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On Hamiltonian Formalism for Dressing Chain Equations of Even Periodicity

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Abstract

We propose a Hamiltonian formalism for N periodic dressing chain with the even number N. The formalism is based on Dirac reduction applied to the $N+1$ periodic dressing chain with the odd number $N+1$ for which the Hamiltonian formalism is well known. The Hamilton dressing chain equations in the N even case depend explicitly on a pair of conjugated Dirac constraints and are equivalent to $A^{(1)}_{N-1}$ invariant symmetric Painlevé equations.

1 Introduction

The N periodic dressing chain has emerged in a study of Schrödinger operators interconnected by Darboux transformations [12]. Its equivalence to $A^{(1)}_{N-1}$ invariant Painlevé equations has been subject of a number of papers, see e.g. [1, 9, 10, 11, 13]. The system can naturally be realized as a self-similarity limit of the t_2 flow equations of several two-dimensional integrable models, e.g. (1) the integrable mKdV hierarchy on the loop algebra of $sl(N) = A_{N-1}$ endowed with a principal gradation [8, 2], (2) a class of constrained KP hierarchy referred to as $2n$-boson integrable models generalizing AKNS hierarchy [5, 4] that in a self-similarity limit are equivalent to the $N = 2n + 1$ periodic dressing chain equations.

The Hamiltonian formalism for N periodic dressing chain [12] is straightforward as long as N is odd. However the same structure for even N requires for consistency an additional relation [10]. Here, we propose a coherent formalism based on Dirac approach where the consistency condition naturally emerges as a secondary constraint for even N and does not need to be set to zero. We illustrate our formalism by explicitly employing the reduction from the Hamiltonian formalism of $N = 5$ to $N = 4$ dressing chains. We show equivalence between the $N = 4$ dressing chain equations with explicit dependence on two Dirac constraints and the $A_3^{(1)}$ symmetric Painlevé V equations.

Furthermore, we observe in this paper that generalization of the method to $2n+1 \to 2n$ reductions for n integer and $n \geq 2$ follows the same steps and we provide a formula describing all such Hamilton equations. We establish the invariance of the two Dirac constraints under extended affine $A_{2n-1}^{(1)}$ Weyl symmetry that ensures invariance of the dressing chain equations. Moreover the $A_{2n-1}^{(1)}$ basic Weyl algebra relations hold despite the presence of additional (Dirac) terms in fundamental brackets between currents $j_i, i = 1, \ldots, 2n$ and the Hamiltonian.

The paper starts by revisiting basic facts about periodic dressing chain equations and the corresponding Hamiltonian formalism in section 2. We stress a different nature of commutation relations satisfied by the quantity $\sum_{i=1}^{N} j_i$ with respect to the basic Poisson bracket $\{\cdot, \cdot\}$ for N odd and even.

The main results of the paper are presented in section 3 and derived from Dirac reduction procedure in subsection 3.1. The subsection 3.2 is devoted to proving that despite explicit presence of Dirac constraints in Hamilton equation of N periodic dressing chain equations they remain invariant under extended affine Weyl symmetry group $A_{N-1}^{(1)}$.

We describe a mechanism to deform the $A_4^{(1)}$ symmetric Painlevé equations in section 4. The reduction formalism $A_4^{(1)} \to A_3^{(1)}$ we have introduced offers new ways to explore how the $A_3^{(1)}$ Weyl symmetry can be broken down to a partial Bäcklund symmetry by addition of quadratic terms to the Hamiltonian of $N = 5$ symmetric Painlevé equations.

In the case of $2n$-boson constrained KP hierarchy [5, 4] the self-similarity reduction yields Hamiltonians of the $A_{2n}^{(1)}$ Painlevé models expressed in terms of n canonical pairs $e_i, Y_i, i = 1, 2, \ldots, n$ satisfying brackets $\{e_i, Y_j\} = \delta_{ij},\ i = 1, \ldots, n$. The Dirac formalism was defined in [4] to further reduce this system to $A_{2n-1}^{(1)}$ Painlevé models by setting $Y_n = 0$. This construction was illustrated by reduction of $A_4^{(1)}$ Painlevé equations down to the $A_3^{(1)}$ symmetric Painlevé V equations. Although the reference [4] employed a different set of variables than the conventional periodic dressing chain variables $j_i, i = 1, \ldots, 2n+1$ of this paper in section 5 we are able to show that the reduction of [4] is in complete agreement with the framework put forward in this paper. The advantage of the current formalism is that is more intuitive and therefore simpler to generalize from the $n = 2$ four boson model to higher n models.

2 Background

Our starting point is the set of N periodic dressing chain equations:

$$(j_i + j_{i+1})_z = -(j_i - j_{i+1})(j_i + j_{i+1}) + \alpha_i, \qquad i = 1, \ldots, N\,, \tag{1}$$

where periodicity implies that $j_{N+i} = j_i, \alpha_{N+i} = \alpha_i, i = 1, \ldots, N-1$.

Let us first work with the odd number $N = 2n+1$ and recall the Hamiltonian formalism for such case [12]. Note that for the odd N one can invert the relation

$$f_i = j_i + j_{i+1}\,, \tag{2}$$

by providing j_i in terms of $f_k, k = 1, \ldots, N$:

$$j_i = \frac{1}{2} \sum_{k=0}^{N-1} (-1)^k f_{i+k} \,. \tag{3}$$

For example for $N = 3$:

$$j_1 = \frac{1}{2}(f_1 - f_2 + f_3),\; j_2 = \frac{1}{2}(f_2 - f_3 + f_1),\; j_3 = \frac{1}{2}(f_3 - f_1 + f_2)\,.$$

Thus as long as N is odd one is able to invert relation (2) and pass effortlessly from f_i-formalism to j_i-formalism establishing full equivalence between the bracket structure

$$\{f_i, f_k\} = \begin{cases} 0 & k \neq i+1, k+1 \neq i \\ 1 & k = i+1 \\ -1 & k+1 = i\,, \end{cases} \tag{4}$$

and the bracket structure:

$$\{j_i, j_k\} = (-1)^{k-i+1},\, 1 \leq i < k \leq N\,. \tag{5}$$

For a sum $\sum_{k=1}^{N} j_k$ it holds from (5) that

$$\{j_i, j_1 + j_2 + j_3 + j_4 + j_5 + \ldots + j_N\} = \begin{cases} (-1)^{i+1} & N \;\text{ even} \\ 0 & N \;\text{ odd} \end{cases}\,. \tag{6}$$

Thus setting the quantity $\Phi = \sum_{k=1}^{N} j_k$ to be equal to a constant is consistent with the underlying Poisson bracket structure for N-odd but for N-even it will be imposed below through the Dirac system of primary/secondary constraints.

