\widehat{\phi}_{m-1}, \phi_{m+1}; \widehat{\psi}_{m-1}, \psi_{m+1}|f - 4|\widehat{\phi}_{m-1}, \phi_{m+1}; \widehat{\psi}_m||\widehat{\phi}_m; \widehat{\psi}_{m-1}, \psi_{m+1}| \\ &\quad +2q_0|\widehat{\phi}_{m-1}; \widehat{\psi}_{m+1}|f - 2r_0|\widehat{\phi}_{m+1}; \widehat{\psi}_{m-1}|f \\ &= 4|\widehat{\phi}_{m-1}; \widehat{\psi}_{m+1}||\widehat{\phi}_{m+1}; \widehat{\psi}_{m-1}| + 2q_0|\widehat{\phi}_{m-1}; \widehat{\psi}_{m+1}|f - 2r_0|\widehat{\phi}_{m+1}; \widehat{\psi}_{m-1}|f \\ &= -gh - q_0 hf - r_0 gf, \end{aligned}$$

where we have made use of

$$f[(\mathrm{Tr}A)f] = [(\mathrm{Tr}A)f]^2$$

and the identity

$$\begin{aligned}&|\widehat{\phi}_{m-1}, \phi_{m+1}; \widehat{\psi}_{m-1}, \psi_{m+1}|f - 4|\widehat{\phi}_{m-1}, \phi_{m+1}; \widehat{\psi}_m||\widehat{\phi}_m; \widehat{\psi}_{m-1}, \psi_{m+1}| \\ &+|\widehat{\phi}_{m-1}; \widehat{\psi}_{m+1}||\widehat{\phi}_{m+1}; \widehat{\psi}_{m-1}| = 0,\end{aligned}$$

which is derived from Lemma 1. Thus, (14a) is proved.

For (14b), we first derive the following relations using Lemma 2,

$$\begin{aligned}(\mathrm{Tr}A)|\widehat{\phi}_{m+1}; \widehat{\psi}_{m-1}| =& |\widehat{\phi}_m, \phi_{m+2}; \widehat{\psi}_{m-1}| - |\widehat{\phi}_{m+1}; \widehat{\psi}_{m-2}, \psi_m|, \\ (\mathrm{Tr}A)^2|\widehat{\phi}_{m+1}; \widehat{\psi}_{m-1}| =& |\widehat{\phi}_{m-1}, \phi_{m+1}, \phi_{m+2}; \widehat{\psi}_{m-1}| + |\widehat{\phi}_m, \phi_{m+3}; \widehat{\psi}_{m-1}| - 2|\widehat{\phi}_m, \phi_{m+2}; \widehat{\psi}_{m-2}, \psi_m| \\ &+ |\widehat{\phi}_{m+1}; \widehat{\psi}_{m-3}, \psi_{m-1}, \psi_m| + |\widehat{\phi}_{m+1}; \widehat{\psi}_{m-2}, \psi_{m+1}|.\end{aligned}$$

