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On the parametrization of solutions of the Yang–Baxter equations

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Abstract

We study all five-, six-, and one eight-vertex type two-state solutions of the Yang-Baxter equations in the form $A_{12}B_{13}C_{23} = C_{23}B_{13}A_{12}$, and analyze the interplay of the 'gauge' and 'inversion' symmetries of these solution. Starting with algebraic solutions, whose parameters have no specific interpretation, and then using these symmetries we can construct a parametrization where we can identify global, color and spectral parameters. We show in particular how the distribution of these parameters may be changed by a change of gauge.

1 Introduction

The Yang-Baxter equations appeared in the study of two-dimensional integrable models of statistical mechanics [1], and in the quantization of 1+1 dimensional integrable equations (see [2, 3]). They are an over-determined system of equations on three matrices $[A, B, C]$ of size $n^2 \times n^2$ (n is, e.g, the number of spin states), and read:

$$\sum_{\alpha_1,\alpha_2,\alpha_3} A^{i_1 i_2}_{\alpha_1\alpha_2} B^{\alpha_1 i_3}_{j_1\alpha_3} C^{\alpha_2\alpha_3}_{j_2 j_3} = \sum_{\beta_1,\beta_2,\beta_3} C^{i_2 i_3}_{\beta_2\beta_3} B^{i_1\beta_3}_{\beta_1 j_3} A^{\beta_1\beta_2}_{j_1 j_2}, \tag{1a}$$

$\forall i_1, i_2, i_3, j_1, j_2, j_3 = 1 \dots n$, or in a shorthand notation

$$A_{12}B_{13}C_{23} = C_{23}B_{13}A_{12}. \tag{1b}$$

Here the matrices act on a direct product of three (identical) vector spaces $V_1 \otimes V_2 \otimes V_3$, and the subscripts tell on which spaces the matrix acts non-trivially, e.g., A_{12} means that A acts as $A_{12} \otimes 1$ etc. We choose to write the equations with three different matrices

A, B, C to emphasize the possible dependence on parameters, but without prejudice on the nature of these parameters.

The main issue of the present work is precisely to discuss, by examples, questions related to the *parametrization* of the solutions of (1b). A natural objective is to write the solution triplet $[A, B, C]$ in the form of a parametrized family:

$$A = R(\vec{u}),\ B = R(\vec{u} \oplus \vec{v}),\ C = R(\vec{v}), \tag{2}$$

using some "universal" function R. Here a privileged set of parameters has been identified, they are the so-called spectral parameters, and have some kind of addition rule $\oplus$. The name of spectral parameter has its origin in the quantum inverse scattering theory [2], and relies on its interpretation as eigenvalue of a spectral problem. The spectral parameters play a crucial role in the quantum inverse scattering approach, and especially in the Bethe Ansatz construction. This is the reason why they are singled out in the parametrization of the solutions. In Baxter's model [1] the operation $\oplus$ in (2) is the addition on some elliptic curve, the uniformization of this curve brings forward elliptic functions and the moduli of these functions are additional parameters of the solution: we will call them 'moduli parameters'. [It should be noted that (1b) has many solutions which cannot be given in the form (2). Take for example $A = B$ arbitrary and $C = P$, the permutation matrix. If this were to be interpreted according to (2) we should take $\vec{v} = 0$, $R(0) = P$, and then $R(\vec{u})$ remains completely arbitrary.]

In this work we take a closer look on the process by which a good parametrization can be given to a solution of (1b). We show how the inversion symmetries can be used for this purpose. Of particular interest is the effect of gauge choice on the nature and distribution of the parameters.

With reference to the parameter dependence it should be noted that there are also the so-called 'constant Yang–Baxter equations', where $A = B = C = R$, i.e:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{3}$$

[For $n = 2$ the complete solution of this equation was presented in [4].] Going from (1b) with (2) to (3), although simple in terms of the parameters (it amounts to setting them to some value for which $\vec{u} = \vec{v} = \vec{u} \oplus \vec{v}$), leads among other things to the successful notion of the quantum group. The reverse move, that is to say obtaining solutions of (1b) starting with solutions of (3), is sometimes called the 'baxterization problem' [5], and is naturally more difficult. In some cases baxterization is obtained from group theory [6]: There exists a discrete group of symmetries of equation (1b), the 'group of inversions', which we denote by $\mathcal{A}ut$. This group acts by non-linear transformations on the solution triplet and moves it to another solution. This can be precisely interpreted as the effect of moving the spectral parameters. The 'baxterization' is essentially the action of $\mathcal{A}ut$, if it covers densely the manifold of spectral parameters, but if $\mathcal{A}ut$ produces only a finite set of points we do not yet have a true baxterization. The group $\mathcal{A}ut$ is the statistical mechanical equivalent of the unitarity and crossing symmetries of S-matrix theory: the generators of these symmetries form a group similar to $\mathcal{A}ut$ [7, 8, 9].

It is important to note that the symmetry $\mathcal{A}ut$ is not the gauge symmetry. The latter is believed to bring in only inessential parameters. One of the results presented in this paper is that gauge may also change the distribution of the true parameters (see section 4.2).

The paper is organized as follows: In Sec. 2 we discuss in general the groups of gauge and inversion transformations and their interplay. In Sec. 3 we give all the five- and six-vertex solutions to equation (1b) and and show how the invariants of the inversion group can be used to construct a meaningful parametrization for them. In Sec. 4 we give a seven parameter symmetric eight-vertex solution to equation (1b) and discuss its parametrization. By allowing some gauge freedom (for later fixing) we get the solution first in a rational form. We then show that a choice of gauge does not change the nature and number of parameters of the solution triple $[A, B, C]$ but—by its interplay with $\mathcal{A}ut$—affects their distribution between A, B and C, and leads, for example, to Baxter's elliptic solution.

2 Two transformation groups

Suppose $[A, B, C]$ is a triplet of matrices, not necessarily verifying (1b). We may define two groups acting on such triplets, respectively the continuous group of gauge transformations $\mathcal{G} = SL(n) \otimes SL(n) \otimes SL(n)$, and a discrete group denoted $\mathcal{A}ut$.