Defining, like Shabat and Veselov in [12], the cubic Hamiltonian

$$H_N = \frac{1}{3} \sum_{k=1}^{N} j_k^3 + \sum_{k=1}^{N} \beta_k j_k\,, \tag{7}$$

one obtains

$$\{j_i + j_{i+1}\,,\, H_N\} = j_{i+1}^2 - j_i^2 + \beta_{i+1} - \beta_i, \quad i = 1, 2, \ldots, N = 2n+1\,, \tag{8}$$

using the bracket (5) for N being odd. We recognize in equations (8) the dressing chain equations (1) with

$$\alpha_i = \beta_{i+1} - \beta_i, \qquad i = 1, \ldots, N\,. \tag{9}$$

The above relation implies that $\sum_{k=1}^{N} \alpha_k = 0$.

Identifying in the Hamilton equations the expression $\{f, H\}$ with f_z makes it possible to allow for $\sum_{k=1}^{N} \alpha_k \neq 0$ by redefining j_i by e.g. $J_i = j_i - \frac{1}{4} z$ (see e.g. [12, 2]) and working with the corresponding Hamiltonian :

$$H_N = \frac{1}{3} \sum_{k=1}^{N} J_k^3 + \frac{z}{4} \sum_{k=1}^{N} J_k^2 + \sum_{k=1}^{N} \beta_k J_k\,. \tag{10}$$

For simplicity we will continue to work here with the Hamiltonian (7).

3 Reduction from $N+1$ periodic dressing chain to even N periodic dressing chain

The main result of the paper is development of the Hamiltonian formalism for even $N = 2n$ case of equation (8), which in general case enters the Hamiltonian formalism in the following form :

$$\{j_i+j_{i+1},H\} = j_{i+1}^2-j_i^2+\beta_{i+1}-\beta_i+(-1)^{i+1}\frac{(j_i+j_{i+1})}{\Phi}\Psi, \quad i=1,2,\ldots,N=2n\,, \tag{11}$$

with $\Phi \neq 0, \Psi \neq 0$, where Φ and Ψ are defined as

$$\Phi = \sum_{k=1}^{N=2n} j_k, \quad \Psi = \sum_{k=1}^{N=2n} (-1)^{k+1}(j_k^2+\beta_k)\,. \tag{12}$$

The Hamiltonian H in relation (11) will be derived below from the Dirac formulation and the bracket $\{\cdot,\cdot\}$ always refers to the Poisson bracket (5).

We can cast the equations (11) into symmetric $A^{(1)}_{N-1}$ Painlevé equations. In particular for $N = 4$ we obtain $A^{(1)}_3$ symmetric Painlevé V equations:

$$\begin{aligned}
\Phi\,\{f_1,H\} &= f_1f_3\,(f_2-f_4)-(\alpha_1+\alpha_3)\,f_1+\alpha_1(f_1+f_3)\,,\\
\Phi\,\{f_2,H\} &= f_2f_4\,(f_3-f_1)+(\alpha_1+\alpha_3)\,f_2+\alpha_2(f_2+f_4)\,,\\
\Phi\,\{f_3,H\} &= f_1f_3\,(f_4-f_2)-(\alpha_1+\alpha_3)\,f_3+\alpha_3(f_1+f_3)\,,\\
\Phi\,\{f_4,H\} &= f_4f_2\,(f_1-f_3)+(\alpha_1+\alpha_3)\,f_4+\alpha_4(f_2+f_4)\,,
\end{aligned} \tag{13}$$

as follows by verification after substitution of f_i, α_i from equations (2), (9). Remarkably after inserting values Φ, Ψ from the definition (12) the right hand sides of equations (11) can be expressed entirely by variables $f_i, i = 1,\ldots,N$ without any need to invert relations (2). This observation is crucial for consistency of the proposed Hamiltonian formalism for even N and arbitrary non-zero Ψ.

3.1 Defining Hamiltonian formalism in terms of Dirac reduction

In section 2 we have been working with the Hamiltonian system (8) for odd $N = 2n+1$ with $\Phi \equiv \sum_{k=1}^{N=2n+1} j_k$ that commutes with all j_i and H_N and can naturally be chosen to be a constant. We will show how to obtain the same result for $\Phi \equiv \sum_{k=1}^{N} j_k$ with N even by working with Dirac formalism.

We now present the Dirac reduction formalism leading to the dressing chain equation (11). For illustration we perform all the steps for the $N = 5 \to N = 4$ case.

The first step is to eliminate j_5 from equation the initial cubic Hamiltonian (7) expressing it in terms of the remaining four objects $j_i, i = 1,2,3,4$. Elimination of j_5 involves setting the condition:

$$\psi_1 = j_5 = -j_1-j_2-j_3-j_4 \sim \text{const}\,. \tag{14}$$

In this limit the Hamiltonian $H_{N=5}$ becomes

$$H_R = H_N|_{j_5=-j_1-j_2-j_3-j_4} = \frac{1}{3}\sum_{i=1}^{4} j_i^3 - \frac{1}{3}(j_1+j_2+j_3+j_4)^3 + \sum_{i=1}^{4}\beta_i j_i - \beta_5(j_1+j_2+j_3+j_4)\,. \tag{15}$$

Condition (14) needs to be accompanied by secondary Dirac condition

$$\psi_2 = \Psi = \{j_5, H_{N=5}\} = \{\psi_1, H_{N=5}\} = -\sum_{k=1}^{4}\Big((-1)^k j_k^2 + \beta_k(-1)^k\Big)\,. \tag{16}$$

Ψ can also be rewritten as

$$\Psi = \left(j_1^2 + j_3^2 - j_2^2 - j_4^2\right) + \frac{1}{2}(-\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4)\,, \tag{17}$$

in terms of α's. The Dirac constraints $\psi_1 = -\Phi, \psi_2 = \Psi$ satisfy the bracket

$$\{\psi_1, \psi_2\} = 2(j_1+j_2+j_3+j_4) = 2\Phi\,, \tag{18}$$

that for $\Phi \neq 0$ gives rise to a regular matrix

$$D_{\alpha\beta} = \{\psi_\alpha, \psi_\beta\} = \begin{pmatrix} 0 & 2\Phi \\ -2\Phi & 0 \end{pmatrix}, \quad D^{-1}_{\alpha\beta} = \frac{1}{2\Phi}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{19}$$

that will be used to calculate the Dirac bracket:

$$\{j_i, j_k\}_D = \{j_i, j_k\} - \{j_i, \psi_\alpha\}\, D^{-1}_{\alpha\beta}\{\psi_\beta, j_k\}\,. \tag{20}$$