Substituting the derivatives of f and g into equation (14b) we have

$$\begin{aligned}&(f_{xx}g + fg_{xx} - 2f_x g_x - ig_t f + if_t g)/2 \\ =&(-|\widehat{\phi}_{m-2}, \phi_m, \phi_{m+1}; \widehat{\psi}_m| + 3|\widehat{\phi}_{m-1}, \phi_{m+2}; \widehat{\psi}_m| + 2|\widehat{\phi}_{m-1}, \phi_{m+1}; \widehat{\psi}_{m-1}, \psi_{m+1}| \\ &+ 3|\widehat{\phi}_m; \widehat{\psi}_{m-2}, \psi_m, \psi_{m+1}| - |\widehat{\phi}_m; \widehat{\psi}_{m-1}, \psi_{m+2}|)|\widehat{\phi}_{m+1}; \widehat{\psi}_{m-1}| + 4q_0|\widehat{\phi}_{m-1}; \widehat{\psi}_{m+1}||\widehat{\phi}_{m+1}; \widehat{\psi}_{m-1}| \\ &+ (3|\widehat{\phi}_{m-1}, \phi_{m+1}, \phi_{m+2}; \widehat{\psi}_{m-1}| - |\widehat{\phi}_m, \phi_{m+3}; \widehat{\psi}_{m-1}| + 2|\widehat{\phi}_m, \phi_{m+2}; \widehat{\psi}_{m-2}, \psi_m| \\ &- |\widehat{\phi}_{m+1}; \widehat{\psi}_{m-3}, \psi_{m-1}, \psi_m| + 3|\widehat{\phi}_{m+1}; \widehat{\psi}_{m-2}, \psi_{m+1}|)f \\ &- q_0(|\widehat{\phi}_{m-2}, \phi_m, \phi_{m+1}; \widehat{\psi}_m| + |\widehat{\phi}_{m-1}, \phi_{m+2}; \widehat{\psi}_m| + 2|\widehat{\phi}_{m-1}, \phi_{m+1}; \widehat{\psi}_{m-1}, \psi_{m+1}| \\ &+ |\widehat{\phi}_m; \widehat{\psi}_{m-2}, \psi_m, \psi_{m+1}| + |\widehat{\phi}_m; \widehat{\psi}_{m-1}, \psi_{m+2}|)f - q_0 f_{xx} f + 2q_0 r_0|\widehat{\phi}_{m+1}; \widehat{\psi}_{m-1}|f \\ &- 2(|\widehat{\phi}_{m-1}, \phi_{m+1}; \widehat{\psi}_m| + |\widehat{\phi}_m; \widehat{\psi}_{m-1}, \psi_{m+1}|)(|\widehat{\phi}_m, \phi_{m+2}; \widehat{\psi}_{m-1}| + |\widehat{\phi}_{m+1}; \widehat{\psi}_{m-2}, \psi_m| - q_0 f_x) \\ =&- 4(|\widehat{\phi}_{m-1}, \phi_{m+1}; \widehat{\psi}_m||\widehat{\phi}_m, \phi_{m+2}; \widehat{\psi}_{m-1}| + |\widehat{\phi}_m; \widehat{\psi}_{m-1}, \psi_{m+1}||\widehat{\phi}_{m+1}; \widehat{\psi}_{m-2}, \psi_m|) \\ &+ 4(|\widehat{\phi}_{m-1}, \phi_{m+1}, \phi_{m+2}; \widehat{\psi}_{m-1}| + |\widehat{\phi}_{m+1}; \widehat{\psi}_{m-2}, \psi_{m+1}|)f \\ &+ 4(|\widehat{\phi}_{m-1}, \phi_{m+2}; \widehat{\psi}_m| + |\widehat{\phi}_m; \widehat{\psi}_{m-2}, \psi_m, \psi_{m+1}|)|\widehat{\phi}_{m+1}; \widehat{\psi}_{m-1}| \\ &- 2q_0(f_{xx}f - f_x^2) + 2q_0^2|\widehat{\phi}_{m-1}; \widehat{\psi}_{m+1}|f + 4q_0|\widehat{\phi}_{m-1}; \widehat{\psi}_{m+1}||\widehat{\phi}_{m+1}; \widehat{\psi}_{m-1}| \\ =&- 2q_0(f_{xx}f - f_x^2) - q_0^2 hf - q_0 gh \\ =&- q_0(f_{xx}f - f_x^2) + q_0 r_0 gf,\end{aligned}$$

in which the identity

$$f[(\mathrm{Tr}A)^2 g] = g[(\mathrm{Tr}A)^2 f] = [(\mathrm{Tr}A)f][(\mathrm{Tr}A)g]$$

