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Full-deautonomisation of a class of second-order mappings in ancillary form

Basil Grammaticos[1] *and Ralph Willox*[2]

[1] *Université Paris-Saclay, CNRS/IN2P3, IJCLab, 91405 Orsay, France and Université de Paris, IJCLab, 91405 Orsay France*

[2] *Graduate School of Mathematical Sciences, the University of Tokyo, 3-8-1 Komaba, Meguro-ku, 153-8914 Tokyo, Japan*

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Abstract

We present an application of the full-deautonomisation method to a class of second-order mappings which, using an ancillary variable, can be cast into a form that greatly facilitates the study of their singularities. The ancillary approach was originally introduced to make it possible to construct discrete Painlevé equations associated with the affine Weyl group $\mathrm{E}_8^{(1)}$ by deautonomising a QRT mapping. The full-deautonomisation method has been shown to offer a practical technique for calculating the exact dynamical degree of a mapping, whereby allowing the detection of discrete integrability using only singularity analysis. We study the confinement property for a given singularity, for a wide class of mappings that includes the autonomous limit of the standard additive Painlevé equation with $\mathrm{E}_8^{(1)}$ symmetry. This leads to a class of non-autonomous mappings, which can be integrable or not, for which we obtain their exact dynamical degrees. The case of a non-confining singularity is also analysed and again we obtain the corresponding dynamical degrees.

Keywords: integrable mappings, singularity confinement, deautonomisation, integrability tests

1 Introduction: singularity confinement

The term singularity confinement [1] describes a situation where a singularity, appearing spontaneously during the iteration of a mapping because of a particular choice of initial conditions, disappears again after a certain number of iterations. What is meant here by ‘singularity’ is when the inverse mapping becomes undefined, which is tantamount to the solution of the mapping losing some degree of freedom. Since it was observed that the mappings which were integrable through spectral methods do possess the singularity confinement property [2], it was surmised that the existence or not of the latter could offer a discrete integrability criterion.

In this sense singularity confinement is the discrete analogue of the Painlevé property which characterises differential systems that are integrable through spectral methods. (When integrability is obtained through other methods, like linearisability or even direct solvability, the relation to singularity confinement or the Painlevé property ceases, as shown in [3]). Before proceeding further let us give an example of the workings of singularity confinement in the case of a second-order mapping. We consider a mapping that belongs to Class I of the Quispel-Roberts-Thompson (QRT) family of mappings [4], according to the classification in [5]. (An easily consultable and detailed list of the canonical forms of the QRT mappings in this classification is given in the Appendix). The mapping has the form

$$x_{n+1} + x_{n-1} = 1 + \frac{z}{x_n}, \tag{1}$$

where z is a non-zero constant. Now suppose that, due to a specific choice of initial conditions, at some iteration m, x_m takes the value 0, while x_{m-1} is finite and non-zero. This leads to the value $x_{m+1} = \infty$ which obviously does not depend on x_{m-1} and therefore corresponds to a singularity for this mapping. Iterating further, one obtains the values $x_{m+2} = 1$, $x_{m+3} = \infty$, $x_{m+4} = 0$ whereupon x_{m+5} is indeterminate since its value would involve $\infty - \infty$. The confinement of the singularity consists in the removal of this indeterminacy. To this end one invokes continuity with respect to the initial conditions, introducing a small quantity ϵ and assuming $x_m = \epsilon$ instead of exactly 0. It is then straightforward to compute $x_{m+1}, \cdots, x_{m+4}$ and obtain x_{m+5}, whereupon taking the limit $\epsilon \to 0$ it turns out that x_{m+5} is no longer indeterminate and, in fact, takes exactly the value x_{m-1}. Thus the mapping has recovered the lost degree of freedom.

The usefulness of the singularity confinement criterion became apparent when it was combined with what is called the deautonomisation procedure [6]. The latter consists in assuming that the parameters that appear in a mapping are functions of the independent variable. Applying the singularity confinement criterion one can, in principle, obtain integrable non-autonomous extensions of a given integrable mapping. The main bulk of the discrete Painlevé equations has been discovered in this way [7], [8]. In the case of mapping (1), we assume that z is a function of n and, again requiring that x_{m+5} be well defined in the sense described above, we find for z_n the constraint:

$$z_{n+4} - z_{n+3} - z_{n+1} + z_n = 0. \tag{2}$$

The solution of equation (2) is $z_n = \alpha n + \beta + \phi_3(n)$ for generic values of α and β and where, for positive integers k, $\phi_k(n)$ denotes a non-constant function with period k. The resulting non-autonomous system is a well-known [9] discrete Painlevé equation.

2 Late confinement and the full-deautonomisation approach

A natural question to ask is what happens if one chooses not to implement the constraint (2) for the parameter in equation (1) but, instead, pursues the iterations. In this case x_{m+5} takes the value ∞ and the succession of singularities becomes $\{0, \infty, 1, \infty, 0, \infty, 1, \infty, 0\}$ and it turns out that x_{m+9} can have a finite value and can even depend on x_{m-1}, provided that the following constraint holds:

$$z_{n+8} - z_{n+7} - z_{n+5} + z_{n+4} - z_{n+3} - z_{n+1} + z_n = 0. \tag{3}$$

We call this situation a 'late' confinement (and in fact, infinitely many possibilities for late confinement do exist if one pursues the iterations even further). It turns out however that the solution of (3) does not have a nice secular plus periodic solution as was the case for (2): its characteristic polynomial has six complex and two real roots, which can in principle be expressed in terms of radicals, the largest real root being $1.425005268\cdots$. Moreover the non-autonomous equation constructed with a z_n obeying (3) is *not* integrable. The dynamical degree of its solution can be computed using the method introduced by Halburd [10], under the name of Diophantine approximation. (We remind that the dynamical degree of a rational mapping, the type of mapping we will be dealing with in this paper, is computed from the homogeneous degree d_n of the iterates of the mapping, as the limit $\lambda = \lim_{n\to\infty} d_n^{1/n}$; this limit always exists and cannot take values less than 1). The dynamical degree for this mapping, obtained numerically from the Diophantine approximation method, converges to approximately 1.42501 after 50 iterations. A first remark is that since this dynamical degree is clearly greater than 1 the degree of the successive iterates grows exponentially, a feature which indicates non-integrability for the mapping. But most important is the observation that the value of the dynamical degree coincides quite nicely with that of the largest root of the characteristic polynomial of the constraint (3). This is not a simple coincidence, as was shown in [11]. In that paper, the present authors in collaboration with T. Mase and A. Ramani, addressed the question of deautonomisation through an algebro-geometric approach. The main result we obtained was that when a mapping like (1) is regularised through exactly 8 blow-ups, one obtains the constraint given by the singularity confinement requirement *at the first confinement opportunity*. Moreover, the characteristic polynomial associated with this constraint is in fact the same as that for the isomorphism induced by the mapping on the Picard group for the surface obtained after blow-up. When more than 8 blow-ups are necessary for the regularisation of the system, the confinement constraints that are obtained give rise to non-integrable mappings, but the characteristic polynomials for the parameter constraints still coincide with those for the isomorphisms on the Picard group of the surfaces obtained through blow-up. Hence, the dynamical degrees for those mappings can be read off directly from the confinement conditions obtained from singularity confinement: they coincide with the largest roots for the characteristic polynomials for the constraints on the parameter.