2.1 The group of gauge transformations

Let $g = (g_1, g_2, g_3)$ be an element of $\mathcal{G}$, acting linearly on the triplet $[A, B, C]$ by similarity transformations:

$$g : [A, B, C] \mapsto [(g_1 \otimes g_2)^{-1} A\, (g_1 \otimes g_2),\ (g_1 \otimes g_3)^{-1} B\, (g_1 \otimes g_3),\ (g_2 \otimes g_3)^{-1} C\, (g_2 \otimes g_3)]. \quad (4)$$

Here the subscript indicates the vector space where the similarity transformation takes place, and in different spaces the g matrix can be different. The group $\mathcal{G}$ is known to take solutions of (1b) into solutions of (1b), but its action is defined everywhere, even outside the space of solutions.

2.2 The group of inversions $\mathcal{A}ut$

Let us first define some elementary operations on a $n^2 \times n^2$ matrix R, with matrix elements R^{ij}_{kl} [6]:

1. the (projective) matrix inverse I:

$$\sum_{\alpha\beta} (IR)^{ij}_{\alpha\beta} R^{\alpha\beta}_{kl} = \mu\ \delta^i_k \delta^j_l, \qquad i, j, k, l = 1, \ldots, n, \quad (5)$$

   with μ an arbitrary multiplicative factor.

2. the transposition t:

$$(tR)^{ij}_{kl} = R^{kl}_{ij}, \qquad i, j, k, l = 1, \ldots, n, \quad (6)$$

3. left and right *partial transpositions* t_l and t_r:

$$(t_l R)^{ij}_{kl} = R^{kj}_{il}, \qquad (t_r R)^{ij}_{kl} = R^{il}_{kj}, \qquad i, j, k, l = 1, \ldots, n. \quad (7)$$

Of course

$$t = t_l\, t_r = t_r\, t_l, \qquad I^2 = t^2 = t_l^2 = t_r^2 = 1, \qquad \text{and} \qquad I\, t = t\, I. \tag{8}$$

However,

$$t_l\, I \neq I\, t_l, \qquad \text{and} \qquad t_r\, I \neq I\, t_r, \tag{9}$$

i.e, the two partial transpositions *do not* commute with the inversion, while their product t does. The transformations $t_l I$ and $t_r I$ are generically of *infinite order*, we shall denote by Γ the group generated by I, t_l, t_r. [Of course we must assume that all matrices we are dealing with are nonsingular.]

We may now define the three generators of the 'inversion group' as follows:

$$\begin{array}{rcllll} K_a : [A,B,C] & \mapsto & [tI\ A, & t_l\ B, & t_l\ C], \\ K_b : [A,B,C] & \mapsto & [t_l\ A, & t_r I t_l\ B, & t_r\ C], \\ K_c : [A,B,C] & \mapsto & [t_r\ A, & t_r\ B, & tI\ C]. \end{array} \tag{10}$$

The three involutions K_a, K_b, K_c act non–linearly (by birational transformations). They generate an infinite discrete group of transformations of triplets, which we denote by $\mathcal{A}ut$.

Proposition 1. The group $\mathcal{A}ut$ generated by (10) is an invariance group of the nonsingular solutions of (1b).

2.3 The compatibility of $\mathcal{A}ut$ with $\mathcal{G}$

Clearly the action of the two groups $\mathcal{A}ut$ and $\mathcal{G}$ do not commute. However, their actions are compatible, in the sense that $\mathcal{A}ut$ respects the equivalence classes of triplets $[A,B,C]$ modulo $\mathcal{G}$. This can be seen as follows: Suppose that $T = [A,B,C]$, and that T' is gauge equivalent to T by $T' = g(T)$, where g acts as defined in (4) with $g = (g_1, g_2, g_3)$. Then from (10) we get

$$K_a(T') = [({}^t g_1 \otimes {}^t g_2) tIA({}^t g_1^{-1} \otimes {}^t g_2^{-1}), ({}^t g_1 \otimes g_3^{-1}) t_l B({}^t g_1^{-1} \otimes g_3), ({}^t g_2 \otimes g_3^{-1}) t_l C({}^t g_2^{-1} \otimes g_3)]$$

where ${}^t g_1$ denotes the transpose of g_1. This can be written in the form

$$K_a \cdot g = g' \cdot K_a \text{ with } g' = ({}^t g_1^{-1}, {}^t g_2^{-1}, g_3),$$

and there are similar relations for K_b and K_c. They show that

Proposition 2. If two triplets are gauge related, so are their images by any element of $\mathcal{A}ut$.

Note that the previous proposition applies even if the triplet $[A,B,C]$ does not solve (1b).

2.4 The moduli space of solutions

Let $\mathcal{S}$ be any continuously parametrized family of solutions $[A,B,C]$ of (1b). We will call *orbit space* the quotient

$$\eta = \mathcal{S}/\mathcal{G}$$

with $\mathcal{G}$ possibly replaced by some of its subgroups. By dividing out the gauge transformations we obtain the true solution space. Next we will define the *moduli space* of $\mathcal{S}$ by the double quotient

$$\mathcal{M} = (\mathcal{S}/\mathcal{G})/\mathcal{A}ut = \eta/\mathcal{A}ut,$$

not caring about the differentiability nor regularity properties of this quotient. The action of $\mathcal{A}ut$ moves the spectral parameters, so $\mathcal{M}$ is basically the space of non-spectral parameters. The second quotient might be extremely singular. The situation described in this paper is particularly simple in that respect, since η is foliated by $\mathcal{A}ut$-invariant algebraic subvarieties.

Note that, if A, B, C have a definite form, as is the case for the five-, six- and eight-vertex Ansatz, we may have to restrict ourselves to some subgroups of $\mathcal{G}$ and $\mathcal{A}ut$ in order to preserve this form. We shall in particular need the diagonal subgroup $\mathcal{G}_d$ of $\mathcal{G}$, with elements

$$g = \left(\begin{bmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{bmatrix}, \begin{bmatrix} t_2 & 0 \\ 0 & t_2^{-1} \end{bmatrix}, \begin{bmatrix} t_3 & 0 \\ 0 & t_3^{-1} \end{bmatrix} \right). \tag{11}$$

3 The five- and six-vertex solutions

3.1 General considerations

The six vertex Ansatz for the matrices A, B and C is

$$X = \begin{pmatrix} X_{11} & 0 & 0 & 0 \\ 0 & X_{22} & X_{23} & 0 \\ 0 & X_{32} & X_{33} & 0 \\ 0 & 0 & 0 & X_{44} \end{pmatrix}, \tag{12}$$

For the five-vertex model we take $X_{32} \equiv 0$, with all other five entries nonzero, while for the six-vertex model all the six entries are assumed to be nonzero in each matrix.