and relations

$$\begin{aligned}
&|\widehat{\phi}_{m-1}, \phi_{m+2}; \widehat{\psi}_m| g/2 - |\widehat{\phi}_{m-1}, \phi_{m+1}; \widehat{\psi}_m||\widehat{\phi}_m, \phi_{m+2}; \widehat{\psi}_{m-1}| + |\widehat{\phi}_{m-1}, \phi_{m+1}, \phi_{m+2}; \widehat{\psi}_{m-1}| f = 0,\\
&|\widehat{\phi}_m; \widehat{\psi}_{m-2}, \psi_m, \psi_{m+1}| g/2 - |\widehat{\phi}_m; \widehat{\psi}_{m-1}, \psi_{m+1}||\widehat{\phi}_{m+1}; \widehat{\psi}_{m-2}, \psi_m| + |\widehat{\phi}_{m+1}; \widehat{\psi}_{m-2}, \psi_{m+1}| f = 0
\end{aligned}$$

have also been used. Thus, (14b) has been proved. The third equation can be proved in a similar way.

Suppose that $A = P^{-1}JP$, i.e. A is similar to J. We introduce $\phi' = P\phi$, $\psi' = P\psi$, which satisfy (18) and (19) with J in place of A. Then, for the quasi double Wronskians, we have $f(\phi', \psi') = |P| f(\phi, \psi)$, $g(\phi', \psi') = |P| g(\phi, \psi)$ and $h(\phi', \psi') = |P| h(\phi, \psi)$, which means A and any matrix that is similar to it can generate same q and r.

B Proof of Theorem 2

For simplicity we denote

$$F = |\widehat{m}; \widehat{m}|, \quad s = |\widehat{m+1}; \widehat{m-1}|, \quad H = |\widehat{m-1}; \widehat{m+1}|.$$

Direct calculation yields

$$\begin{aligned}
F_x =& |\widehat{m-1}, m+1; \widehat{m}| + |\widehat{m}; \widehat{m-1}, m+1|,\\
F_{xx} =& |\widehat{m-2}, m, m+1; \widehat{m}| + |\widehat{m-1}, m+2; \widehat{m}| + 2|\widehat{m-1}, m+1; \widehat{m-1}, m+1|\\
&+ |\widehat{m}; \widehat{m-2}, m, m+1| + |\widehat{m}; \widehat{m-1}, m+2|,\\
s_x =& |\widehat{m}, m+2; \widehat{m-1}| + |\widehat{m+1}; \widehat{m-2}, m|,\\
s_{xx} =& |\widehat{m-1}, m+1, m+2; \widehat{m-1}| + |\widehat{m}, m+3; \widehat{m-1}| + 2|\widehat{m}, m+2; \widehat{m-2}, m|\\
&+ |\widehat{m+1}; \widehat{m-3}, m-1, m| + |\widehat{m+1}; \widehat{m-2}, m+1|,\\
iF_t =& -2|\widehat{m-2}, m, m+1; \widehat{m}| + 2|\widehat{m-1}, m+2; \widehat{m}| + 2|\widehat{m}; \widehat{m-2}, m, m+1|\\
&- 2|\widehat{m}; \widehat{m-1}, m+2| + 2q_0(-1)^m|\widehat{m-1}; \widehat{(m+1)}| + 2r_0(-1)^m|\widehat{m+1}; \widehat{(m-1)}|,\\
is_t =& -2q_0r_0|\widehat{m+1}; \widehat{(m-1)}| - 2|\widehat{m-1}, m+1, m+2; \widehat{m-1}| + 2|\widehat{m}, m+3; \widehat{m-1}|\\
&+ 2|\widehat{m+1}; \widehat{m-3}, m-1, m| - 2|\widehat{m+1}; \widehat{m-2}, m+1|\\
&+ 2q_0(-1)^m(-|\widehat{m-2}, m, m+1; \widehat{m}| + |\widehat{m-1}, m+1; \widehat{m-1}, m+1|\\
&- |\widehat{m+1}; \widehat{m-2}, m+1|).
\end{aligned}$$