The result presented above, together with the analysis of [11] led to the introduction of what is called the 'full-deautonomisation' method as an enhanced (with respect to singularity confinement) discrete integrability criterion. The necessity of an integrability criterion going beyond singularity confinement became apparent after the discovery of mappings which were non-integrable despite the fact that they possess only confined singularities. We know by now that infinitely many such mappings exist [12], so let us choose an explicit example that is slightly different from the classical one given in [13]:

$$x_{n+1} + x_{n-1} = x_n + \frac{1}{x_n^2 - 1}. \tag{4}$$

The singularity patterns of (4) are $\{\pm 1, \infty, \infty, \mp 1\}$ which are both confining patterns, but the dynamical degree of this mapping is greater than one. Its value can be calculated exactly and turns out to be equal to $\frac{3+\sqrt{5}}{2}$. Deautonomising the mapping by assuming that the right-hand side is written as $a_n x_n + b_n/(x_n^2 - 1)$ and requiring that the singularity patterns remain the same, we find the constraints $a_n = 1$ and $b_{n+3} = b_n$ which clearly do not have any connection to the dynamical degree of the mapping. However, as explained in [14], the spirit of the full-deautonomisation method is to explore all possible extensions of the initial mapping which would preserve the singularity patterns. Thus we can (must) consider adding extra terms to the right-hand side of (4) which preserve the initial singularity patterns. To make a long story short: the proper extension of (4) to consider is

$$x_{n+1} + x_{n-1} = x_n + \frac{1 + f_n x_n}{x_n^2 - 1}. \tag{5}$$

By requiring that the singularity pattern of the original mapping be preserved, we obtain the constraint

$$f_{n+3} - 2f_{n+2} - 2f_{n+1} + f_n = 0. \tag{6}$$

It can be easily shown that the largest root of the characteristic polynomial for (6) is $\frac{3+\sqrt{5}}{2}$, i.e. exactly the dynamical degree of the mapping (4).

Several examples presented in [2], [15] have established the usefulness of the full-deautonomisation approach as a discrete integrability criterion. Most of the examples presented in those publications concern rather simple mappings belonging to the classes I and II of the classification of the QRT canonical forms in [5]. In the present paper we shall apply the full-deautonomisation method to a different class of mappings, class VII in [5] (which upon integrable deautonomisation leads to discrete Painlevé equations associated to the affine Weyl group $E_8^{(1)}$, i.e. the highest one in the Sakai classification [16]), in order to show that the full-deautonomisation approach can still be made to work even for these highly non-trivial and intricate systems.

3 Trihomographic mappings and the ancillary representation

As we mentioned in the introduction, the singularity confinement criterion has been most useful in the derivation of discrete Painlevé equations using the deautonomisation procedure. Starting (usually) from a QRT mapping and assuming that the parameters that appear in the mapping are functions of the independent variable n, one can apply the confinement criterion in order to obtain the precise n-dependence in the parameters that ensures integrability. One difficulty lies in the initial mapping, namely: how does one choose the mapping to deautonomise in order to obtain the discrete analogue of a given Painlevé equation? What proved to be immensely useful in tackling this conundrum was the classification of the canonical forms of the QRT mappings. Nine forms were identified initially in [5], one of which was strictly asymmetric (in the QRT terminology), but it turned out that two more forms do exist [17] and which correspond to asymmetric mappings as well. (As pointed out in the introduction, a complete and detailed list of the QRT canonical forms can be found in the Appendix).

Thanks to the canonical form classification it was possible to construct the discrete analogue of the Painlevé VI equation [18], in QRT-symmetric form, deautonomising a mapping belonging to the canonical class VI (the numbering here is purely coincidental). It was thus expected that mappings from the classes VII and VIII would give rise to discrete Painlevé equations associated to the affine Weyl group $\mathrm{E}_8^{(1)}$ of the Sakai classification [16]. In particular, class VII mappings would give rise to additive equations while class VIII ones would correspond to multiplicative ones. (Unfortunately no canonical form could be proposed which upon deautonomisation would lead to the elliptic discrete Painlevé equations that were discovered by Sakai). While the approach sketched in the introduction appears straightforward, there exist technical difficulties which make the application of deautonomisation quite arduous. For example, starting form an autonomous expression such as

$$\frac{(x_{n+1}-x_n-z^2)(x_{n-1}-x_n-z^2)+4z^2x_n}{x_{n+1}-2x_n+x_{n-1}-2z^2}=f(x_n), \tag{7}$$

it is a priori not clear at all by which combinations of z_{n+1}, z_n and z_{n-1} the occurences of the parameter z should be replaced to make the calculations involved in the deautonomisation tractable. Thus the derivation of $\mathrm{E}_8^{(1)}$-associated discrete Painlevé equations had to wait until their explicit geometrical construction [19], when it became clear that the generic symmetric, additive, equation has the form

$$\frac{(x_n-x_{n+1}+(z_n+z_{n+1})^2)(x_n-x_{n-1}+(z_n+z_{n-1})^2)+4x_n(z_n+z_{n+1})(z_n+z_{n-1})}{(z_n+z_{n-1})(x_n-x_{n+1}+(z_n+z_{n+1})^2)+(z_n+z_{n+1})(x_n-x_{n-1}+(z_n+z_{n-1})^2)}=R(x_n), \tag{8}$$

with R given by

$$R(x_n)=2\frac{x_n^4+S_2x_n^3+S_4x_n^2+S_6x_n+S_8}{S_1x_n^3+S_3x_n^2+S_5x_n+S_7}, \tag{9}$$

where S_k ($k\leq 8$) is the elementary symmetric function of degree k in the 8 variables $z_n+\kappa_n^i$ ($i=1,\dots,8$), and where the parameters κ^i are, generically, functions of the independent variable as well.

Another important milestone in the solution of this problem was the derivation in [19] of the basic Miura transformations in $\mathrm{E}_8^{(1)}$, the form of which actually turned out to be suitable for the representation of *all* discrete Painlevé equations [20]. This form was dubbed 'trihomographic' and in the simplest, additive and symmetric, case it has the form

$$\frac{x_{n+1}-(z_n+z_{n-1}+k_n)^2}{x_{n+1}-(z_n+z_{n-1}-k_n)^2}\,\frac{x_{n-1}-(z_n+z_{n+1}+k_n)^2}{x_{n-1}-(z_n+z_{n+1}-k_n)^2}\,\frac{x_n-(2z_n+z_{n-1}+z_{n+1}-k_n)^2}{x_n-(2z_n+z_{n-1}+z_{n+1}+k_n)^2}=1. \tag{10}$$

Most importantly, it turned out that the trihomographic form (10) is, in fact, strictly equivalent to a mapping of the form (8) with a right-hand side given by

$$R(x_n)=\frac{x_n-k_n^2}{2z_n+z_{n-1}+z_{n+1}}+2z_n+z_{n-1}+z_{n+1}. \tag{11}$$

Note however that this result does not cover the case of general additive $\mathrm{E}_8^{(1)}$-associated equations since, as explained in [20], these can only be expressed in terms of a system of four coupled trihomographic equations. While this is perfectly acceptable in principle, it is not very convenient from a practical (computational) point of view, when one would like to apply the deautonomisation procedure. Fortunately a solution to this final hurdle also exists.