The form (12) is not strictly stable by $\mathcal{A}ut$, since the partial transpositions exchange the non-zero off-diagonal elements with the vanishing upper-right and lower-left entries. However, the subgroup $\mathcal{A}ut_2$ of elements of $\mathcal{A}ut$, which are products of squares, respects the ansatz.

The action of Γ (defined in section (2.2)) on a generic 4×4 matrix was analyzed in [10], where it was shown that the invariants of $\mathcal{A}ut_2$ are ratios of some quadratic polynomials in the entries of the matrix. Out of the 18 polynomials p_i of [10], only five are non-vanishing when evaluated on a matrix of the form (12), they are

$$\begin{aligned} p_1(X) &= X_{11}X_{22} + X_{33}X_{44}, \\ p_2(X) &= X_{11}X_{22} - X_{33}X_{44}, \\ p_5(X) &= X_{11}X_{33} + X_{22}X_{44}, \\ p_6(X) &= X_{11}X_{33} - X_{22}X_{44}, \\ p_9(X) &= X_{11}X_{44} + X_{22}X_{33} - X_{23}X_{32}. \end{aligned} \tag{13}$$

Invariants of $\mathcal{A}ut_2$ can then be obtained by taking ratios of the form $p_i(A)/p_j(A)$, resp. $(B), (C)$. One should notice that, in the case under study, all these polynomials are *independent of the gauge parameters*, contrary to what happens for the general (16-vertex) case.

The rank of the system of the four invariant ratios constructed from the generic matrix (12) is only 3, the additional relation being

$$p_1^2 - p_2^2 = p_5^2 - p_6^2. \tag{14}$$

A solution for which $p_9(X) \equiv 0$ is called "free-fermion type" [11, 12, 13].

3.2 The five-vertex solutions

For the five-vertex solution we take the matrix elements $A_{32} = B_{32} = C_{32} = 0$. In addition let us scale so that $A_{11} = B_{11} = C_{11} = 1$. Substituting this Ansatz into (1b) leads easily to precisely two solutions.

3.2.1 The first solution

The first solution is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_2 & a & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_2 & b & 0 \\ 0 & 0 & y_3 & 0 \\ 0 & 0 & 0 & \frac{x_4 y_3}{x_3} \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{x_2}{x_4} & c & 0 \\ 0 & 0 & y_3 & 0 \\ 0 & 0 & 0 & \frac{y_3}{x_3} \end{pmatrix}, \tag{15}$$

with the constraint

$$ac = b\,(1 - x_2 x_3/x_4). \tag{16}$$

Since now $p_9 \neq 0$ let us consider

$$\Delta := \frac{p_1^2 - p_2^2}{p_9^2}, \tag{17}$$

we find

$$\Delta(A) = \Delta(B) = \Delta(C) = \frac{4x_4/(x_2x_3)}{[1 + x_4/(x_2x_3)]^2}\,, \tag{18}$$

and thus we have found a global invariant, which we may exchange for $d := \sqrt{x_2x_3/x_4}$ [and in terms of d, the constraint becomes $ac = b(1-d^2)$]. Note that d may be constructed from the covariants $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ of [10], while the modular invariant J of [10] vanishes.

From the other ratios let us look at the following:

$$\delta := \frac{p_1 + p_2}{p_1 - p_2}, \quad \delta' := p_5 - p_6 p_5 + p_6. \tag{19}$$

We find

$$\begin{aligned} q_1^2 &:= \delta'(A) = \delta'(B) = (x_2/d)^2, \\ q_2^2 &:= \delta(A) = \delta'(C) = (d/x_3)^2, \\ q_3^2 &:= \delta(B) = \delta(C) = (d/y_3)^2. \end{aligned} \tag{20}$$

Since the common index between A and B is 1, between A and C is 2 and between B and C is 3, we have also introduced new 'color' variables q_i in (20). After this let us define

$$R_{5a}(i,j) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & d\,q_i & (1-d^2)\,g_i g_j^{-1} & 0 \\ 0 & 0 & d\,q_j^{-1} & 0 \\ 0 & 0 & 0 & q_i\,q_j^{-1} \end{pmatrix}, \tag{21}$$

and then the solution (15) can be written as $A = R_{5a}(1,2)$, $B = R_{5a}(1,3)$, $C = R_{5a}(2,3)$. In this form g_i are the variables that are changed by gauge, we could fix them by putting $g_i = 1$.

Thus, starting with the solution (15) without any particular structure we were able to put it into a form in which the variables were either global (d) or associated with the vector spaces (q_i). [These latter ones are often called 'color' variables [14].] This was accomplished by looking at what remains from the generic invariants of $\mathcal{A}ut_2$ for the specific solution. The action of $\mathcal{A}ut_2$ on R_{5a} is given as follows:

$$(K_a K_b)^2 : g_1 \mapsto g_1 d^2, \quad (K_b K_c)^2 : g_3 \mapsto g_3 d^2, \quad (K_c K_a)^2 : g_2 \mapsto g_2 d^2 \tag{22}$$

As a consequence the action of $\mathcal{A}ut_2$ cannot be distinguished from the one of the gauge transformations, i.e. $\mathcal{A}ut_2$ acts as unity on the orbit space η.

At this point let us introduce a notation for the *parameter content* of a solution triple: we say that the parameter content of (A, B, C) is (n_A, n_B, n_C), if fixing A fixes n_A parameters, then fixing B fixes n_B of the *remaining* parameters and so on. The parameter content of this solution is clearly $(3, 1, 0)$, since from A we get q_1, q_2, d and from B the remaining q_3.

Finally we note that in the constant limit we must take all q_i equal and obtain the well know solution of (3)

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 1-pq & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{23}$$

where $p = dq_i$, $q = d/q_i$.

3.2.2 The second solution (free fermion type)

By inspection we can write the second solution in terms of

$$R_{5b}(i,j) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p_i & g_{ij} & 0 \\ 0 & 0 & q_j & 0 \\ 0 & 0 & 0 & -p_i q_j \end{pmatrix}, \tag{24}$$

as $A = R_{5b}(1,2)$, $B = R_{5b}(1,3)$, $C = R_{5b}(2,3)$, together with the constraint

$$g_{12} g_{23} = g_{13}(1 - p_2 q_2). \tag{25}$$

The resolution of this last constraint is more problematic and the choice of gauge less trivial than in the first solution. [Recall that under a gauge transformation g (see (11)), $g_{ij} \to g_{ij}t_i/t_j$, thus we can fix two of g_{ij}'s.]