Taking $\Xi = |\widehat{m}; \widehat{m}|$ and (for $1 \le j \le 2m+2$)

$$P_{js} = \begin{cases} \partial_x - q_0(\partial_x^s\psi) \circ (\partial_x^s\phi)^{-1}\circ, & 0 \le s \le m,\\ -\partial_x + r_0(\partial_x^s\phi) \circ (\partial_x^s\psi)^{-1}\circ, & m+1 \le s \le 2m+2, \end{cases}$$

using (130) we can have

$$(\mathrm{Tr}A)F = |\widehat{m-1}, m+1; \widehat{m}| - |\widehat{m}; \widehat{m-1}, m+1|,$$

and

$$
\begin{aligned}
(\mathrm{Tr}A)s =&|\widehat{m}, m+2; \widehat{m-1}| - |\widehat{m+1}; \widehat{m-2}, m| \\
&+ q_0(-1)^m(|\widehat{m-1}, m+1; \widehat{m}| - |\widehat{m}; \widehat{m-1}, m+1|), \\
(\mathrm{Tr}A)^2F =&|\widehat{m-2}, m, m+1; \widehat{m}| + |\widehat{m-1}, m+2; \widehat{m}| - 2|\widehat{m-1}, m+1; \widehat{m-1}, m+1| \\
&+ |\widehat{m}; \widehat{m-2}, m, m+1| + |\widehat{m}; \widehat{m-1}, m+2| + 2q_0(-1)^m|\widehat{m-1}; \widehat{m+1}| \\
&- 2r_0(-1)^m|\widehat{m+1}; \widehat{m-1}|, \\
(\mathrm{Tr}A)^2s =&|\widehat{m-1}, m+1, m+2; \widehat{m-1}| + |\widehat{m}, m+3; \widehat{m-1}| - 2|\widehat{m}, m+2; \widehat{m-2}, m| \\
&+ |\widehat{m+1}; \widehat{m-3}, m-1, m| + |\widehat{m+1}; \widehat{m-2}, m+1| \\
&+ 2q_0(-1)^m(|\widehat{m-1}, m+2; \widehat{m}| + |\widehat{m}; \widehat{m-2}, m, m+1| - |\widehat{m-1}, m+1; \widehat{m-1}, m+1|) \\
&+ 2q_0(q_0|\widehat{m-1}; \widehat{m+1}| - r_0|\widehat{m+1}; \widehat{m-1}|).
\end{aligned}
$$

Substituting the derivatives of f into the left-hand side of (14a) one obtains

$$
\begin{aligned}
& ff_{xx} - f_x^2 && (131)\\
=\ & FF_{xx} - F_x^2 && (132)\\
=\ & 4(|\widehat{m}; \widehat{m}||\widehat{m-1}, m+1; \widehat{m-1}, m+1| - |\widehat{m-1}, m+1; \widehat{m}||\widehat{m}; \widehat{m-1}, m+1|) \\
& -2q_0(-1)^m|\widehat{m}; \widehat{m}||\widehat{m-1}; \widehat{m+1}| + 2r_0(-1)^m|\widehat{m}; \widehat{m}||\widehat{m+1}; \widehat{m-1}| \\
=\ & 4|\widehat{m-1}; \widehat{m+1}||\widehat{m+1}; \widehat{m-1}| - 2q_0(-1)^m|\widehat{m}; \widehat{m}||\widehat{m-1}; \widehat{m+1}| \\
& +2r_0(-1)^m|\widehat{m}; \widehat{m}||\widehat{m+1}; \widehat{m-1}| && (133)\\
=\ & 4Hs - 2q_0Hf + 2r_0sf, && (134)
\end{aligned}
$$

where we have made use of the equality

$$F[(\mathrm{Tr}A)F] = [(\mathrm{Tr}A)F]^2$$

and the relation

$$
\begin{aligned}
&|\widehat{m-1}; \widehat{m+1}||\widehat{m+1}; \widehat{m-1}| - |\widehat{m}; \widehat{m}||\widehat{m-1}, m+1; \widehat{m-1}, m+1| \\
&+|\widehat{m-1}, m+1; \widehat{m}||\widehat{m}; \widehat{m-1}, m+1| = 0.
\end{aligned}
$$