It was observed in [21] that by introducing an ancillary variable ξ as

$$x_n = \xi_n^2, \tag{12}$$

it is possible to write the right-hand side (9) as

$$R(x_n) = 2\xi_n \frac{\Pi(\xi_n) + \Pi(-\xi_n)}{\Pi(\xi_n) - \Pi(-\xi_n)}, \tag{13}$$

where $\Pi(\xi)$ is given by

$$\Pi(\xi_n) = \prod_{i=1}^{8} (z_n + \kappa_n^i + \xi_n). \tag{14}$$

It is then straightforward to show that equation (8) with r.h.s. (9) can be written as

$$\frac{x_{n+1} - (\xi_n - z_n - z_{n+1})^2}{x_{n+1} - (\xi_n + z_n + z_{n+1})^2} \frac{x_{n-1} - (\xi_n - z_n - z_{n-1})^2}{x_{n-1} - (\xi_n + z_n + z_{n-1})^2} = \frac{\prod_{i=1}^{8} (\kappa_n^i + z_n - \xi_n)}{\prod_{i=1}^{8} (\kappa_n^i + z_n + \xi_n)}. \tag{15}$$

As we will see in Section 4, this factorised 'ancillary' form – obtained at the price of introducing the ancillary variable ξ – greatly simplifies the application of the singularity confinement criterion. It is also important to point out here that, thanks to the introduction of the appropriate ancillary variable it was possible to propose what is probably the simplest form [22] of the discrete elliptic Painlevé equation. Furthermore, when in Section 5 we shall use the full-deautonomisation method to study general confining, additive, mappings with the same left-hand sides as the class VII mappings (i.e. equations of the form (7) with general right-hand sides but which have the singularity confinement property) it will become clear that the ancillary forms of these mappings are in fact a crucial and necessary ingredient, without which there would be no hope at all of being able to implement it.

4 Singularity confinement in the ancillary representation

A first type of singularity of equation (15) is obvious: x_{n+1} does not depend on the value of x_{n-1} (for generic x_{n-1}) when the right-hand side of (15) is equal to zero or infinity. Given the structure of (15) the difference between these two cases corresponds to a simple change of sign in ξ_n which, however, is a change that leaves the equation invariant and it therefore suffices to study just one of these two possibilities. Here we shall choose the case in which we encounter a zero in the right-hand side of (15). So let us assume that at some iteration m we have $\xi_m = \kappa_m^j + z_m$, for some specific j. Balancing this with the left-hand side we find $x_{m+1} = (\kappa_m^j - z_{m+1})^2$ and taking the square root we find $\xi_{m+1} = \pm(\kappa_m^j - z_{m+1})$. Iterating further we obtain, in the left-hand side, a rational factor containing x_{m+2} multiplied by the expression

$$\frac{x_m - (\xi_{m+1} - z_{m+1} - z_m)^2}{x_m - (\xi_{m+1} + z_{m+1} + z_m)^2} \equiv \frac{(\kappa_m^j + z_m)^2 - (\xi_{m+1} - z_{m+1} - z_m)^2}{(\kappa_m^j + z_m)^2 - (\xi_{m+1} + z_{m+1} + z_m)^2},$$

which is either zero or infinity depending on the choice of sign in ξ_{m+1}. In either case it must be balanced by a zero factor or an infinity on the right-hand side of the equation if we want to have an opportunity for the singularity to be confined at this stage. This balance can of course be achieved using any of the eight factors in the right-hand side of (15) at $n = m + 1$ but, for the sake of simplicity, let us decide that it occurs in the factor that contains κ^j_{m+1}. This factor takes the form

$$\frac{\kappa^j_{m+1} + z_{m+1} - \xi_{m+1}}{\kappa^j_{m+1} + z_{m+1} + \xi_{m+1}} \equiv \frac{\kappa^j_{m+1} + z_{m+1} \mp (\kappa^j_m - z_{m+1})}{\kappa^j_{m+1} + z_{m+1} \pm (\kappa^j_m - z_{m+1})}.$$

It is straightforward to show that whatever the choice of sign for ξ_{m+1} the condition for the two sides of (15) to balance (either through the numerator or the denominator) is always

$$\kappa^j_{m+1} + \kappa^j_m = 0. \tag{16}$$

Repeating this analysis after introducing a small parameter ϵ, just as we did for (1), shows that ξ_{m+2} indeed depends on x_{n-1} if we require κ^j to obey the constraint (16), and therefore that with this choice of κ^j this particular singularity is confined immediately after it occurs. As a result, since there are eight ways in which this type of singularity can arise, we have to have all eight κ^i change sign at each iteration for the singularity to be confined in all possible cases:

$$^\forall i = 1, \ldots, 8: \ \kappa^i_n = \kappa^i_0 \, (-1)^n. \tag{17}$$

However, the singularity examined above is not the only possible one for (15). Another singularity might arise when ξ_m becomes infinite but since the ancillary variable ξ is present on both sides of the equation, one can in fact make sure that these infinities balance out precisely. This is tantamount to infinity not becoming a singularity for (15), meaning that x_{m+1} does indeed depend on x_{m-1}. For this to happen the following condition for the parameter z must be satisfied:

$$z_{m+1} - 2z_m + z_{m-1} = \frac{1}{2} \sum_{i=1}^{8} \kappa^i_m. \tag{18}$$

Since only the combinations $z_n + z_{n+1}$ and $z_n + z_{n-1}$ appear in the left-hand side of (15), it is possible to introduce a gauge $z_n \mapsto z_n + (-1)^n \delta$ on z_n, with $\delta = -\frac{1}{8} \sum_{i=1}^{8} \kappa^i_0$, which leaves the equation invariant if we redefine the κ^i as $\kappa^i_n \mapsto \kappa^i_n - (-1)^n \delta$, thereby cancelling the right-hand side in (18) if all κ^i are of the form (17). We are thus left with the constraint

$$z_{n+1} - 2z_n + z_{n-1} = 0, \tag{19}$$

the solution of which is $z_n = \alpha n + \beta$, which together with (17) under the condition that

$$\sum_{i=1}^{8} \kappa^i_0 = 0, \tag{20}$$

ensures that equation (15) does not possess a singularity at infinity and that all other singularities are confined as soon as they arise. The equation thus obtained corresponds to the additive $\mathrm{E}_8^{(1)}$-associated discrete Painlevé equation which was first obtained in [19] from purely geometrical arguments.

A few comments are in order here. First of all, given the form of equation (15) one might have the impression that $\xi_m = 0$ also corresponds to a singularity. A straightforward calculation however shows that this is not the case: x_{m+1} still depends on x_{m-1} even when $\xi_m = 0$. This is also easily assessed on the equivalent form (8) with right-hand side as in (9), for which $x_n = 0$ is clearly not a singularity. Furthermore, given the relation of the ancillary equation (15) to the non-ancillary form (8)-(9), it is natural to wonder whether the singularity structures we found in the case of the ancillary representation are exactly the same in the non-ancillary one. While this is easily checked for the case where $x_m = \xi_m^2 = \infty$, for which a straightforward calculation yields condition (18) if one requires that the equation in non-ancillary form does not possess a singularity at $x_m = \infty$, this is not at all obvious for the singularities at finite positions. Although, with sufficient hindsight, it can be checked that $x_m = (\kappa_m^j + z_m)^2$ is indeed a singularity of (8)-(9) and that the conditions (17) ensure that the ensuing singularity patterns are the same as for the ancillary case, this seems to be very difficult to ascertain directly on (8)-(9). While finding the singularities is certainly possible, performing the singularity confinement analysis quickly leads to highly intractable calculations. This is the reason why the ancillary form is in our opinion crucial in the singularity analysis of equations such as (8) or more generally (7).

A last comment concerns the type of singularity patterns that we have investigated here. These involved clearly very special choices of confinement (or even 'non-singularity' for the behaviour at infinity). While it is certainly possible to investigate other, more general, singularity patterns that involve longer singularity patterns and/or confinement conditions that mix different κ^i, such analysis quickly becomes quite lengthy and complicated. Faced with the profusion of the possible singular behaviours we decided to focus on a situation where a single singularity would control integrability. In the following we shall therefore limit ourselves, for singularities at finite positions, to the study of 'minimal' length singularity patterns such as those above: entering and exiting immediately, at the same iteration. However, for the singularity that might arise at $\xi_m = \infty$ we shall allow for singularity patterns of arbitrary length.