For a uniform representation in which each $R_{5b}(i,j)$ depends only on two variables we should have $g_{ij} = g(p_i, q_j)$, but it is easy to see that (25) does not have solutions of this type. We must then relax the condition, and if we instead allow $g_{ij} = g(p_i, p_j, q_i, q_j)$ then a family of solutions can be constructed:

$$g_{ij} = (1 - p_i q_i)^{\alpha}(1 - p_j q_j)^{1-\alpha}. \tag{26}$$

The total number of parameters of (24) is four, as in the previous case, but now they are all 'color' parameters (p_1, p_2, q_2, q_3). If now we choose, for example, the non-uniform gauge $g_{12} = g_{13} = 1$ the parameter content of the solution is $(2, 1, 1)$. In the uniform gauge (26) with $\alpha = 0$ or 1 we must introduce extra parameters and get parameter content $(3, 2, 0)$ and $(3, 1, 1)$, respectively, and in other cases $(4, 2, 0)$. Thus the price we have to pay for uniformity is the introduction of extra spurious parameters, that could in principle be gauged away.

In this case the action of $\mathcal{A}ut_2$ is:

$$\begin{aligned}
(K_a K_b)^2 \quad &: g_{12} \mapsto -g_{12}, \quad g_{13} \mapsto -g_{13}, \quad g_{23} \mapsto g_{23},\\
(K_b K_c)^2 \quad &: g_{12} \mapsto g_{12}, \quad g_{13} \mapsto -g_{13}, \quad g_{23} \mapsto -g_{23},\\
(K_c K_a)^2 \quad &: g_{12} \mapsto -g_{12}, \quad g_{13} \mapsto g_{13}, \quad g_{23} \mapsto -g_{23},
\end{aligned}$$

which again is indistinguishable from simple gauge transformations of square one.

The constant limit of this solution is obtained from (24) with $p_i = p$, $q_i = q$, $g_{ij} = 1 - pq$, $\forall i, j$.

3.3 The six-vertex solutions

In this case all entries in (12) are nonzero. We use a gauge transformation with a diagonal g_i and overall scaling to make $X_{23} = X_{32} = 1$ for A and B, say. From equation (1b) we then get $C_{23} = C_{32}$, which can be scaled to 1. We will therefore only write down the diagonal elements as $dp(X) := [X_{11}, X_{22}, X_{33}, X_{44}]$, and simplify notation by using $x_i := X_{ii}$, e.g., $dp(A) = [a_1, a_2, a_3, a_4]$.

When the Ansatz (12) and $X_{23} = X_{32} = 1$ is used in (1b) we get 6 equations,

$$\begin{aligned}
b_2 a_1 - b_1 a_2 - c_2 &= 0,\\
-b_1 - c_2 a_3 + c_1 a_1 &= 0,\\
-a_3 - c_3 b_1 + c_1 b_3 &= 0,\\
a_2 - c_4 b_2 + c_2 b_4 &= 0,\\
b_4 - c_4 a_4 + c_3 a_2 &= 0,\\
b_4 a_3 - b_3 a_4 + c_3 &= 0.
\end{aligned}$$

We solve a_1 from the second equation, a_2 from the fourth, a_3 from the third and a_4 from the fifth, and then b_4 from the first. Since the parameters are assumed to be nonzero there is no ambiguity in doing this and what then remains is one equation which factors as (all computations were done using REDUCE [15] and Maple [16]):

$$(b_1 b_2 b_3 + c_3 c_4 b_1^2 b_2 - c_2 c_3 b_1 b_2 b_3 - c_1 c_4 b_1 b_2 b_3 - c_1 c_2 b_3 + c_1 c_2 b_2 b_3^2)(1 - c_2 c_3 - c_1 c_4) = 0. \tag{27}$$

We thus recover the two known six-vertex-type solutions:

3.3.1 The first solution: the asymmetric six-vertex

The first solution is obtained when we use the first factor of (27). After solving for b_2 and some simple parameter changes we can write the solution as follows:

$$\begin{aligned} dp(C) &= [a, b, c, d], \\ dp(B) &= [ae, bf/h, cf, de/h], \\ dp(A) &= [e + bc(f - e), bd(f - e))/h, ca(f - e), (e + bc(f - e))/h], \end{aligned} \tag{28}$$

where

$$h := ef + (ade - bcf)(e - f). \tag{29}$$

In this form the solution has a rational expression, but one cannot yet identify any spectral (or color) parameters.

Before analyzing the solution in detail let us just note that the choice of C fixes four parameters, and the further choice of B fixes the two remaining ones, so the parameter content is (4,2,0).

The group of gauge transformations $\mathcal{G}$ consists of the diagonal transformations $\mathcal{G}_d$ as above, and they only change the off-diagonal elements of the matrices. The orbit space η is thus six dimensional and we may take a, b, c, d, e, f as the coordinates on η.

In order to construct a good parametrization we start with (17) and in this case get

$$\Delta(A) = \Delta(B) = \Delta(C) = \frac{4abcd}{(ad + bc - 1)^2}, \tag{30}$$

which is a 'global invariant'. Here again, the modular invariant J of [10] vanishes.

From (19) we find

$$\begin{aligned} q_3^4 &:= \delta(C) = \delta(B) = \frac{ab}{cd}, \\ q_2^4 &:= \delta'(C) = \delta(A) = \frac{bd}{ac}, \\ q_1^4 &:= \delta'(B) = \delta'(A) = \frac{bd}{ach^2}, \end{aligned} \tag{31}$$

which defines three new (color) parameters. Thus we have been able to identify four of the six parameters. To study the remaining ones we note that any matrix of the form (12) with $X_{23} = X_{32} = 1$ (solution or not) can be parametrized equally well with the four parameters Δ, δ, δ', λ, (where Δ, δ, δ' were defined in (17, 19) and the nature of λ is left open at the moment) and may be written as $R(\Delta, \delta', \delta, \lambda)$. For the present solution we have

$$A = R(\Delta, q_1, q_2, \lambda_A), \qquad B = R(\Delta, q_1, q_3, \lambda_B), \qquad C = R(\Delta, q_2, q_3, \lambda_C) \tag{32}$$

where $\lambda_A, \lambda_B, \lambda_C$ must verify an additional relation, which will appear as we clarify the λ dependence. For this purpose let us write the matrix elements of the solution matrix $R(\Delta, \delta', \delta, \lambda)$ as

$$R_{11} = u\left(\frac{\delta}{\delta'}\right)^{1/4}, \quad R_{22} = v\left(\delta'\delta\right)^{1/4}, \quad R_{33} = v\left(\frac{1}{\delta'\delta}\right)^{1/4}, \quad R_{44} = u\left(\frac{\delta'}{\delta}\right)^{1/4}. \tag{33}$$