Let us calculate $\{j_1, j_2\}_D$ as illustration of the Dirac bracket technique in this context. Using brackets:

$\{j_1, \psi_1\} = -\{j_1, \Phi\} = -1,\ \{\psi_2, j_2\} = 2(j_1 - j_3 - j_4).\ \{j_1, \psi_2\} = -2(j_2 + j_3 + j_4),\ \{\psi_1, j_2\} = -1\,,$

we find

$$\{j_1, j_2\}_D = 1 - \{j_1, \psi_1\}D^{-1}_{12}\{\psi_2, j_2\} - \{j_1, \psi_2\}D^{-1}_{21}\{\psi_1, j_2\} = 1 - \frac{j_1 - j_3 - j_4}{\Phi} - \frac{j_2 + j_3 + j_4}{\Phi} = \frac{\Phi}{\Phi} - \frac{j_1 + j_2}{\Phi} = \frac{j_3 + j_4}{\Phi} = \frac{f_3}{\Phi}\,. \tag{21}$$

Proceeding in same manner with calculations for all the other non-vanishing brackets $\{j_i, j_j\}_D$ one obtains

$$\begin{aligned} &\{j_1, j_4\}_D = -\frac{f_2}{\Phi}, \quad \{j_3, j_4\}_D = \frac{f_1}{\Phi}, \quad \{j_1, j_2\}_D = \frac{f_3}{\Phi}, \\ &\{j_2, j_4\}_D = -\frac{f_4}{\Phi} + \frac{f_3}{\Phi}, \quad \{j_2, j_3\}_D = \frac{f_4}{\Phi}, \quad \{j_1, j_3\}_D = -\frac{f_3}{\Phi} + \frac{f_2}{\Phi} \end{aligned} \tag{22}$$

and it follows indeed that $\{j_i, \Phi\}_D = 0$ for any i.

Takasaki in [10] obtained the brackets (22) by first assuming that the periodic dressing chain equations (1) hold for $N = 4$ and then inserting equations (1) into the alternating sum $\sum_{k=1}^{N}(-1)^k\,(j_k + j_{k+1})_z$ that is identically zero as long as N is even. Imposing consistency of those two equations amounts to setting $\Psi = 0$. This in turn makes it possible to invert equations (2) and find expressions for j_i in terms of f_i. The brackets $\{j_i, j_j\}$ can in such way be derived from brackets (4) for f_i leading to an alternative derivation of (22) [10].

Here we will instead follow the Dirac procedure and use the original Poisson bracket $\{j_i, j_k\} = (-1)^{k-i+1}$ with the Hamiltonian:

$$H = H_R + \lambda_1\psi_1 + \lambda_2\psi_2\,, \tag{23}$$

where H_R is given in (15) and $\lambda_\alpha, \alpha = 1, 2$ are Lagrange multipliers that can be derived from

$$0 = \{\psi_\alpha, H_R\} + \sum_{\beta=1}^{2}\lambda_\beta\{\psi_\alpha, \psi_\beta\} = \{\psi_\alpha, H_R\} + \sum_{\beta=1}^{2} D_{\alpha\,\beta}\lambda_\beta\,, \tag{24}$$

or

$$\lambda_\alpha = -\sum_{\beta=1}^{2} D^{-1}_{\alpha\,\beta}\{\psi_\beta, H_R\} \;\to\; \lambda_1 = \frac{\{\psi_2, H_R\}}{2\Phi},\;\; \lambda_2 = -\frac{\psi_2}{2\Phi}\,.$$

Thus

$$\{j_i, H\} = \{j_i, H_R\} + \lambda_\alpha\{j_i, \psi_\alpha\} = \{j_i, H_R\} - \sum_{\alpha,\beta=1}^{2}\{j_i, \psi_\alpha\}D^{-1}_{\alpha\,\beta}\{\psi_\beta, H_R\} = \{j_i, H_R\}_D\,. \tag{25}$$

Especially, it follows that

$$\{\psi_\gamma, H\} = \{\psi_\gamma, H_R\} - \sum_{\alpha,\beta=1}^{2}\{\psi_\gamma, \psi_\alpha\}D^{-1}_{\alpha\,\beta}\{\psi_\beta, H_R\} = 0, \qquad \gamma = 1, 2\,.$$

From equations (25) we obtain

$$\begin{aligned}\{j_i + j_{i+1}, H\} &= \{j_i + j_{i+1}, H_R\} - \{j_i + j_{i+1}, \psi_1\}D^{-1}_{1\,2}\{\psi_2, H_R\} - \{j_i + j_{i+1}, \psi_2\}D^{-1}_{2\,1}\{\psi_1, H_R\}\\ &= \{j_i + j_{i+1}, H\} - \{j_i + j_{i+1}, \psi_2\}D^{-1}_{2\,1}\{\psi_1, H_R\}\,,\end{aligned} \tag{26}$$

where we used that

$$\{j_i + j_{i+1}, \psi_1\} = -\{j_i + j_{i+1}, \Phi\} = 0\,,$$

ensuring that the cubic term $\{\psi_2, H_R\}$ never appears in the dressing chain formulas.

After some algebra (and use of equation (17) in the bottom equation below) we obtain:

$$\begin{aligned}
\{j_1+j_2,H\} &= -j_1^2+j_2^2-\beta_1+\beta_2+\frac{(j_1+j_2)\Psi}{\Phi}\,,\\
\{j_2+j_3,H\} &= -j_2^2+j_3^2-\beta_2+\beta_3-\frac{(j_2+j_3)\Psi}{\Phi}\,,\\
\{j_3+j_4,H\} &= -j_3^2+j_4^2-\beta_3+\beta_4+\frac{(j_3+j_4)\Psi}{\Phi}\,,\\
\{j_4+j_1,H\} &= -j_1^2+j_4^2+2j_2^2-2j_3^2-\beta_1+2\beta_2-2\beta_3\\
&\quad+\beta_4+\frac{j_1+2j_2+2j_3+j_4}{\Phi}\Psi\\
&= j_1^2-j_4^2-\beta_4+\beta_1-\frac{(j_4+j_1)\Psi}{\Phi}\,.
\end{aligned} \tag{27}$$

The above results (27) can be summarized as the result (11) for $N = 4$. Summing over $i = 1, \ldots, 4$ gives

$$\{\Phi, H\} = 0\,,$$

which of course does not contradict $\{\Phi, H_R\} = \Psi$ found earlier.