Meanwhile, a direct calculation of the right-hand side of (14a) gives rise to $4(2Hs - q_0Hf + r_0sf)$. Thus, equation (14a) is proved.

For equation (14b), let us first consider $(D_x^2 - iD_t)s \cdot F$. We have

$$
\begin{aligned}
& s_{xx}F - 2s_xF_x + sF_{xx} - i(s_tF - sF_t) \\
=\ & (F(s_{xx} - is_t) + s(F_{xx} + iF_t) - 2s_xF_x)) \\
=\ & (3|\widehat{m-1}, m+1, m+2; \widehat{m-1}| - |\widehat{m}, m+3; \widehat{m-1}| + 2|\widehat{m}, m+2; \widehat{m-2}, m| \\
& -|\widehat{m+1}; \widehat{m-3}, m-1, m| + 3|\widehat{m+1}; \widehat{m-2}, m+1|)F + 2q_0r_0sF \\
& +2q_0(-1)^m(|\widehat{m-2}, m, m+1; \widehat{m}| - |\widehat{m-1}, m+1; \widehat{m-1}, m+1| + |\widehat{m}; \widehat{m-1}, m+2|)F \\
& +(-|\widehat{m-2}, m, m+1; \widehat{m}| + 3|\widehat{m-1}, m+2; \widehat{m}| + 2|\widehat{m-1}, m+1; \widehat{m-1}, m+1| \\
& +3|\widehat{m}; \widehat{m-2}, m, m+1| - |\widehat{m}; \widehat{m-1}, m+2|)s + 2q_0(-1)^m|\widehat{m-1}; \widehat{m+1}|s + 2r_0(-1)^ms^2 \\
& -2(|\widehat{m-1}, m+1; \widehat{m}| + |\widehat{m}; \widehat{m-1}, m+1|)(|\widehat{m}, m+2; \widehat{m-1}| + |\widehat{m+1}; \widehat{m-2}, m|).
\end{aligned}
$$

Utilizing identity

$$F[(\mathrm{Tr}A)^2 s] = s[(\mathrm{Tr}A)^2 F] = [(\mathrm{Tr}A)F][(\mathrm{Tr}A)s],$$

(14b) gives rise to

$$\begin{aligned}
&-4(|\widehat{m-1}, m+1; \widehat{m}||\widehat{m}, m+2; \widehat{m-1}| + |\widehat{m}; \widehat{m-1}, m+1||\widehat{m+1}; \widehat{m-2}, m|) \\
&+4F(|\widehat{m-1}, m+1, m+2; \widehat{m-1}| + |\widehat{m+1}; \widehat{m-2}, m+1|) \\
&+4s(|\widehat{m-1}, m+2; \widehat{m}| + |\widehat{m}; \widehat{m-2}, m, m+1|) \\
&-2q_0^2 HF + 4q_0 r_0 sf + 4q_0(-1)^m Hs \\
= \;&-2q_0^2 HF + 4q_0 r_0 sf + 4q_0(-1)^m Hs.
\end{aligned}$$

Then we have

$$\begin{aligned}
&(D_x^2 - iD_t)s \cdot f \\
= \;&(-1)^m(-2q_0^2 HF + 4q_0 r_0 sf + 4q_0(-1)^m Hs) \\
= \;&-q_0^2 hf - 2q_0 r_0 gf - q_0 gh,
\end{aligned}$$

which proves (14b). Equation (14c) can be verified similarly.

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