This form is compatible with the previous assignments, and the λ dependence is entirely inside the u's and the v's. What remains are the following relations:

$$\begin{aligned} \sqrt{\Delta} &= \frac{2u_A v_A}{u_A^2+v_A^2-1} = \frac{2u_B v_B}{u_B^2+v_B^2-1} = \frac{2u_C v_C}{u_C^2+v_C^2-1}, && (34) \\ v_A &= v_B u_C - v_C u_B, \qquad u_B = u_A u_C - v_A v_C, && (35) \end{aligned}$$

which are resolved by

$$\begin{aligned} u_I &= \frac{\sin(\gamma-\lambda_I)}{\sin(\gamma)}, \quad v_I = \frac{\sin(\lambda_I)}{\sin(\gamma)}, \quad I = A, B, C && (36) \\ \lambda_B &= \lambda_A + \lambda_C, \quad \sqrt{\Delta} = -1/\cos(\gamma). && (37) \end{aligned}$$

After changing from Δ to γ we get

$$R(\gamma, q', q, \lambda) := \begin{pmatrix} \frac{q}{q'}\frac{\sin(\gamma-\lambda)}{\sin(\gamma)} & 0 & 0 & 0 \\ 0 & qq'\frac{\sin(\lambda)}{\sin(\gamma)} & 1 & 0 \\ 0 & 1 & \frac{1}{qq'}\frac{\sin(\lambda)}{\sin(\gamma)} & 0 \\ 0 & 0 & 0 & \frac{q'}{q}\frac{\sin(\gamma-\lambda)}{\sin(\gamma)} \end{pmatrix} \tag{38}$$

and the solution is given by

$$A = R(\gamma, q_1, q_2, \lambda_A), \quad B = R(\gamma, q_1, q_3, \lambda_A + \lambda_C), \quad C = R(\gamma, q_2, q_3, \lambda_C). \tag{39}$$

Thus we have been able to introduce a good parametrization to the algebraic solution (28). This is the asymmetric six-vertex solution of [11, 12, 13].

With this parametrization we see that the spectral parameters of A, B, and C are points on the circle (34), with its simple addition law. It also clarifies the action of $\mathcal{A}ut_2$. Since

$$t_l I t_l I : \; R(\gamma, q, q', \lambda) \longmapsto R(\gamma, q, q', \lambda + 2\gamma), \tag{40}$$

the action of $\mathcal{A}ut_2$ is just a shift of the spectral parameter. Moreover, among the moduli parameters, only γ is global, while the q_i's are attached to the vector spaces on which the matrix operates ('color parameters'). It is interesting to note how the periodic orbits of $\mathcal{A}ut$ appear: they correspond to the values of γ which are commensurate to π. The special case $\gamma = \pi/2$ is included in the following.

The first known parametrized solution of (2), $R(u) = P + uI$, is obtained as a singular limit of (38): take $q_k = \sqrt{-1}$, $\lambda = -\gamma u$ and then let $\gamma \to 0$. The case 6V(I) of [19] is sub-case $q = q'$ of (38). For other special cases, see [20].

3.3.2 The second solution (free fermion type)

If we solve for c_4 from the second factor of (27), we get C and B as follows:

$$\begin{aligned} dp(C) &= [c_1, c_2, c_3, (1 - c_2 c_3)/c_1], \\ dp(B) &= [b_1, b_2, b_3, (1 - b_2 b_3)/b_1], \end{aligned} \tag{41}$$

In this case it is natural to put the diagonal elements of (12) into a 2×2 matrix as

$$\hat{X} = \begin{pmatrix} X_{11} & -X_{22} \\ X_{33} & X_{44} \end{pmatrix}, \tag{42}$$

and then (41) implies $\det \hat{B} = \det \hat{C} = 1$. Furthermore, we find that the remaining matrix $\hat{A}$ is given by a matrix product (note the order)

$$\hat{A} = \hat{C}^{-1}\hat{B}. \tag{43}$$

Thus in this case the natural parametrization is through the group $SL(2)$: For any $SL(2)$ matrix $\hat{X}$ let $R(\hat{X})$ be the 4×4 matrix of type (12) obtained by putting the elements of $\hat{X}$ on the diagonal as discussed above ($X_{23} = X_{32} = 1$). Then, according to the above, we can write the result in the form $A = R(\hat{A})$, $B = R(\hat{A} \oplus \hat{C})$, $C = R(\hat{C})$, where now $\hat{A} \oplus \hat{C} = \hat{C}\hat{A}$ (in particular, here $\oplus$ is not Abelian). The parameter content of this solution is clearly $(3, 3, 0)$. This already shows the difference with the first solution. This solution is the one of [17], see also [18]. Its constant limit is the permutation matrix.

There are no gauge parameters in our presentation of the solution. The action of $\mathcal{A}ut_2$ is

$$(K_a K_b)^2 : \hat{A} \mapsto -\hat{A}, \quad \hat{B} \mapsto -\hat{B}, \quad \hat{C} \mapsto \hat{C} \tag{44}$$

and so on. It is equivalent to discrete gauge transformations of square one.

This solution allows many reductions with one-dimensional spectral parameters. For example, we may take the solution (38), with $\gamma = \pi/2$. Consider the polynomials p_i introduced above. In this case p_9 vanishes, and the rank of the remaining invariant ratios is 2. Fixing the value of $\delta = q^4$ and $\delta' = q'^4$ determines a curve on $SL(2)$, leading to an Euler type of parametrization:

$$\hat{X} = \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} q'^{-1} & 0 \\ 0 & q' \end{bmatrix} \tag{45}$$

From (40), the action of $(t_l I)^2$ is a shift of θ by π and is indeed of order 2. The parameters q and q' are free for two of the matrices $[A, B, C]$, say B and C. If $\delta(B) = \delta(C)$, then $\delta(A) = \delta'(C)$ and $\delta'(A) = \delta'(B)$, and the composition law (43) coincides with the addition on θ, and we have a special case of solution (39). The solution (3.1) of [14] corresponds to a slightly different splitting:

$$\hat{X} = \frac{1}{2} \begin{bmatrix} e^{-q} & e^{q} \\ -e^{-q} & e^{q} \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} e^{q'} & -e^{q'} \\ e^{-q'} & e^{-q'} \end{bmatrix} \tag{46}$$

As a summary we can state that any six vertex solution is one of the two presented here, depending on whether p_9 vanishes or not.