Also, one can form an alternating sum of expressions of both sides of (11) to obtain

$$\begin{aligned}
\sum_{i=1}^{4}(-1)^i\{j_i+j_{i+1},H\} &= -\sum_{k=1}^{4}\Big((-1)^k j_k^2+\beta_k(-1)^k\Big)+\sum_{k=1}^{4}\Big((-1)^k j_{k+1}^2+\beta_{k+1}(-1)^k\Big)\\
&\quad-\frac{f_1+f_2+f_3+f_4}{\Phi}\Psi = 2\Psi-2\Psi=0\,,
\end{aligned} \tag{28}$$

consistently with that the left hand side is identically zero for N even! Thus the system of equations (11) does not need imposition of the condition $\Psi = 0$ for consistency.

Moreover, one can alternatively calculate

$$\{j_i+j_{i+1}\,,\,\frac{1}{3}\sum_{i=1}^{4} j_i^3+\sum_{i=1}^{4}\beta_i j_i\}_D\,,\quad i=1,2,3,4\,,$$

with the Dirac bracket $\{\cdot,\cdot\}_D$ as given in equations (22) instead of using bracket (5) and Hamiltonian H to reproduce the same result as in (27).

We now present a simple observation on how to explicitly transform any dressing chain of even cyclicity to the one with $\Psi = 0$ by shifting $j_i, i = 1, \ldots, 2n$ by terms proportional to Ψ/Φ. Let us namely introduce

$$\bar{j}_i = j_i + \frac{(-1)^i}{2}\frac{\Psi}{\Phi} \tag{29}$$

and notice that quantities $f_i = j_i + j_{i+1} = \bar{j}_i + \bar{j}_{i+1}$ remain invariant under the shift in (29). Accordingly, we can rewrite equation (11) (and its particular case in equation (27)) as

$$\{\bar{j}_i+\bar{j}_{i+1},H\} = \bar{j}_{i+1}^2-\bar{j}_i^2+\beta_{i+1}-\beta_i,\quad i=1,2,\ldots,N=2n\,, \tag{30}$$

removing explicit dependence on Ψ and Φ in the commutation relations. One can indeed check that $\bar{\Psi} = \bar{j}_1^2 + \bar{j}_3^2 - \bar{j}_2^2 - \bar{j}_4^2 + \frac{1}{2}(-\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4)$ is identically zero for $\bar{j}_i$ defined in relation (29) and so the system of equations (30) is consistent.

Since $\{j_i, \Psi\}_D = 0$ for any i the new quantities $\bar{j}_i$ will satisfy the same Dirac brackets (22) as the original quantities j_i.

3.2 Invariance under the $A_{N-1}^{(1)}$ transformations

In this subsection we show that equations (11) exhibit invariance under the $A_{N-1}^{(1)}$ transformations. For illustration we here consider the $N = 4$ example and s_i, $i = 1, 2, 3, 4$ transformations of $A_3^{(1)}$ [1]:

$$
\begin{aligned}
&j_i \xrightarrow{s_i} \tilde{j}_i = j_i - \frac{\kappa_i}{j_i + j_{i+1}}, \quad j_{i+1} \xrightarrow{s_i} \tilde{j}_{i+1} = j_{i+1} + \frac{\kappa_i}{j_i + j_{i+1}}, \\
&j_k \xrightarrow{s_i} j_k, \; k \neq i, k \neq i+1,
\end{aligned}
\tag{31}
$$

for $\kappa_i = \alpha_i = \beta_{i+1} - \beta_i$, when it is accompanied by transformations of coefficients $\alpha_i \to -\alpha_i, \alpha_{i\pm1} \to \alpha_{i\pm1} + \alpha_i$. This is accomplished by the following s_i transformation:

$$
\beta_i \xrightarrow{s_i} \beta_{i+1}, \quad \beta_{i+1} \xrightarrow{s_i} \beta_i, \tag{32}
$$

Explicit calculations show that the system of equations (11) is invariant under the transformations (31) and (32). The proof utilizes the fact that the objects Ψ, Φ (and obviously also f_i with the same index i as in equation (31)) are invariant under transformations (31) and (32). For Ψ this requires a small calculation which for e.g. s_1 goes as follows:

$$
\begin{aligned}
j_1^2 - j_2^2 + \beta_1 - \beta_2 \xrightarrow{s_1} \left(j_1 - \frac{\kappa_1}{f_1}\right)^2 - \left(j_2 + \frac{\kappa_1}{f_1}\right)^2 + \bar{\beta}_1 - \bar{\beta}_2 = j_1^2 - j_2^2 - 2\kappa_1 + \bar{\beta}_1 - \bar{\beta}_2 \\
= j_1^2 - j_2^2 - 2(\beta_2 - \beta_1) + \beta_2 - \beta_1 = j_1^2 - j_2^2 + \beta_1 - \beta_2 .
\end{aligned}
\tag{33}
$$

Obviously the $i-$th component of equations (11) is invariant as f_i remains unchanged. Explicitly, the transformation of the right hand side of the $i-$th component of equation (11) is as follows:

$$
\begin{aligned}
&- j_i^2 + j_{i+1}^2 - \beta_i + \beta_{i+1} = f_i(j_{i+1} - j_i) - \beta_i + \beta_{i+1} \\
&\xrightarrow{s_i} (j_{i+1} + j_i)(j_{i+1} - j_i) + 2\kappa_i \frac{f_i}{f_i} - \beta_{i+1} + \beta_i = (j_{i+1} + j_i)(j_{i+1} - j_i) + \beta_{i+1} - \beta_i .
\end{aligned}
$$

However since f_{i-1} (and f_{i+1}) is are not invariant under the transformation (31) we need to separately consider the $(i+1)-$th (and the similar case of the $(i-1)-$th) component of equations (11) in order to explicitly prove their invariance. The left hand side becomes after s_i transformation:

$$
\begin{aligned}
\text{LHS} &= \{j_{i+1} + j_{i+2}, H\} - \kappa_i \frac{1}{(j_i + j_{i+1})^2} \{j_{i+1} + j_{i+2}, H\} = \\
&= -j_{i+1}^2 + j_{i+2}^2 - \beta_{i+1} + \beta_{i+2} + (-1)^i \frac{f_{i+1}}{\Phi} \Psi \\
&- \frac{\kappa_i}{f_i}(j_{i+1} - j_i) + \frac{\kappa_i(\beta_i - \beta_{i+1})}{f_i^2} + (-1)^i \frac{\kappa_i}{f_i \Phi} \Psi ,
\end{aligned}
$$

while the right hand side becomes

$$\mathrm{RHS} = -(j_{i+1} + \frac{\kappa_i}{f_i})^2 + j_{i+2}^2 - s_i(\beta_{i+1}) + s_i(\beta_{i+2}) + (-1)^i (f_{i+1} + \frac{\kappa_i}{f_i})\frac{\Psi}{\Phi} = -j_{i+1}^2 - 2j_{i+1}\frac{\kappa_i}{f_i} - \frac{\kappa_i^2}{f_i^2} + j_{i+2}^2 - \beta_i + \beta_{i+2} + (-1)^i \frac{f_{i+1} + \kappa_i/f_i}{\Phi}\Psi\,.$$

One can now easily show that the above two equations are equal to each other using that $\kappa_i = \beta_{i+1} - \beta_i$ so that

$$-\frac{\kappa_i^2}{f_i^2} = \frac{\kappa_i(\beta_i - \beta_{i+1})}{f_i^2}, \quad -2j_{i+1}\frac{\kappa_i}{f_i} = -\frac{\kappa_i}{f_i}(j_{i+1} - j_i) - \beta_{i+1} + \beta_i\,,$$

etc.