5 A general class VII mapping and its integrability properties

Having set the frame, we now proceed to examine the confinement structure and possible integrability properties of a class VII mapping where the right-hand side can be a rational expression involving polynomials of degree higher (but also lower) than in (9). As explained above, since we want to be able to ensure confinement for the singularities of the mapping in the easiest way possible (at least for singularities that arise at finite positions) we are going to work within the ancillary representation and thus we will consider equations of the form

$$\frac{x_{n+1} - (\xi_n - z_n - z_{n+1})^2}{x_{n+1} - (\xi_n + z_n + z_{n+1})^2} \, \frac{x_{n-1} - (\xi_n - z_n - z_{n-1})^2}{x_{n-1} - (\xi_n + z_n + z_{n-1})^2} = \frac{\prod_{i=1}^k (\kappa_n^i + z_n - \xi_n)}{\prod_{i=1}^k (\kappa_n^i + z_n + \xi_n)}. \tag{21}$$

where k is a positive integer, not necessarily equal to 8. The general, non-ancillary, form of these mappings is similar to that of (8),

$$\frac{(x_n-x_{n+1}+(z_n+z_{n+1})^2)(x_n-x_{n-1}+(z_n+z_{n-1})^2)+4x_n(z_n+z_{n+1})(z_n+z_{n-1})}{(z_n+z_{n-1})(x_n-x_{n+1}+(z_n+z_{n+1})^2)+(z_n+z_{n+1})(x_n-x_{n-1}+(z_n+z_{n-1})^2)} = R_k(x_n), \tag{22}$$

but now with a right-hand side that, depending on the parity of k, takes the form

$$^{\forall}k \text{ odd}: \quad R_k(x) = 2\,\frac{e_1^k\,x^{\frac{k-1}{2}} + e_3^k\,x^{\frac{k-3}{2}} + \cdots + e_{k-2}^k\,x^1 + e_k^k}{e_0^k\,x^{\frac{k-1}{2}} + e_2^k\,x^{\frac{k-3}{2}} + \cdots + e_{k-3}^k\,x^1 + e_{k-1}^k}, \tag{23}$$

or

$$^{\forall}k \text{ even}: \quad R_k(x) = 2\,\frac{e_0^k\,x^{\frac{k}{2}} + e_2^k\,x^{\frac{k-2}{2}} + \cdots + e_{k-2}^k\,x^1 + e_k^k}{e_1^k\,x^{\frac{k-2}{2}} + e_3^k\,x^{\frac{k-4}{2}} + \cdots + e_{k-1}^k}, \tag{24}$$

where e_m^k denotes the degree m elementary symmetric polynomial in k variables $z_n+\kappa_n^i$ ($i = 1,\ldots,k$),

$$e_m^k = \sum_{1\le j_1<\cdots<j_m\le k} (z_n+\kappa_n^{j_1})\cdots(z_n+\kappa_n^{j_m}), \tag{25}$$

and where we define e_0^k to be identically 1. All the mappings defined by equation (21), for general k, possess singularities at $\xi_m = \pm(\kappa_m^j + z_m)$ ($i = 1,\ldots,k$) and it should be clear from the analysis carried out in section 4 that the confinement of these singularities works in exactly the same way as in the special case $k = 8$: requiring that all parameters κ_n^i have a period 2 dependence in the independent variable n,

$$^{\forall}i = 1,\ldots,k: \ \kappa_n^i = \kappa_0^i\,(-1)^n, \tag{26}$$

ensures that all singularities that arise at $\xi_m = \pm(\kappa_m^j + z_m)$ are immediately confined. This then only leaves the behaviour at $\xi_m = \infty$ to be considered in detail.

Expanding both sides of equation (21) in powers of $\frac{1}{\xi_n}$ we obtain

$$1 - \frac{4}{\xi_n}(z_{n+1}+2z_n+z_{n-1}) + o(\frac{1}{\xi_n}) = (-1)^k - \frac{2(-1)^k}{\xi_n}\left(kz_n + \sum_{i=1}^k \kappa_n^j\right) + o(\frac{1}{\xi_n}), \tag{27}$$

which clearly cannot hold at the limit $\xi_n \to \infty$ when k is odd. Hence, for odd values of k, equation (21) necessarily always has a singularity at $\xi_n = \infty$. However, when k is even, the condition

$$z_{n+1} + (2-\frac{k}{2})z_n + z_{n-1} = \frac{1}{2}\sum_{i=1}^k \kappa_n^i, \tag{28}$$

ensures that the equation does not become singular at $\xi_n = \infty$. As before, a suitable gauge on the parameters, $z_n \mapsto z_n + (-1)^n\delta_k$, $\kappa_n^i \mapsto \kappa_n^i - (-1)^n\delta_k$ with $\delta_k = -\frac{1}{k}\sum_{i=1}^k \kappa_0^i$, leaves the equation invariant and allows one to put the sum of all κ^i equal to zero for κ^i of the form (26). In this particular gauge we are then left with the constraints

$$z_{n+1} + (2-\frac{k}{2})z_n + z_{n-1} = 0. \tag{29}$$

and

$$\sum_{i=1}^{k} \kappa_0^i = 0. \tag{30}$$

We remark that for $k = 8$ we find, as expected, the conditions (19) and (20).

Next we turn to the case where infinity is indeed a singularity. Let us first, as an example, consider the case where this singularity confines after one step, leading to the pattern $\{\infty, \infty\}$. It turns out that this pattern can be confined both for even as well as for odd k. Starting from a value $\xi_m = 1/\epsilon$, we find that $\xi_{m+1} \sim 1/\epsilon$ as well but that ξ_{m+2} can take finite values that depend on x_{m-1} when

$$z_{m+3} + 2z_{m+2} + 2z_{m+1} + z_m = \frac{k}{2}(z_{m+2} + z_{m+1}) + \frac{1}{2}\sum_{i=1}^{k}(\kappa_{n+1}^i + \kappa_{n+2}^i), \tag{31}$$

for even k and

$$z_{m+3} + 2z_{m+2} + 2z_{m+1} + z_m = \frac{k+1}{2}(z_{m+2} + z_{m+1}) + \frac{1}{2}\sum_{i=1}^{k}(\kappa_{n+1}^i + \kappa_{n+2}^i), \tag{32}$$

for odd k. Note that for κ^i that satisfy the confinement conditions (26), the sums $\sum_{i=1}^{k}(\kappa_{n+1}^i + \kappa_{n+2}^i)$ in the right-hand sides of these equations are always zero, irrespective of the gauge one chooses for the parameters, and the two equations can thus be cast in the form,

$$z_{m+3} + \left(2 - \left[(k+1)/2\right]\right)(z_{m+2} + z_{m+1}) + z_m = 0, \tag{33}$$

for all integers $k \geq 1$, where $\left[(k+1)/2\right]$ stands for the integer part of the fraction $\frac{k+1}{2}$.

We shall not examine any more special cases and immediately give the confinement condition for the general case where the singularity pattern at infinity consists in a succession of infinities $\{\infty, \infty, \cdots, \infty\}$ of length $\ell + 1$, which for $\ell = 0$ includes the case where $\xi = \infty$ is not a genuine singularity of the equation, in case k is even. In fact, it turns out that a pattern corresponding to an even value for ℓ can only exist in case k is even as well. Singularity patterns with odd ℓ on the other hand exist for both even as well as for odd values of k.