4 An eight-vertex Ansatz

We take next a particular eight-vertex Ansatz: the three matrices A, B, C are assumed to be symmetric with respect to the secondary diagonal:

$$X = \begin{pmatrix} X_{11} & 0 & 0 & X_{14} \\ 0 & X_{22} & X_{23} & 0 \\ 0 & X_{32} & X_{22} & 0 \\ X_{41} & 0 & 0 & X_{11} \end{pmatrix} \tag{47}$$

There are many solutions of (1) having this form; here we will analyze the solution that has the maximum number of parameters, which is seven after the scalings have been fixed by putting $X_{11} = 1$.

4.1 General observations

The solution $\mathcal{S} := (A, B, C)$ can be given in terms of seven independent parameters a, b, c, x, y, z, v:

$$A = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & x & \dfrac{b\,(v-x)}{cy} & 0 \\ 0 & \dfrac{c\,(v-xyz)}{bz} & x & 0 \\ \dfrac{(v-y)\,(v-z)}{ayz} & 0 & 0 & 1 \end{pmatrix} \tag{48}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & b \\ 0 & y & \dfrac{a\,(v-x)}{cx} & 0 \\ 0 & \dfrac{c\,(v-z)}{az} & y & 0 \\ \dfrac{(v-y)\,(v-xyz)}{bxz} & 0 & 0 & 1 \end{pmatrix} \tag{49}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & z & \dfrac{a\,(v-xyz)}{bx} & 0 \\ 0 & \dfrac{b\,(v-z)}{ay} & z & 0 \\ \dfrac{(v-y)\,(v-x)}{cxy} & 0 & 0 & 1 \end{pmatrix} \tag{50}$$

This solution is globally stable by the diagonal group $\mathcal{G}_d$, and the action of $g \in \mathcal{G}_d$ (11) moves only a, b, c as follows:

$$a \mapsto a\, t_1^{-2} t_2^{-2}, \quad b \mapsto b\, t_1^{-2} t_3^{-2}, \quad c \mapsto c\, t_2^{-2} t_3^{-2} \tag{51}$$

The remaining four parameters are gauge invariant and are therefore coordinates on the orbit space $\eta = \mathcal{S}/\mathcal{G}_d$. The choice of a gauge amounts to the choice of three functions $a(x, y, z, v)$, $b(x, y, z, v)$, $c(x, y, z, v)$, and we will later see the effect of choosing a particular form.

The action of the generators of $\mathcal{A}ut$ is a birational transformation of the parameters and reads:

$$
\begin{aligned}
K_a \quad : \quad & [a,b,c] \mapsto [-\frac{(v-y)\,(v-z)}{ayz}, \frac{c\,(v-z)}{az}, \frac{b\,(v-z)}{ay}] \\
& [x,y,z,v] \mapsto [\frac{(v-z-y)\,x}{v-x-xyz}, y, z, z-v+y]
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
K_b \quad : \quad & [a,b,c] \mapsto [\frac{c\,(v-xyz)}{bz}, -\frac{(v-z-x)\,(v-y)\,(v-xyz)}{bxz\,(v-xyz-y)}, \frac{a\,(v-xyz)}{bx}] \\
& [x,y,z,v] \mapsto [x, \frac{(v-z-x)\,y}{v-xyz-y}, z, z-v+x]
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
K_c \quad : \quad & [a,b,c] \mapsto [\frac{b\,(v-x)}{cy}, \frac{a\,(v-x)}{cx}, -\frac{(v-y)\,(v-x)}{cxy}] \\
& [x,y,z,v] \mapsto [x, y, \frac{(v-x-y)\,z}{v-xyz-z}, x-v+y]
\end{aligned} \tag{54}
$$

One verifies here that $\mathcal{A}ut$ indeed acts on the orbit space η, since the transformation of x, y, z and v does *not* depend on a, b, c.

There are two invariants of $\mathcal{A}ut$ on η:

$$
\Delta_1 = \frac{v(2v-xyz-x-y-z)}{xyz} \tag{55}
$$

$$
\Delta_2 = \frac{(v-x)(v-y)(v-z)(v-xyz)}{x^2y^2z^2} \tag{56}
$$

There is a canonical way to find these invariants [21]. It consists of first calculating the squares of the generators K_a, K_b, K_c in homogeneous coordinates. Such squares appear as the multiplication by some polynomial (Φ_a, Φ_b, Φ_c). For example using the homogenizing variable t:

$$
\begin{aligned}
K_a : \quad & t && \mapsto t\left(t^2x - t^2v + xyz\right) \\
& x && \mapsto xt^2\,(z-v+y) \\
& y && \mapsto y\left(t^2x - t^2v + xyz\right) \\
& z && \mapsto z\left(t^2x - t^2v + xyz\right) \\
& v && \mapsto (z-v+y)\left(t^2x - t^2v + xyz\right) \\
K_a^2 \simeq \Phi_a \quad & = && \; t^4\left(t^2x - t^2v + xyz\right)^3 v\,(z-v+y)
\end{aligned}
$$

Any rational invariant is the ratio of two polynomials which have the same covariance properties under K_a (resp. K_b and K_c). The covariance factors are known to be the factors appearing in Φ_a, (resp. Φ_b, Φ_c), and for a given degree, there are only a finite number of possible covariance factors. It is thus possible to find all algebraic invariants of a given degree. This algorithm is unfortunately unbounded, since we do not know any bound on the degree of the invariant. However, it proves quite efficient in practice.

The two invariants are 'global', as they can be calculated from any of the three matrices A, B or C using the polynomials p_5 and p_9 as given in [10]:

$$\Delta_1 = -2\,\frac{p_9}{p_5}, \quad \Delta_2 = \frac{\text{product of anti-diagonal entries}}{\text{product of diagonal entries}}.$$

The surfaces Δ_1 = constant, Δ_2 = constant, in η are preserved by the induced action of $\mathcal{A}ut$. They are of generic dimension 2 and define the varieties where the spectral parameters live. The invariants Δ_1 and Δ_2 are the coordinates on the moduli space of the solution $\mathcal{S}$. Note that they are invariant by any permutation of x, y, z. Note also that the free fermion condition [11] is just $\Delta_1 = 0$.