This establishes that the equations (11) are invariant under Bäcklund transformations $s_i, i = 1, \ldots, 4$.

Note that the automorphism π such that $\pi(j_i) = j_{i+1}, \pi(\alpha_i) = \alpha_{i+1}$ transforms Φ, Ψ as follows

$$\pi(\Phi) = \Phi, \qquad \pi(\Psi) = -\Psi$$

and thus the equations (11) are invariant under the automorphism π as well, which completes the proof of the $A_3^{(1)}$ invariance. Due to invariance of Φ, Ψ under Backlund symmetries it follows easily that system of $\bar{j}_i, i = 1, \ldots, 4$ defined in relation (29) will transform under $s_i, \pi, i = 1, \ldots, 4$ exactly as $j_i, i = 1, \ldots, 4$.

The above result is consistent with the fact that the dressing equations (11) can be cast in form of the Noumi-Yamada equations (13) and these equations are known to be invariant under the extended affine Weyl group. However for the dressing chain of even cyclicity there is no equivalent expression (3) that would give variables j_i in terms of variables f_i of the Noumi-Yamada system. Thus the necessity to establish separately the invariance under $A_{N-1}^{(1)}$ transformations for even dressing chains even more so because the invariance would not held for the dressing chain (1) with N even but only for the equation (11) augmented by the Ψ/Φ terms.

4 Deformation of the $N = 4$ periodic dressing chain

Having established the reduction from $N = 5$ to $N = 4$ periodic dressing chains we will take advantage of the formalism to explore how to break the extended affine Weyl symmetry $A_3^{(1)}$ symmetry by explicit addition of extra terms in the Hamiltonian. We first perform a deformation of $N = 5$ symmetric Painlevé equations

$$f_{i\,,z} = f_i(f_{i+1} + f_{i-2} - f_{i+2} - f_{i-1}) + \alpha_i,\ i = 1, 2, 3, 4, 5\,. \tag{34}$$

Equations (34) are invariant under $s_i, i = 1, 2, 3, 4, 5$ and π transformations of the extended affine $A_4^{(1)}$ Weyl group [1, 7]. Equations (34) are also invariant under additional automorphisms

$$\begin{aligned} \pi_i &: f_i \to -f_i,\ f_{i-1} \to -f_{i+1},\ f_{i+1}, \to -f_{i-1},\ f_{i-2} \to -f_{i+2},\ f_{i+2} \to -f_{i-2}, \\ &: \alpha_i \to -\alpha_i,\ \alpha_{i-1} \to -\alpha_{i+1},\ \alpha_{i+1} \to -\alpha_{i-1},\ \alpha_{i-2} \to -\alpha_{i+2},\ \alpha_{i+2} \to -\alpha_{i-2}, \end{aligned} \tag{35}$$

with $i = 1, 2, 3, 4, 5$.

As the next step we augment the Hamiltonian $H_5 = \sum_{k=1}^{5}(j_k^3/3 + \beta_k j_k)$ by additional quadratic term:

$$\begin{aligned} H_{\text{deform}} &= \frac{1}{2}\eta_1(f_2+f_3+f_4+f_5)^2 + \frac{1}{2}\eta_2(f_1+f_3+f_4+f_5)^2 + \frac{1}{2}\eta_3(f_1+f_2+f_4+f_5)^2 \\ &+ \frac{1}{2}\eta_4(f_1+f_2+f_3+f_5)^2 + \frac{1}{2}\eta_5(f_1+f_2+f_3+f_4)^2 \,, \end{aligned} \tag{36}$$

with $\eta_i, i = 1, \ldots, 5$ being deformation parameters.

Inserting $H = H_5 + H_{\text{deform}}$ into Hamilton equations leads to :

$$\begin{aligned} f_{i\,,z} &= f_i(f_{i+1} + f_{i-2} - f_{i+2} - f_{i-1}) + \alpha_i + \eta_{i+1}(\sum_{k\neq i+1} f_k) \\ &+ \eta_{i-3}(\sum_{k\neq i-3} f_k) - \eta_{i+3}(\sum_{k\neq i+3} f_k) - \eta_{i-1}(\sum_{k\neq i-1} f_k)\,, \end{aligned} \tag{37}$$

where e.g. $\sum_{k\neq 2} f_k = f_1 + f_3 + f_4 + f_5$ and $i = 1, \ldots, 5$.

Remarkably equations (37) are still invariant under $\pi_i, i = 1, \ldots, 5$ automorphisms (35) now extended to also act on the parameters η_i as follows :

$$\pi_i \;:\; \eta_i \to -\eta_i,\; \eta_{i-1} \to -\eta_{i+1},\; \eta_{i+1}, \to -\eta_{i-1},\; \eta_{i-2} \to -\eta_{i+2},\; \eta_{i+2} \to -\eta_{i-2}\,. \tag{38}$$

Equations (37) are also invariant under the extended automorphism $\pi : f_i \to f_{i+1}, \alpha_i \to \alpha_{i+1}, \eta_i \to \eta_{i+1}$ such that $\pi_4\pi_3\pi_2\pi_1 = \pi^{-1}$.

However equations (37) are not symmetric under Bäcklund transformations $s_i, i = 1, \ldots, 5$ (31). Setting some of the parameters $\eta_i = 0$ will, as we will see, restore invariance under some of the Bäcklund transformations s_i.