The general confinement condition for a singularity pattern $\{\infty, \infty, \cdots, \infty\}$ of length $\ell + 1$ is

$$z_{m+\ell+2} + \left(2 - \left[(k+1)/2\right]\right)(z_{m+\ell+1} + \cdots + z_{m+1}) + z_m = \frac{1}{2}\sum_{i=1}^{k}\left(\kappa_{m+1}^i + \cdots + \kappa_{m+\ell+1}^i\right), \tag{34}$$

for any non-negative value of ℓ if k is even, or only for odd ℓ when k is odd. As before, when ℓ is odd the right-hand side in this constraint is automatically zero when the κ^i satisfy the confinement conditions (26). However, for even values of ℓ the right-hand side of the constraint can still be put to zero in the aforementioned gauge $z_n \mapsto z_n + (-1)^n\delta_k$, $\kappa_n^i \mapsto \kappa_n^i - (-1)^n\delta_k$ with $\delta_k = -\frac{1}{k}\sum_{i=1}^{k}\kappa_0^i$. Hence, in this gauge the confinement conditions can be summarized as

$$z_{m+\ell+2} + \left(2 - \left[(k+1)/2\right]\right)(z_{m+\ell+1} + \cdots + z_{m+1}) + z_m = 0, \tag{35}$$

for odd k and ℓ, or for any non-negative ℓ if k is even, and where the bracket notation is again used to indicate the integer part of the argument.

Under the hypothesis that the full-deautonomisation method also works for mappings of this particular class, the largest root of the characteristic equation associated to the constraint (35),

$$\lambda^{\ell+2} + \big(2 - \big[(k+1)/2\big]\big)\,(\lambda^{\ell+1} + \cdots + \lambda) + 1 = 0, \tag{36}$$

should coincide with the dynamical degree of the mapping. It is easy to verify that whenever the inequality

$$2 + \frac{2}{\ell+1} = 4 - \frac{2\ell}{\ell+1} < \left[\frac{k+1}{2}\right] \tag{37}$$

is satisfied, the characteristic polynomial (36) must have a root that is greater than 1 and hence, under the assumption that the full-deautonomisation method indeed works, the corresponding mapping should be non-integrable. The inequality (37) is always satisfied when $k \geq 9$, for any value of ℓ, and all corresponding mappings should have dynamical degrees greater than 1. On the other hand, inequality (37) can never be satisfied for $k = 1, 2, 3$ or 4, whatever ℓ one chooses, and the resulting mappings should be integrable regardless of the length of the singularity pattern at ∞.

These results are summarized in Table 1, in which the largest roots of the characteristic polynomial (36) are listed for the basic cases: white squares correspond to integrable mappings, grey ones to non-integrable mappings and black squares indicate impossible combinations of k and ℓ.

ℓ \ k	1	2	3	4	5	6	7	8	9	…
0		1		1		1		1		…
1	1	1	1	1	1	1	2.618	2.618	3.732	…
2		1		1		1.722		2.890		…
3	1	1	1	1	1.883	1.883	2.966	2.966	3.985	…
4		1		1		1.947		2.989		…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 1. *Largest roots of the characteristic polynomial (36)*

Interpreting the values in Table 1 as the dynamical degrees of the mappings that correspond to those particular combinations of k and ℓ, it is clear that the mappings for $k = 1, 2, 3$ and 4 must be very special since they should be integrable for any length of the singularity pattern at infinity, and hence also when that singularity does not confine at all. This is only possible if the corresponding mappings are linearisable. It turns out that this is indeed the case, but also that the cases $k = 1, 2$ and $k = 3, 4$ are in fact quite different.

For $k = 1$ we have

$$\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)} = 2(z_n + \kappa_n^1), \quad (38)$$

a mapping which turns out to have bounded degree growth when κ^1 obeys (26), for arbitrary z_n. As was shown in [23], under the constraint (26), this equation, is just one obtained in [24] where it was shown that it can be transformed to the Gambier-type [25] mapping $(y_n + y_{n+1})(y_n + y_{n-1}) = f_n y_n$, where f_n is a free fuction of n. Moreover, at $x = \infty$ it has an 'anticonfined' singularity rather than a genuinely unconfined one:

$$\ldots, \infty, \infty, \infty, h(x_0), x_0, \infty, \infty, \infty, \ldots, \quad (39)$$

where the function $h(x_0)$ is obtained from the inverse map of (38) from the initial conditions $(x_0, x_1 = \infty)$. Note that such anticonfined singularities are perfectly compatible with the linearisable and integrable character of these mappings [26]. Furthermore, it is known that mappings with bounded degree growth are either periodic or can be transformed into projective mappings on $\mathbb{P}^2(\mathbb{C})$ [27]. In fact, when z_n is not free but obeys the constraint

$$z_{n+\ell+2} + z_{n+\ell+1} + \cdots + z_{n+1} + z_n = 0, \quad (40)$$

i.e. when the singularity at infinity in (38) indeed confines after ℓ steps (ℓ odd), the mapping becomes periodic with period $\ell + 3$ (which is the same period as for the z_n).

The same result holds for the mapping at $k = 2$, for which we find the equation:

$$\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)} = z_n + \frac{x_n - (\kappa_0^1)^2}{z_n}, \quad (41)$$

which has bounded degree growth as well for arbitrary z_n, when the κ^i satisfy (26). Equation (41) is also linearisable: it was first identified in [28] where it was shown that it can be integrated through a transformation to the "standard" Gambier equation $(y_n + y_{n+1})(y_n + y_{n-1}) = f_n(y_n^2 - 1)$, where f_n is again a free function of n. As for the case $k = 1$, when z_n is indeed a free function of n, this mapping also has an anticonfined singularity at $x = \infty$ of the form (39). On the other hand, when z_n obeys the confinement constraint (40) (but now for any non-negative integer ℓ) the mapping becomes periodic with period $\ell + 3$.

The mappings obtained at $k = 3$ and $k = 4$ on the contrary have linear (unbounded) degree growth and possess a genuine unconfined singularity at $x = \infty$ which, however, is still compatible with their linearisable and therefore integrable character.

For $k = 3$ we find

$$\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)}$$

$$= 2\frac{e_1^3 x_n + e_3^3}{x_n + e_2^3}. \quad (42)$$

When the κ^i ($i = 1, 2, 3$) satisfy (26) this is a linearisable mapping of what has been called the third kind, the integration of which was presented in [23]. When the z_n do obey the confinement condition at $x = \infty$,

$$z_{n+\ell+2} + z_n = 0, \quad (43)$$

for odd ℓ, the mapping becomes periodic with period $2(\ell + 2)$, i.e. the periodicity of the z_n.

For $k = 4$ we find

$$\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)}$$
$$= 2\frac{x_n^2 + e_2^4 x_n + e_4^4}{e_1^4 x_n + e_3^4}, \quad (44)$$

which is linearisable for κ^i that obey the constraint (26). Equation (44) was first identified in [24] and [28] and its detailed integration was given in [29] (equation (83) in that reference). When the z_n satisfy the condition (43) the mapping becomes periodic with period $2(\ell + 2)$, for any non-negative integer value for ℓ.

The cases $k = 5, 6$ are more interesting. For both cases the constraint on z_n, for the singularity at infinity to confine, reads

$$z_{n+\ell+2} - (z_{n+\ell+1} + \cdots + z_{n+1}) + z_n = 0, \quad (45)$$

for odd ℓ in the case $k = 5$ and for general non-negative integer ℓ for $k = 6$, in the gauge (30). For $\ell = 1$ this leads to a solution $z_n = \alpha n + \beta + \gamma(-1)^n$ and the corresponding equations are discrete Painlevé equations associated with the affine Weyl groups $E_6^{(1)}$,

$$\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)}$$
$$= 2\frac{e_1^5 x_n^2 + e_3^5 x_n + e_5^5}{x_n^2 + e_2^5 x_n + e_4^5}, \quad (46)$$

for $k = 5$ and $E_7^{(1)}$,

$$\frac{(x_n - x_{n+1} + (z_n + z_{n+1})^2)(x_n - x_{n-1} + (z_n + z_{n-1})^2) + 4x_n(z_n + z_{n+1})(z_n + z_{n-1})}{(z_n + z_{n-1})(x_n - x_{n+1} + (z_n + z_{n+1})^2) + (z_n + z_{n+1})(x_n - x_{n-1} + (z_n + z_{n-1})^2)}$$
$$= 2\frac{x_n^3 + e_2^6 x_n^2 + e_4^6 x_n + e_6^6}{e_1^6 x_n^2 + e_3^6 x_n + e_5^6}, \quad (47)$$

for $k = 6$ respectively. Both equations were first derived in [23].