For later discussions let us introduce the parameters q_i by

$$q_1 = \frac{v-x}{\sqrt{xyz}},\; q_2 = \frac{v-xyz}{\sqrt{xyz}},\; q_3 = \frac{v-z}{\sqrt{xyz}},\; q_4 = \frac{v-y}{\sqrt{xyz}}. \tag{57}$$

For the inverse relations define first Λ by

$$\prod_{i=1}^{4}(\Lambda - q_i) = 1, \tag{58}$$

and then

$$x = (\Lambda-q_1)(\Lambda-q_2),\; y = (\Lambda-q_4)(\Lambda-q_2),\; z = (\Lambda-q_3)(\Lambda-q_2),\; v = \Lambda(\Lambda-q_2), \tag{59}$$

and for the Δ's we get

$$\Delta_2 = q_1q_2q_3q_4, \tag{60}$$

$$\begin{aligned}\Delta_1 &= -2\Lambda^2 + \Lambda(q_1+q_2+q_3+q_4)\\ &= q_1q_2 + q_3q_4 - x - \frac{1}{x} = q_1q_3 + q_2q_4 - y - \frac{1}{y} = q_2q_3 + q_1q_4 - z - \frac{1}{z}.\end{aligned} \tag{61}$$

With these definitions the anti-diagonal entries of our solution can be written as

$$ad(A) = \left\{a, \frac{b}{c}\sqrt{\frac{xz}{y}}\,q_1, \frac{c}{b}\sqrt{\frac{xy}{z}}\,q_2, \frac{x}{a}\,q_3q_4\right\}, \tag{62}$$

$$ad(B) = \left\{b, \frac{a}{c}\sqrt{\frac{yz}{x}}\,q_1, \frac{c}{a}\sqrt{\frac{xy}{z}}\,q_3, \frac{y}{b}\,q_2q_4\right\}, \tag{63}$$

$$ad(C) = \left\{c, \frac{a}{b}\sqrt{\frac{yz}{x}}\,q_2, \frac{b}{a}\sqrt{\frac{xz}{y}}\,q_3, \frac{z}{c}\,q_1q_4\right\}. \tag{64}$$

Note that the q_i behave almost like the color parameters.

4.2 Specific gauges and related parametrizations

We will show here how the gauge condition, i.e. the choice of a, b, c as functions of x, y, z, v affects the distribution of the parameters among the three members of the solution triplet. The solution we have is a four parameter solution, once the gauge is fixed, as is Baxter's solution [1, 22]. Note that for us, 'fixing the gauge' means preventing continuous residual gauge freedom but leaves room for discrete transformations.

As a possible simple gauge we could take $a = 1, b = 1, c = 1$. With this choice, choosing C uses up *all* four parameters of the solution, and B and A are completely determined once C is known, thus the parameter content in this case is (4,0,0). The solution is fully rational but does not lead to a parametrized family of commuting transfer matrices. It leads to another very interesting — although apparently less constrained — situation: we have an infinite sequence of commuting transfer matrices with commutation between successors. We shall not explore this possibility here.

An important question now is the following: How should we choose the gauge condition in order to get a parametrized family of solutions? Clearly the minimum requirement is to choose a, b, c in such a way that the knowledge of C, for example, uses only *three* of the four available parameters on η, leading to the parameter content (3,1,0). In other words, the gauge choice must lower the rank of the set of anti-diagonal elements

$$\Sigma := \left\{ z, c, \frac{a}{b}\sqrt{\frac{yz}{x}}\, q_2, \frac{b}{a}\sqrt{\frac{xz}{y}}\, q_3, \frac{z}{c}\, q_1 q_4 \right\}, \tag{65}$$

from four to three. Instead of Σ we could consider the set $\Sigma' := \{z,\ c,\ \frac{b}{a}\sqrt{\frac{x}{y}}\, q_3,\ q_2 q_3,\ q_1 q_4\}$, or using (61), $\Sigma'' := \{z,\ c,\ \frac{b}{a}\sqrt{\frac{x}{y}}\, q_3,\ \Delta_1,\ \Delta_2\}$. In this last set, z, Δ_1, and Δ_2 are clearly functionally independent, so in order to have no more than these three parameters we must impose the condition

$$c = f(z, \Delta_1, \Delta_2), \quad \frac{a}{b} = \sqrt{\frac{x}{y}}\, q_3\, g(z, \Delta_1, \Delta_2), \tag{66}$$

where f and g are some arbitrary functions.

It was argued earlier that for many applications C and B, say, should have a similar structure, and in particular the same number of free parameters. If we therefore apply the above argument to B we get in a similar way the conditions

$$b = h(y, \Delta_1, \Delta_2), \quad \frac{a}{c} = \sqrt{\frac{x}{z}}\, q_3\, k(y, \Delta_1, \Delta_2), \tag{67}$$

where h and k are free functions. The compatibility of (66, 67) (solving for a in two ways) implies

$$k = \omega(\Delta_1, \Delta_2) h(y, \Delta_1, \Delta_2)/\sqrt{y}, \quad g = \omega(\Delta_1, \Delta_2) f(z, \Delta_1, \Delta_2)/\sqrt{z}, \tag{68}$$

where ω is an arbitrary function, so that

$$a = \sqrt{\frac{x}{yz}}\, q_3\, h(y)\, f(z)\, \omega = \frac{v - z}{yz}\, h(y)\, f(z)\, \omega. \tag{69}$$

[From now on we do not write out the Δ_1, Δ_2 dependence.]

In order to get a true one parameter family of commuting transfer matrices we want the matrices B and C to be in the same parametrized family, i.e: $B = R(y, \Delta_1, \Delta_2)$ and $C = R(z, \Delta_1, \Delta_2)$ for some $R(\tau, \Delta_1, \Delta_2)$. This can be done, the condition is $h = f$ and yields:

$$R(\tau,\Delta_1,\Delta_2)=\begin{pmatrix} 1 & 0 & 0 & f(\tau) \\ 0 & \tau & \omega\, f(\tau)\, Q & 0 \\ 0 & \dfrac{\tau}{\omega\, f(\tau)} & \tau & 0 \\ \dfrac{\tau\Delta_2}{f(\tau)\, Q} & 0 & 0 & 1 \end{pmatrix} \tag{70}$$

with Q a root of

$$\tau Q^2-\tau^2 Q-\tau Q\Delta_1-Q+\tau\Delta_2=0. \tag{71}$$

This result shows that the elliptic curve (71) must be introduced even if we just want to write two entries of the solution as a parametrized family. This still leaves considerable freedom in choosing the gauge. Simple-looking results are obtained e.g. if we take $\omega=1$ and $f=1$ or τ, but these are no longer rational in τ since $Q(\tau)$ will involve square roots.