We will use invariance under π_i to guide us with regard to which η_i parameters to set to zero and proceed by choosing a model that is invariant under π^2 and π_2. The π_2 invariant deformation is given by

$$H^{(2)}_{\text{deform}} = \frac{1}{2}\eta_1(f_2+f_3+f_4+f_5)^2 + \frac{1}{2}\eta_2(f_1+f_3+f_4+f_5)^2 + \frac{1}{2}\eta_3(f_1+f_2+f_4+f_5)^2\,, \tag{39}$$

using the fact that $\pi_2 : \eta_1 \to -\eta_3, \eta_3 \to -\eta_1, \eta_2 \to -\eta_2$. For $N = 4$ reduction this gives:

$$H^{(2)}_{\text{deform}} = \frac{1}{2}\eta_1(j_1+j_2)^2 + \frac{1}{2}\eta_2(j_2+j_3)^2 + \frac{1}{2}\eta_3(j_3+j_4)^2\,, \tag{40}$$

which maintains its invariance under the reduction of π_2:

$$\begin{aligned} \hat{\pi}_2 &: j_2 \to -j_3,\; j_1 \to -j_4,\; j_3 \to -j_2,\; j_4 \to -j_1, \\ &: \beta_2 \to \beta_3,\; \beta_1 \to \beta_4,\; \beta_3 \to \beta_2,\; \beta_4 \to \beta_1, \\ &: \eta_2 \to -\eta_2,\; \eta_1 \to -\eta_3,\; \eta_3 \to -\eta_1\,, \end{aligned} \tag{41}$$

Note that $H = \sum_{k=1}^{4}(j_k^3/3 + \beta_k j_k) + H_{\text{deform}}$ satisfies $\pi_i(H) = -H, i = 1, \ldots, 5$. Further, setting $\eta_2 = 0$ we get $\pi_2(H) = -H$ for $H = \sum_{k=1}^{4}(j_k^3/3 + \beta_k j_k) + H^{(2)}_{\text{deform}}$, from which we obtain a system of equations (with $\{f, H\}$ replaced by f_z):

$$\begin{aligned}
(j_1+j_2)_z &= -j_1^2+j_2^2-\beta_1+\beta_2+\frac{(j_1+j_2)\Psi}{\Phi}\\
(j_2+j_3)_z &= -j_2^2+j_3^2-\beta_2+\beta_3-\eta_1(j_1+j_2)+\eta_3(j_3+j_4)-\frac{(j_2+j_3)\Psi}{\Phi}\\
(j_3+j_4)_z &= -j_3^2+j_4^2-\beta_3+\beta_4+\frac{(j_3+j_4)\Psi}{\Phi}\\
(j_4+j_1)_z &= -j_1^2+j_4^2+2j_2^2-2j_3^2-\beta_1+2\beta_2-2\beta_3+\beta_4\\
&\quad +\eta_1(j_1+j_2)-\eta_3(j_3+j_4)+\frac{j_1+2j_2+2j_3+j_4}{\Phi}\Psi\,,
\end{aligned} \tag{42}$$

that will not only maintain the π_2 symmetry from (41) but also be invariant under s_1, s_3 transformations. It will also be invariant under π^2 extended by $\pi^2 : \eta_1 \to \eta_3$ and under $\hat{\pi}_2$ from (41).

The symmetry operations of equations (42) satisfy the relations

$$\hat{\pi}_2 s_1 = s_3\hat{\pi}_2,\ \pi^2 s_1 = s_3\pi^2,\ \pi^2\hat{\pi}_2 = \hat{\pi}_2\pi^2$$

where s_1, s_3 were defined in equations (31).

The constraint Ψ in equation (42) does not depend on η's and will satisfy: $\hat{\pi}_2(\Psi) = -\Psi$ and $\pi^2(\Psi) = \Psi$. As in equation (13) we can rewrite equations (42) in a compact way

$$\Phi\, f_{i,z} = f_i f_{i+2}\,(f_{i+1}-f_{i-1})+(-1)^i\,(\alpha_1+\alpha_3)\,f_i+\alpha_i(f_i+f_{i+2})+(\delta_{i,2}-\delta_{i,4})(-\eta_1 f_1+\eta_3 f_3)$$

for $i = 1, 2, 3, 4$.

We recognize in the above equations a model proposed earlier [3] within the framework of Painlevé equations and shown to pass the Painlevé test due to the presence of the remaining Bäcklund symmetries. Generally, the deformed models studied in the literature, [2, 3], pass the Painléve test if the deformation maintains invariance under at least one of the $s_i, i = 1,, 2, ..., N-1$ of the original Bäcklund transformations but fail to pass the test if all of $s_i, i = 1,, 2, ..., N-1$ are broken. The model described by equations (42) is invariant under the two Bäcklund symmetries s_1, s_3. Since it is fully equivalent to the model considered in [2] in the setting of Painlevé equations it falls within a class of models that pass the Painlevé test.

5 Connection to $2n$ boson models

In [5, 4] we proposed self-similarity reductions of the constrained KP hierarchy with symmetry structure defined by Bäcklund transformations induced by a discrete structure of Volterra type lattice. The models and their self-similarity reductions were referred to as "$2n$ boson models".

The $2n$ boson models models were conveniently expressed in [4] in terms of n canonical pairs $e_i, Y_i, i = 1, 2, \ldots, n$ satisfying the bracket $\{e_i, Y_j\} = \delta_{ij},\ i = 1, \ldots, n$, which enter the Hamiltonian (here for simplicity given for $n = 2$):

$$\mathcal{H}_{A_4^{(1)}} = -\sum_{j=1}^{2} e_j\,(Y_j-2x)\,(Y_j-e_j)+2e_1\,(Y_1-2x)\,(Y_2-e_2)+\sum_{j=1}^{2}\bar{k}_j Y_j-\sum_{j=1}^{2}\kappa_j e_j$$

The reference [4] presented an explicit symplectic map from canonical variables

$$p_i = f_{2i}, \quad q_i = \sum_{k=1}^{i} f_i, \quad i = 1, 2, \dots, n$$

from $A_{2n}^{(1)}$ Noumi-Yamada system [7] in terms of f_i to the $2n$ boson model represented by $e_i, Y_i, i = 1, 2, \dots, n$ variables. This construction has established equivalence of $2n$ boson model to symmetric Painlevé systems with $A_{2n}^{(1)}$ Weyl symmetry and therefore also equivalence to $N = 2n + 1$ periodic dressing chain systems.

To derive the reduced system invariant under $A_{2n-1}^{(1)}$ Weyl symmetry the paper [4] adopted the Dirac technique with the primary constraint:

$$\psi_1 = Y_2 = 0 \,. \tag{43}$$

In what follows we compare the approach of [4] to the method proposed in this paper.