For $k = 6$ and $\ell = 0$ on the other hand, z_n must satisfy $z_{n+2} - z_{n+1} + z_n = 0$ and is therefore expressed in terms of the cubic roots of unity. The corresponding equation is of the form (47), but where we must impose $\sum_{i=1}^{6} \kappa^i = 0$, and the mapping turns out to be periodic with period 6.

When $\ell > 1$, for the cases listed in Table 1 for $k = 5$ and $k = 6$, we verified that numerical estimates of the dynamical degree given by Halburd's Diophantine method converge to the largest root of the characteristic polynomial for condition (45), thus showing that our full-deautonomisation approach indeed yields the correct value for the dynamical degree for those mappings.

The same holds for the cases $k = 7$ and $k = 8$ when $\ell \geq 1$. The confinement condition on z_n is

$$z_{n+\ell+2} - 2(z_{n+\ell+1} + \cdots + z_{n+1}) + z_n = 0, \tag{48}$$

which always has a root greater than 1 if $\ell \geq 1$. Note that for $k = 8, \ell = 0$ however, z_n takes the form $z_n = \alpha n + \beta$ and the equation one obtains is of course nothing but the generic additive $\mathrm{E}_8^{(1)}$-associated discrete Painlevé equation (8)-(9) encountered in the previous subsection.

Numerical calculations for the dynamical degrees of the mappings with $k = 7, 8$ and 9 for $\ell \geq 1$ again converge to the values indicated in Table 1, thus vindicating the full-deautonomisation method even for such highly complicated mappings. Since all the values for the dynamical degrees thus obtained are greater than 1, the corresponding mappings are all non-integrable. In fact, for $k > 8$ the equations obtained with z_n given by (47) are always non-integrable. We have computed the dynamical degree of the first few cases, up to $k = 16$ and the computed degrees are in perfect agreement with the value of the largest root of the characteristic polynomial (36).

Note that among all equations (21) that have confining singularities, only three have autonomous limits: the only cases in which $z_n =$ constant (combined with κ^i being zero) is a solution to (35) are the cases of equations (46), (47) and (8)-(9). All other equations necessarily require a non-trivial n-dependence in z_n in order to confine. However, one can consider the case of an autonomous mapping where z_n, being constant, cannot satisfy the condition (35) and hence, for which the singularity at $x = \infty$ is unconfined when $k = 7$ or $k \geq 9$. In this case the singularity pattern corresponds to the unconfined sequence $\{\infty, \infty, \cdots\}$, i.e. the limit $\ell \to \infty$ of the confined pattern. In this limit, the largest root of (36) converges to a value that can be computed exactly. It suffices to rewrite (36) as

$$1 + \big(2 - \big[(k+1)/2\big]\big)\,\frac{1 - \lambda^{-\ell-1}}{\lambda - 1} + \lambda^{-\ell-2} = 0$$

and take the limit $\ell \to \infty$ supposing that there exists a $\lambda > 1$, which is true for $k \geq 5$. This yields

$$\lambda = \left[\frac{k+1}{2}\right] - 1, \tag{49}$$

which is precisely the value for the dynamical degree obtained for all mappings with $k \geq 5$ that do not confine at infinity, and for all autonomous mappings at $k = 7$ and $k \geq 9$ in particular.

6 Conclusions

In this paper we have combined the full-deautonomisation approach with the ancillary representation of mappings. The latter was introduced in order to palliate a difficulty encountered in the derivation of discrete Painlevé equations by deautonomising QRT mappings. All QRT mappings associated with classes II to VI have a factorised left-hand side of the form $F(x_{n+1}, x_n)F(x_{n-1}, x_n)$, while class I is even simpler. Studying the behaviour of their singularities is therefore straightforward. However this is not the case with class VII and VIII mappings. Here the study of the singularity is complicated and, in practice, prohibitively so. The introduction of the ancillary representation, which consists in just replacing the dependent variable by some more convenient one solves this problem, since it provides a factorised form of the equation. Moreover it obviates the distinction between classs VII and VIII since the only difference between the two is a different choice of ancillary variable: instead of the relation $x = \xi^2$ for additive, class VII, equations we have $x = \xi + 1/\xi$ for the multiplicative, class VIII, ones. And, as a bonus, the ancillary representation, through the relation $x = \theta_1^2(\xi)/\theta_0^2(\xi)$, where the θ are theta functions, offers a natural way to represent the elliptic discrete Painlevé equations [22]. With the introduction of this new parametrisation a host of new, $\mathrm{E}_8^{(1)}$ associated discrete Painlevé equations, were derived.

The full-deautonomisation approach was introduced in order to palliate another difficulty. While all discrete equations integrable through spectral methods possess confined singularities, the singularity confinement property is not sufficient for the integrability of rational mappings. The latter is associated to the low growth of some characteristic (typically the degree growth of some initial condition) and while singularity confinement is associated with factorisations and simplifications that do lower the degree, it turns out that in some cases this does not suffice in order to curb the exponential degree growth. To this end the full-deautonomisation approach was introduced, requiring confinement of the singularities of the most general non-autonomous form. (Sometimes this necessitates the extension of the mapping by introducing new terms which do not alter the confining singularity pattern, but finding the proper terms to add is often non trivial). As was shown in [11] the deautonomisation constraints allow one to calculate the dynamical degree of the mapping, thus making it possible not only to ascertain the integrable character but, in the presence of non-integrability, to have a precise knowledge of the growth properties of the system.

Since the first confining non-integrable systems that were discovered all took the form of QRT class I mappings, it was natural that most applications of the full-deautomisation method to date had to do with systems belonging to that class (and some of class II). In this paper we have radically changed our perspective and decided to investigate the validity of the method in the case of the mappings from class VII (and as we just explained our results apply without any change apart from that of the ancillary variable to those of class VIII). Given the richness of this class we decided to choose a special type of parametrisation of the mappings to analyse, one where a single singularity would control integrability. Our analysis identified all integrable cases and for the non-integrable ones furnished the dynamical degree which was in perfect agreement with a direct calculation thereof. We should point out here that the other known method for the obtention of the dynamical degree, due to Halburd [30], cannot be applied in order to identify the integrable cases since the integrability is controlled by a single singularity [31]. Moreover, it is important to stress that the ancillary form not only made the calculations involved in the full-deautonomisation method technically feasible, but also that the ancillary form is actually special in the sense that it automatically gives the correct "full-deautonomisation" for all the mappings in the class we investigated. No extra terms had to be introduced in order to obtain the dynamical degrees of the mappings we investigated.

Having, once more, affirmed the power of the full-deautonomisation method we can ask ourselves what are the possible next steps in our investigation. One possibility would be to apply the method to higher-order mappings for which little is known to date as far as integrability is concerned. (A first application in this direction can be found in a previous work of the authors in [15]). Given the fact that full-deautonomisation seems to work in every case, a different path would be to try to cast the method in some rigorous framework or, at least, explain its uncanny effectiveness. We may address either of these questions in some future work of ours.