The above expressions for a, b, c, were obtained by the condition that B and C depend only on three of the four parameters. If we now continue and require the same on A we obtain the additional conditions

$$a=m(x,\Delta_1,\Delta_2),\quad \frac{c}{b}=\sqrt{\frac{z}{y}}\, q_1\; n(x,\Delta_1,\Delta_2). \tag{72}$$

The compatibility of these with (66,67) leads eventually to the symmetric solution

$$a=\sqrt{\frac{x}{q_1q_2}}\,\phi_1\phi_2,\quad b=\sqrt{\frac{y}{q_1q_3}}\,\phi_1\phi_3,\quad c=\sqrt{\frac{z}{q_2q_3}}\,\phi_2\phi_3, \tag{73}$$

where $\phi_i=\phi_i(\Delta_1,\Delta_2)$ is the residual freedom in the choice of gauge and the previously used function ω is related to ϕ_3 by $\omega\phi_3^2=1$.

The gauge choice (73) leads to the following parameter counting: each matrix of the triplet contains three independent entries, and the choice of C, say, fixes three of the four parameters of the solution. One free parameter is left for B and finally A is determined once B is chosen.

4.3 Elliptic parametrization

Baxter solution [22] is actually contained in (48, 49, 50) if we choose the gauge (73) with $\phi_i=\Delta_2^{1/4}$, in other words $f(\tau)=h(\tau)=m(\tau)=\sqrt{\tau\Delta_2/Q(\tau)}$ and $\omega=1/\sqrt{\Delta_2}$. This gauge is also uniquely defined by the requirement that the matrices are symmetric under the usual transposition.

But fixing the gauge is not the end of the story. For a good spectral parameter we need also a good composition rule, in this case it is obtained as follows. Fixing C fixes the values of the invariants $\Delta_1(C)$ and $\Delta_2(C)$, and therefore the elliptic curve (71). C, B, and A will then be given by three points on this same curve. There is a natural addition rule on elliptic curves, and to verify that our parameters satisfy it we have to use the usual uniformization with elliptic functions Baxter [22]. The result is as follows: Let us define γ and k by

$$\Delta_1 = -2cn(\gamma)dn(\gamma), \quad \Delta_2 = sn^4(\gamma)k^2 \tag{74}$$

where sn, cn, dn are the Jacobi elliptic functions of modulus k, and σ, ρ, χ by

$$z = \frac{sn(\sigma)}{sn(\gamma-\sigma)}, \quad y = \frac{sn(\rho)}{sn(\gamma-\rho)}, \quad x = \frac{sn(\chi)}{sn(\gamma-\chi)}, \tag{75}$$

then the relations (55,56) are satisfied when we take

$$v = \frac{sn(\rho)[sn(\sigma)sn(\chi) + sn(\gamma)sn(\gamma-\rho)]}{sn(\gamma-\chi)sn(\gamma-\rho)sn(\gamma-\sigma)} \tag{76}$$

and use the addition rule

$$\rho = \sigma + \chi. \tag{77}$$

To complete the parametrization we note that

$$q_1 = sn(\gamma)\sqrt{\frac{sn(\sigma)sn(\gamma-\sigma)}{sn(\rho)sn(\gamma-\rho)sn(\chi)sn(\gamma-\chi)}}, \tag{78}$$

$$q_2 = sn(\gamma)\sqrt{\frac{sn(\rho)sn(\gamma-\rho)}{sn(\sigma)sn(\gamma-\sigma)sn(\chi)sn(\gamma-\chi)}}, \tag{79}$$

$$q_3 = sn(\gamma)\sqrt{\frac{sn(\chi)sn(\gamma-\chi)}{sn(\sigma)sn(\gamma-\sigma)sn(\rho)sn(\gamma-\rho)}}, \tag{80}$$

$$q_4 = sn(\gamma)k^2\sqrt{sn(\chi)sn(\gamma-\chi)sn(\rho)sn(\gamma-\rho)sn(\sigma)sn(\gamma-\sigma)}, \tag{81}$$

$$Q(x) = \frac{sn(\gamma)^2}{sn(\chi)sn(\gamma-\chi)}, \tag{82}$$

and similarly for y, z.

If we now define (note the overall scaling)

$$R(\alpha,\gamma,k) = \begin{pmatrix} sn(\gamma-\alpha) & 0 & 0 & sn(\alpha)\,sn(\gamma)\,k \\ 0 & sn(\alpha) & sn(\gamma) & 0 \\ 0 & sn(\gamma) & sn(\alpha) & 0 \\ sn(\alpha)\,sn(\gamma)\,k & 0 & 0 & sn(\gamma-\alpha) \end{pmatrix} \tag{83}$$

then $A = R(\chi,\gamma,k), B = R(\rho,\gamma,k), C = R(\sigma,\gamma,k)$ solve (1b), and this is exactly Baxter's solution. [If $k = 0$ we get a special case of the six-vertex solution.]

What (74), (75), (76), together with (77) show is that the two dimensional surface in η given by fixing the values of Δ_1 and Δ_2 is a product of two elliptic curves (or two points on the same elliptic curve). Baxter's parametrization makes it explicit. Furthermore it also allows to visualize the action of $\mathcal{A}ut$, since

$$\begin{array}{ll} K_aK_b : \rho \mapsto \rho+\gamma, & \sigma \mapsto \sigma \\ K_bK_c : \rho \mapsto \rho, & \sigma \mapsto \sigma+\gamma \end{array} \tag{84}$$

5 Conclusion

We have analyzed several two-state solutions of the Yang-Baxter equations, and shown how starting from a rational solution without recognizable structure one can construct spectral and moduli parameters using the symmetries of the equations. For five- and six-vertex ansatze our results are complete.

The effect of gauge transformation on the parametrization is particularly interesting. One of the lessons of the present work is that it can be much easier to solve the YBE when a gauge has not been fixed. Finding a good parametrization is a separate problem, which can be done at leisure, after a solution has been found.

The present method can be used for any solution of the Yang-Baxter equations, whenever they are found. Unfortunately we do not yet have a thorough analysis of these equations with absolutely no a priori assumptions on their form (i.e. no Ansatz at all), in the spirit of the complete resolution of the 'constant' equations obtained earlier by one of the authors [4].

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