The Dirac procedure of [4] went as follows. Once we set the primary constraint (43), the secondary constraint ψ_2 then follows from

$$\psi_2 = -\frac{\partial \mathcal{H}_{A_4^{(1)}}}{\partial e_2} = 4xe_2 + 2e_1(Y_1 - 2x) + \kappa_2 \,. \tag{44}$$

The fundamental bracket is

$$\{\psi_1, \psi_2\} = -4x \neq 0 \,, \tag{45}$$

and can be used to calculate the Dirac brackets:

$$\{e_2, e_1\}_D = \frac{e_1}{2x}, \quad \{e_2, Y_1\}_D = -\frac{Y_1 - 2x}{2x} \,. \tag{46}$$

The other bracket $\{e_1, Y_1\} = \{e_1, Y_1\}_D = 1$ is unchanged. It follows from the Dirac brackets (46) that

$$\{e_1, \psi_\alpha\}_D = 0, \quad \{Y_1, \psi_\alpha\}_D = 0 \qquad \alpha = 1, 2 \,. \tag{47}$$

We can therefore directly implement reduction by substituting

$$Y_2 = \psi_1, \quad e_2 = \left(\psi_2 - 2e_1(Y_1 - 2x) - \kappa_2\right)/4x$$

into $\mathcal{H}_{A_4^{(1)}}$ to obtain the reduced Hamiltonian :

$$2x\bar{\mathcal{H}}_2 = e_1\left(Y_1 - 2x\right)Y_1\left(e_1 - 2x\right) + \kappa_2 e_1 Y_1 + \bar{k}_1 2xY_1 \quad - \left(\kappa_1 + \kappa_2\right)2xe_1 \,, \tag{48}$$

where for simplicity we set $\psi_1 = 0, \psi_2 = 0$, since they both commute with e_1, Y_1. The corresponding Hamilton equations are :

$$\begin{aligned} \{e_1, \bar{\mathcal{H}}_2\} &= 2xe_1 - e_1^2 - 2e_1Y_1 + \bar{k}_1 + \frac{1}{2x}\left(2e_1^2Y_1 + e_1\kappa_2\right) \\ \{Y_1, \bar{\mathcal{H}}_2\} &= -2xY_1 + 2e_1Y_1 + Y_1^2 + \kappa_1 + \kappa_2 - \frac{1}{2x}\left(2Y_1^2e_1 + Y_1\kappa_2\right) \,. \end{aligned} \tag{49}$$

In what follows we will show their complete equivalence to equations (11) for $N = 4$.

Mapping of $N = 5$ brackets for f_i into the e_1, Y_1, e_2, Y_2 goes through the canonical p, q system :

$$p_1 = f_2,\ p_2 = f_4,\ q_1 = f_1,\ q_2 = f_1 + f_3$$

such that $\{q_i, p_j\} = \delta_{ij}$. The symplectic map from p_i, q_j to e_i, Y_j is as follows:

$$\begin{aligned} Y_2 &= q_1 + p_2 + 2x = f_1 + f_4 + 2x, \quad e_2 = q_2 + p_2 + 2x = f_1 + f_3 + f_4 + 2x \\ Y_1 &= -q_2 - p_2 - p_1 = -f_1 - f_3 - f_2 - f_4, \quad e_1 = -q_1 = -f_1\,. \end{aligned} \tag{50}$$

We now impose the constraint $\psi_1 = Y_2$ followed by the secondary constraint $\psi_2 = 4xe_2 + 2e_1(Y_1 - 2x) + \kappa_2$ and show that this is equivalent to $N = 4$ model obtained by reducing the Shabat-Veselov's $N = 5$ dressing model.

From the top equation of (50) and the fact that

$$-2x = f_1 + f_2 + f_3 + f_4 + f_5 = 2(j_1 + j_2 + j_3 + j_4 + j_5)$$

for the $N = 5$ system it follows that

$$Y_2 = f_1 + f_4 - (f_1 + f_2 + f_3 + f_4 + f_5) = -j_1 - j_2 - 2j_3 - j_4 - j_5 = -j_3 + x$$

Thus $Y_2 = 0$ is equivalent to $j_3 = x$ or

$$Y_2 = 0 \longrightarrow j_1 + j_2 + j_4 + j_5 = -2x\,. \tag{51}$$

Based on this relation we now can define the $N = 4$ system as

$$\begin{aligned} \tilde{f}_1 &= j_1 + j_2, \quad \tilde{f}_3 = j_4 + j_5 = \tilde{j}_3 + \tilde{j}_4, \\ \tilde{f}_2 &= j_2 + j_4 = j_2 + \tilde{j}_3, \quad \tilde{f}_4 = j_5 + j_1 = \tilde{j}_4 + j_1, \end{aligned} \tag{52}$$

where we have introduced

$$\tilde{j}_3 = j_4, \quad \tilde{j}_4 = j_5\,, \tag{53}$$

such that

$$\tilde{f}_1 + \tilde{f}_3 = -2x, \quad \tilde{f}_2 + \tilde{f}_4 = -2x\,, \tag{54}$$

equivalent to

$$j_1 + j_2 + \tilde{j}_3 + \tilde{j}_4 = -2x\,. \tag{55}$$

To calculate the secondary constraint ψ_2 we need to express e_1, e_2, Y_1 in terms of $N = 4$ quantities that is as follows:

$$\begin{aligned} e_1 &= -f_1 = -j_1 - j_2 \\ e_2 &= q_2 + p_2 + 2x = f_1 + f_3 + f_4 + 2x = j_1 + j_2 + j_3 + 2j_4 + j_5 + 2x \\ &= -j_1/2 - j_2/2 + \tilde{j}_3/2 - \tilde{j}_4/2 \\ Y_1 &= -f_1 - f_3 - f_2 - f_4 = -j_1 - 2j_2 - 2j_3 - 2j_4 - j_5 = -j_2 - \tilde{j}_3 = -\tilde{f}_2\,. \end{aligned} \tag{56}$$

In the above equation we used identities that hold for $Y_2 = 0$.

We can now calculate the secondary constraint ψ_2:

$$\psi_2 = 4xe_2 + 2e_1(Y_1 - 2x) + \kappa_2 = 2[2xe_2 + e_1(Y_1 - 2x)] + \kappa_2 = -j_1^2 + j_2^2 - \tilde{j}_3^2 + \tilde{j}_4^2 + \kappa_2\,. \quad (57)$$

Recovering our constraint Ψ from section 3 up to an overall sign. Thus the Dirac reduction from [4] agrees with the Dirac reduction we have designed for the periodic dressing chain equations.

On basis of identification (57) we can rewrite the first of Hamilton equations (49) as :

$$\begin{aligned}\{e_1, \bar{\mathcal{H}}_2\} &= 2xe_1 - e_1^2 - 2e_1Y_1 + \bar{k}_1 + \frac{e_1}{2x}\left(2e_1Y_1 + \kappa_2 + 4xe_2 - 4xe_2 - 4xe_1 + 4xe_1\right)\\ &= j_1^2 - j_2^2 + \bar{k}_1 + \frac{e_1}{2x}\psi_2\,,\end{aligned} \quad (58)$$

after we inserted values of e_1, Y_1, e_2 in terms of $j_1, j_2, \tilde{j}_3, \tilde{j}_4$ from equations (56).