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Appendix: the QRT canonical forms

In the main body of the article we referred on several occasions to the canonical forms of the QRT mapping and their classification. These are often referred to in various works of the authors but they have never been collected in a single organised and easily consultable list. In order to remedy this problem we decided to give here the full list, which can serve for future reference to those interested in these questions. We shall not present the derivation of the list since the pertaining details can be found in [17] and we limit ourselves to the presentation of the results. In order to fix the notations we assume that the A_0 matrix for the QRT mapping [4] has the form

$$A_0 = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \zeta \\ \kappa & \lambda & \mu \end{pmatrix}, \tag{50}$$

in the asymmetric case, while in the symmetric case we have $\delta = \beta$, $\kappa = \gamma$ and $\lambda = \zeta$.

In hindsight it is clear that the numbering of the canonical cases is far from optimal but as it was introduced more than 20 years ago, it can no longer be modified. (In fact the same holds for the choice of the symbols for the elements of the QRT matrices. Following the greek alphabet it would have been logical to have in the third line the letters η, θ and κ but, again, introducing such a change at this late date would be a disruption).

We move now to the classification, which also includes the form of the A_1 matrix for the corresponding mapping.

Case I

(I) $$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$x_{n+1} + x_n = -\frac{\delta y_n^2 + \epsilon y_n + \zeta}{\alpha y_n^2 + \beta y_n + \gamma} \tag{51a}$$

$$y_n + y_{n-1} = -\frac{\beta x_n^2 + \epsilon x_n + \lambda}{\alpha x_n^2 + \delta x_n + \kappa} \tag{51b}$$

Case II

(II)
$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_{n+1} x_n = \frac{\kappa y_n^2 + \lambda y_n + \mu}{\alpha y_n^2 + \beta y_n + \gamma} \tag{52a}$$

$$y_n y_{n-1} = \frac{\gamma x_n^2 + \zeta x_n + \mu}{\alpha x_n^2 + \delta x_n + \kappa} \tag{52b}$$

Case III

(III)
$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$(x_{n+1} + y_n)(y_n + x_n) = \frac{\alpha y_n^4 + (\beta - \delta) y_n^3 + (\gamma + \kappa - \epsilon) y_n^2 + (\lambda - \zeta) y_n + \mu}{\alpha y_n^2 + \beta y_n + \gamma} \tag{53a}$$

$$(x_n + y_n)(x_n + y_{n-1}) = \frac{\alpha x_n^4 - (\beta - \delta) x_n^3 + (\gamma + \kappa - \epsilon) x_n^2 - (\lambda - \zeta) x_n + \mu}{\alpha x_n^2 + \delta x_n + \kappa} \tag{53b}$$

Case IV

(IV)
$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(x_{n+1} y_n - 1)(y_n x_n - 1) = \frac{\kappa y_n^4 + (\delta + \lambda) y_n^3 + (\mu + \epsilon + \alpha) y_n^2 + (\beta + \zeta) y_n + \gamma}{\alpha y_n^2 + \beta y_n + \gamma} \tag{54a}$$

$$(y_n x_n - 1)(x_n y_{n-1} - 1) = \frac{\gamma x_n^4 + (\beta + \zeta) x_n^3 + (\mu + \epsilon + \alpha) x_n^2 + (\delta + \lambda) x_n + \kappa}{\alpha x_n^2 + \delta x_n + \kappa} \tag{54b}$$

Case V

(V)
$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 2z \\ 1 & 2z & 0 \end{pmatrix}$$

$$\begin{aligned} &\frac{(x_{n+1} + y_n + 2z)(y_n + x_n + 2z)}{(x_{n+1} + y_n)(y_n + x_n)} \\ &= \frac{\alpha y_n^4 + (4\alpha z + \beta - \delta) y_n^3 + (4\alpha z^2 + 4\beta z - 2\delta z - \epsilon + \gamma + \kappa) y_n^2 + (4\beta z^2 - 2\epsilon z + 4\gamma z + \lambda - \zeta) y_n + 4\gamma z^2 - 2\zeta z + \mu}{\alpha y_n^4 + (\beta - \delta) y_n^3 + (\gamma + \kappa - \epsilon) y_n^2 + (\lambda - \zeta) y_n + \mu} \end{aligned} \tag{55a}$$

$$\frac{(x_n + y_n + 2z)(x_n + y_{n-1} + 2z)}{(x_n + y_n)(x_n + y_{n-1}}$$

$$= \frac{\alpha x_n^4+(4\alpha z-\beta+\delta)x_n^3+(4\alpha z^2-2\beta z+4\delta z-\epsilon+\gamma+\kappa)x_n^2+(4\delta z^2-2\epsilon z+4\kappa z-\lambda+\zeta)x_n+4\kappa z^2-2\lambda z+\mu}{\alpha x_n^4-(\beta-\delta)x_n^3+(\gamma+\kappa-\epsilon)x_n^2-(\lambda-\zeta)x_n+\mu} \quad (55b)$$

The case $z = 0$ in V (sometimes referred to as V_0) is special and leads to the mapping

$$\frac{1}{x_{n+1}+y_n} + \frac{1}{y_n+x_n)} = \frac{2\alpha y_n^3+(2\beta-\delta)y_n^2+(-\epsilon+2\gamma)y_n-\zeta}{\alpha y_n^4+(\beta-\delta)y_n^3+(\gamma+\kappa-\epsilon)y_n^2+(\lambda-\zeta)y_n+\mu} \quad (56a)$$

$$\frac{1}{x_n+y_n} + \frac{1}{x_n+y_{n-1}} = \frac{2\alpha x_n^3+(-\beta+2\delta)x_n^2+(-\epsilon+2\kappa)x_n-\lambda}{\alpha x_n^4-(\beta-\delta)x_n^3+(\gamma+\kappa-\epsilon)x_n^2-(\lambda-\zeta)x_n+\mu} \quad (56b)$$

Case VI

This is the only canonical case where the upper left element of the A_1 matrix is not zero. It was introduced so as to correspond to the first form obtained for the discrete analogue of the $\mathrm{P_{VI}}$ equation.

(VI)
$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -(1+z^2) & 0 \\ 0 & 0 & z^2 \end{pmatrix}$$

$$\frac{(x_{n+1}y_n-z^2)(y_nx_n-z^2)}{(x_{n+1}y_n-1)(y_nx_n-1)} = \frac{\kappa y_n^4+(\delta z^2+\lambda)y_n^3+(\alpha z^4+\epsilon z^2+\mu)y_n^2+(\beta z^4+\zeta z^2)y_n+\gamma z^4}{\kappa y_n^4+(\delta+\lambda)y_n^3+(\alpha+\epsilon+\mu)y_n^2+(\beta+\gamma)y_n+\gamma} \quad (57a)$$

$$\frac{(y_nx_n-z^2)(x_ny_{n-1}-z^2)}{(y_nx_n-1)(x_ny_{n-1}-1)} = \frac{\gamma y_n^4+(\beta z^2+\zeta)y_n^3+(\alpha z^4+\epsilon z^2+\mu)y_n^2+(\delta z^4+\lambda z^2)y_n+\kappa z^4}{\gamma y_n^4+(\beta+\zeta)y_n^3+(\alpha+\epsilon+\mu)y_n^2+(\delta+\lambda)y_n+\kappa} \quad (57b)$$

A alternate form to VI does exist, one where the upper left matrix element is zero. We usually refer to it as VI′.

(VI′)
$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & z+1/z & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\frac{(zx_{n+1}+y_n)(y_n+zx_n)}{(x_{n+1}+zy_n)(zy_n+x_n)} = \frac{\alpha y_n^4+(\beta-\delta z)y_n^3+(\gamma-\epsilon z+\kappa z^2)y_n^2+(\lambda z^2-\zeta z)y_n+\mu z^2}{\alpha z^2y_n^4+(\beta z^2-\delta z)y_n^3+(\gamma z^2-\epsilon z+\kappa)y_n^2+(\lambda-\zeta z)y_n+\mu} \quad (58a)$$

$$\frac{(x_n+zy_n)(x_n+zy_{n-1})}{(zx_n+y_n)(zx_n+y_{n-1})} = \frac{\alpha x_n^4+(-\beta z+\delta)x_n^3+(\gamma z^2-\epsilon z+\kappa)x_n^2+(-\lambda z-\zeta z^2)x_n+\mu z^2}{\alpha z^2x_n^4+(-\beta z+\delta z^2)x_n^3+(\gamma-\epsilon z+\kappa z^2)x_n^2+(-\lambda z+\zeta)x_n+\mu} \quad (58b)$$

Case VII

(VII)
$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & -2z^2 \\ 1 & -2z^2 & z^4 \end{pmatrix}$$

$$\frac{(x_{n+1}-y_n-z^2)(x_n-y_n-z^2)+4y_nz^2}{x_{n+1}-2y_n+x_n-2z^2}$$

$$= -\frac{\alpha y_n^4+(6\alpha z^2+\beta+\delta)y_n^3+(\alpha z^4+6\beta z^2+\delta z^2+\gamma+\epsilon+\kappa)y_n^2+(\beta z^4+\epsilon z^2+6\gamma z^2+\lambda+\zeta)y_n+\gamma z^4+\zeta z^2+\mu}{2\alpha z^2 y_n^3+(2\alpha z^2+2\beta+\delta)y_n^2+(2\beta z^2+\epsilon+2\gamma)y_n+2\gamma z^2+\zeta} \quad (59a)$$

$$\frac{(y_{n-1}-x_n-z^2)(y_n-x_n-z^2)+4x_nz^2}{y_{n-1}-2x_n+y_n-2z^2}$$
$$= -\frac{\alpha x_n^4+(6\alpha z^2+\beta+\delta)x_n^3+(\alpha z^4+\beta z^2+6\delta z^2+\gamma+\epsilon+\kappa)x_n^2+(\delta z^4+\epsilon z^2+6\kappa z^2+\lambda+\zeta)x_n+\kappa z^4+\lambda z^2+\mu}{2\alpha z^2 x_n^3+(2\alpha z^2+\beta+2\delta)x_n^2+(2\delta z^2+\epsilon+2\kappa)x_n+2\kappa z^2+\lambda} \quad (59b)$$

Case VIII

(VIII)
$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -(z^2+1/z^2) & 0 \\ 1 & 0 & (z^2-1/z^2)^2 \end{pmatrix}$$

$$\frac{(z^2x_{n+1}-y_n)(z^2x_n-y_n)-(z^4-1)^2}{(x_{n+1}-y_n/z^2)(x_n-y_n/z^2)-(z^2-1/z^2)^2}$$
$$= z^4\frac{\alpha y_n^4+(\beta+\delta z^2)y_n^3+(-\alpha(z^4-1)^2+\gamma+\epsilon z^2+\kappa z^4)y_n^2+(-\beta(z^4-1)^2+\lambda z^4+\zeta z^2)y_n-\gamma(z^4-1)^2+\mu z^4}{\alpha z^8 y_n^4+(\beta z^8+\delta z^6)y_n^3+(-\alpha(z^4-1)^2+\gamma z^8+\epsilon z^6+\kappa z^4)y_n^2+(-\beta(z^4-1)^2+\lambda z^4+\zeta z^6)y_n-\gamma(z^4-1)^2+\mu z^4} \quad (60a)$$

$$\frac{(z^2y_{n-1}-x_n)(z^2y_n-x_n)-(z^4-1)^2}{(y_{n-1}-x_n/z^2)(y_n-x_n/z^2)-(z^2-1/z^2)^2}$$
$$= z^4\frac{\alpha z^8 x_n^4+(\beta z^2+\delta)x_n^3+(-\alpha(z^4-1)^2+\gamma z^2+\epsilon z^2+\kappa)x_n^2+(-\delta(z^4-1)^2+\lambda z^2+\zeta z^4)x_n-\kappa(z^4-1)^2+\mu z^4}{\alpha z^8 x_n^4+(\beta z^6+\delta z^8 x_n^3+(-\alpha(z^4-1)^2+\gamma z^4+\epsilon z^6+\kappa z^8)x_n^2+(-\delta(z^4-1)^2+\lambda z^6+\zeta z^4)x_n-\kappa(z^4-1)^2+\mu z^4} \quad (60b)$$

This completes the list of the cases that allow for a symmetric form. The corresponding equation can be obtained from the equation for x_{n+1} in the (x, y) system by putting $\delta = \beta$, $\kappa = \gamma$ and $\lambda = \zeta$, and by replacing x_n by x_{n-1} and y_n by x_n, while leaving x_{n+1} unchanged.

The remaining three canonical forms correspond to genuinely asymmetric equations.

Case IX

(IX)
$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_n + x_{n+1} = -\frac{\delta y_n^2+\epsilon y_n+\zeta}{\alpha y_n^2+\beta y_n+\gamma} \quad (61a)$$

$$y_{n-1}y_n = \frac{\gamma x_n^2+\zeta x_n+\mu}{\alpha x_n^2+\delta x_n+\kappa} \quad (61b)$$

Case X

(X)
$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\frac{x_n+x_{n-1}-y_n}{x_nx_{n-1}-1} = \frac{\alpha y_n^3+(\beta+\delta)y_n^2+(\epsilon+\gamma)y_n+\zeta}{(\alpha-\kappa)y_n^2+(\beta-\lambda)y_n+\gamma-\mu} \quad (62a)$$

$$(x_n^2-y_{n+1}x_n+1)(x_n^2-y_nx_n+1)$$

$$= \frac{\alpha x_n^6+(\beta+\delta)x_n^5+(\gamma+\kappa+\epsilon+2\alpha)x_n^4+(\beta+2\delta+\lambda+\zeta)x_n^3+(\epsilon+\alpha+2\kappa+\mu)x_n^2+(\delta+\lambda)x_n+\kappa}{\alpha x_n^2+\delta x_n+\kappa} \quad (62b)$$

Case XI

$$\text{(XI)} \qquad A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\frac{x_n x_{n+1} + y_n}{x_n + x_{n+1}} = -\frac{y_n^3 + (\beta+\kappa)y_n^2 + (\gamma+\lambda)y_n + \mu}{\delta y_n^2 + \epsilon y_n + \zeta}, \quad (63\text{a})$$

$$(y_{n-1} - x_n^2)(y_n - x_n^2) = \frac{\alpha x_n^6 + \delta x_n^5 + (\beta+\kappa)x_n^4 + \epsilon x_n^3 + (\gamma+\lambda)x_n^2 + \zeta x_n + \mu}{x_n^2 + \beta x_n + \gamma} \quad (63b)$$

In [17] it was pointed out that the most general form of the A_1 QRT matrix (once the appropriate homographic transformations have been performed) is

$$A_1 = \begin{pmatrix} 1 & 0 & \gamma \\ 0 & \epsilon & 0 \\ \gamma & 0 & 1 \end{pmatrix}. \quad (64)$$

When deautonomised, the corresponding QRT mapping should in principle lead to the most general discrete Painlevé equation, namely the elliptic equation associated to $\mathrm{E}_8^{(1)}$. However obtaining the latter by direct deautonomisation of this canonical form is a practically impossible task. Thanks to the ancillary representation [21] this formidable task becomes almost elementary.