Recalling from equation (56) that $e_1 = -j_1 - j_2$ and inserting it into the equation (58) we find

$$\{j_1 + j_2, \bar{\mathcal{H}}_2\} = j_2^2 - j_1^2 - \bar{k}_1 + \frac{j_1 + j_2}{-2x}\Psi = j_2^2 - j_1^2 + \alpha_1 + \frac{j_1 + j_2}{-2x}\Psi\,, \quad (59)$$

where $\alpha_1 = -\bar{k}_1$, $-2x = \Phi$ and $\Psi = -\psi_2$ to agree with our convention. We recognize in (59) the first dressing equation of (27).

Let us rewrite the second of Hamilton equations (49) as :

$$\begin{aligned}\{Y_1, \bar{\mathcal{H}}_2\} &= -2xY_1 + 2e_1Y_1 + Y_1^2 + \kappa_1 + \kappa_2 - \frac{1}{2x}\left(2Y_1^2e_1 + Y_1\kappa_2\right)\\ &= j_2^2 - \tilde{j}_3^2 + \kappa_1 + \kappa_2 - \frac{Y_1}{2x}\,\psi_2\,.\end{aligned} \quad (60)$$

Recalling that $Y_1 = -(j_2 + \tilde{j}_3)$ we can rewrite the above as :

$$\{j_2 + \tilde{j}_3, \bar{\mathcal{H}}_2\} = \tilde{j}_3^2 - j_2^2 - \kappa_1 - \kappa_2 - \frac{j_2 + \tilde{j}_3}{-2x}\,\Psi = \tilde{j}_3^2 - j_2^2 + \alpha_2 - \frac{j_2 + \tilde{j}_3}{-2x}\,\Psi\,, \quad (61)$$

where we set $\alpha_2 = -\kappa_1 - \kappa_2$, $-2x = \Phi$ and $\Psi = -\psi_2$ to agree with our earlier convention. We recognize in (61) the second dressing equation of (27).

Using relation (55) we can similarly derive equations for $\{j_1 + \tilde{j}_4, \bar{\mathcal{H}}_2\}$ and $\{\tilde{j}_3 + \tilde{j}_4, \bar{\mathcal{H}}_2\}$ obtaining all the dressing equations (27) from the e-Y system.

Repeating almost verbatim what we have done in reference [4] we can cast equations (49) into into symmetric $A_3^{(1)}$ Painlevé V equations (13) using that from relation (56) it follows that $Y_1 = -f_2$ and $e_1 = -f_1$. Subsequently $f_4 = -2x + Y_1$ and $f_3 = -2x + e_1$ (we we ignore tildes for simplicity). As defined above the constants are :

$$\alpha_1 = -\bar{k}_1, \quad \alpha_2 = -\kappa_1 - \kappa_2, \quad \alpha_3 = \kappa_2 + \bar{k}_1, \; \alpha_4 = \kappa_1\,.$$

6 Conclusion and outlook

We proposed a consistent and systematic approach based on Dirac reduction framework to formulating dressing chain equations for even $2n$ periodicity. Such approach is naturally obtained from the corresponding dressing chain equations of odd $2n+1$ periodicity by reduction. Both chains of even or odd periodicity are equivalent to $A^{(1)}_{2n-1}$, $A^{(1)}_{2n}$ Painlevé systems, respectively. The formalism is in agreement with the previous result obtained in a setup of $2n$ boson models [4] and facilitates further studies of deformations of the original Bäcklund symmetry.

Among subjects deserving further investigation is establishing the bi-Hamiltonian nature of the dressing chains with even periodicity. The bi-Hamiltonian character of the dressing chains with odd periodicity is well known [12] and in further development separation of variables has been developed in such case [6]. Uncovering the bihamiltonian structure for dressing chains of even cyclicity and related development of separation of variables is a proposal to be studied elsewhere.

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References

[1] Adler V E, Recuttings of polygons, *Functional Analysis and Its Applications*, 1993, **27**:2, 141–143

[2] Alves V C C, Aratyn H, Gomes J F and Zimerman A H, Gauge symmetry origin of Bäcklund transformations for Painlevé equations, *J. Phys. A: Math. Gen.*, 2021, **54**, 195701

[3] Aratyn H, Gomes J F, Ruy D V and Zimerman A H, A symmetric system of mixed Painlevé III-V equations and its integrable origin, *J. Phys. A: Math. Theor.*, 2016, **49**, 045201

[4] Aratyn H, Gomes J F and Zimerman A H, A symmetry reduction technique for higher order Painlevé systems, *Physics Letters*, 2012, **A 376**, 1966-1969

[5] Aratyn H, Gomes J F and Zimerman A H, Higher order Painlev e equations and their symmetries via reductions of a class of integrable models, *J. Phys. A: Math. Theor.*, 2011, **44**, 235202

[6] Błaszak M, From Bi-Hamiltonian Geometry to Separation of Variables: Stationary Harry-Dym and the KdV Dressing Chain, *Journal of Nonlinear Mathematical Physics*, 2002, **9**, Supplement 1, 1-13

[7] Noumi M and Yamada Y, Affine Weyl groups, discrete dynamical systems and Painlevé equations, *Commun. Math. Phys.* , 1998 , **199**, 281-195, Higher order Painlevé equations of type $A^{(1)}_l$, *Funkcial.Ekvac.*, 1998, **41**, 483-503, arXiv:math/9808003

[8] Schiff J, Bäcklund transformations of MKdV and Painlevé equations, *Nonlinearity*, 1994, **7**, 305-312

[9] Sen A, Hone A N W and Clarkson P A, On the Lax pairs of the symmetric Painlevé equations, *Studies in Applied Mathematics*, 2006, **117**, 299-319

[10] Takasaki K, Spectral Curve, Darboux Coordinates and Hamiltonian Structure of Periodic Dressing Chains, *Commun. in Mathematical Physics*, 2003, **241**, 111-142

[11] Tsuda T, Universal characters, integrable chains and the Painlevé equations, *Advances in Mathematics* , 2005 , **197**, 587- 606

[12] Veselov A P and Shabat A B, Dressing chains and the spectral theory of the Schrödinger operator, *Funct. Anal. Appl.* , 1993, **27**, 81–96

[13] Willox R and Hietarinta J, Painlevé equations from Darboux chains: I. PIII-PV, *J. Phys. A: Math. Gen.*, 2003, **36**, 10615–10635