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Letter to the Editors

New solvable systems of 2 first-order nonlinearly-coupled ordinary differential equations

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Abstract

In this short communication we introduce a rather simple *autonomous* system of 2 *nonlinearly-coupled* first-order Ordinary Differential Equations (ODEs), whose *initial-values* problem is *explicitly solvable* by *algebraic* operations. Its ODEs feature 2 right-hand sides which are the ratios of 2 *homogeneous* polynomials of *first* degree divided by the same *homogeneous* polynomial of *second* degree. The model features only 4 *arbitrary* parameters. We also report its *isochronous* variant featuring 4 *nonlinearly-coupled* first-order ODEs in 4 dependent variables, featuring 9 *arbitrary* parameters.■

In this short communication we introduce a rather simple *autonomous* system of 2 *nonlinearly-coupled* first-order Ordinary Differential Equations (ODEs) which features solutions obtainable by *algebraic* operations; and we report the explicit solution of its *initial-values* problem. To the best of our knowledge these findings are new; but since the corresponding literature includes an enormous number of entries over at least 2 centuries we cannot be quite certain.

The technique we have used to arrive at these results is fairly simple, but we shall not describe it (the expert reader shall easily guess it from the results reported below); we shall describe it in a separate paper [1] also reporting several other examples of *algebraically solvable* systems of 2 first-order nonlinear ODEs.

The system of ODEs on which we focus reads as follows:

$$\dot{x}_1(t) = \frac{x_1(t) + \alpha_1 x_2(t)}{\beta_1 [x_1(t)]^2 + (\alpha_1\beta_1 + \alpha_2\beta_2)\, x_1(t)\, x_2(t) + \beta_2 [x_2(t)]^2}\ , \tag{1a}$$

$$\dot{x}_2(t) = -\frac{x_2(t) + \alpha_2 x_1(t)}{\beta_1 [x_1(t)]^2 + (\alpha_1\beta_1 + \alpha_2\beta_2) x_1(t) x_2(t) + \beta_2 [x_2(t)]^2} . \tag{1b}$$

Notation: t is the independent variable (one might think of it as *time*); $x_n(t)$ are the 2 *dependent* variables; the superimposed dot denotes t-differentiation; α_n and β_n $(n = 1, 2)$ are 4 (t-independent) *arbitrary* parameters. 2 additional parameters γ_1 and γ_2 might of course be introduced by rescaling the 2 dependent variables $(x_n(t) \Rightarrow \gamma_n x_n(t))$; while no third parameter γ can be additionally introduced by rescaling the independent variable t, since the system (1) is clearly *invariant* under the common rescaling $x_n(t) \Rightarrow \gamma x_n(t)$, $t \Rightarrow \gamma t$. ■

Proposition: the solution of the *initial-values* problem of this system (1) reads as follows:

$$x_n(t) = \left(\gamma_{n1}\sqrt{1 + t/t_1} + \gamma_{n2}\sqrt{1 + t/t_2}\right), \quad n = 1, 2 , \tag{2}$$

with

$$\gamma_{11} = \frac{b_2 [b_1 x_1(0) + a_2 x_2(0)]}{b_1 b_2 - a_1 a_2} , \quad \gamma_{22} = \frac{b_1 [a_1 x_1(0) + b_2 x_2(0)]}{b_1 b_2 - a_1 a_2} , \tag{3a}$$

$$\gamma_{12} = \frac{-a_2 [a_1 x_1(0) + b_2 x_2(0)]}{b_1 b_2 - a_1 a_2} , \quad \gamma_{21} = \frac{-a_1 [b_1 x_1(0) + a_2 x_2(0)]}{b_1 b_2 - a_1 a_2} , \tag{3b}$$

$$b_1 = 2\beta_1 - \alpha_2 (r + \alpha_1\beta_1 + \alpha_2\beta_2) , \quad b_2 = -2\beta_2 + \alpha_1 (r + \alpha_1\beta_1 + \alpha_2\beta_2) , \tag{3c}$$

$$a_n = (-)^{n+1} r + \alpha_1\beta_1 - \alpha_2\beta_2 , \quad n = 1, 2 , \tag{3d}$$

$$r = \sqrt{(\alpha_1\beta_1 + \alpha_2\beta_2)^2 - 4\beta_1\beta_2} , \tag{3e}$$

$$t_1 = -\eta/\eta_1 , \quad t_2 = \eta/\eta_2 , \tag{3f}$$

$$\eta = (\gamma_{12}\gamma_{21} - \gamma_{11}\gamma_{22}) \left\{\beta_1 [x_1(0)]^2 + (\alpha_1\beta_1 + \alpha_2\beta_2) x_1(0) x_2(0) + \beta_2 [x_2(0)]^2\right\} , \tag{3g}$$

$$\eta_1 = 2 [(\alpha_2\gamma_{12} + \gamma_{22}) x_1(0) + (\gamma_{12} + \alpha_1\gamma_{22}) x_2(0)] , \tag{3h}$$

$$\eta_2 = 2 [(\alpha_2\gamma_{11} + \gamma_{21}) x_1(0) + (\gamma_{11} + \alpha_1\gamma_{21}) x_2(0)] . \quad ■ \tag{3i}$$

Remark 1. The formulas written above provide the *explicit* definition (up to the implicit sign ambiguities due to the square-roots appearing in them) of the initial-values problem of the system of ODEs (1), in terms of the 4 parameters α_n and β_n $(n = 1, 2)$ featured by the system of 2 ODEs (1). The skeptical reader who wishes to check that the formulas (2) with (3) provide the solution of the initial-value problem of the system of ODEs (1) is of course welcome to do so! ■

Remark 2. Because of the sign ambiguities due to the square-roots appearing in the formulas (2) and (3e), it might appear that the formulas (2) are *inadequate* to provide the solution of the initial-values problem of the system of ODEs (1). But it is easy to check that the formulas (2) with (3) become *identities* at $t = 0$, with the most obvious assignment of the square-roots; and thereafter they provide a well-defined definition of the solution for all time by *continuity* in the independent variable t. Of course a *singularity* may then be hit if the time-evolution causes the argument of one of the 2 square-roots $\sqrt{1 + t/t_n}$ $(n = 1, 2)$ appearing in the right-hand of the 2 eqs. (2) to vanish; but this is a *natural* feature of *nonlinear* systems of evolution equations. ■

Remark 3. Because the 2 ODEs of the system (1) feature right-hand sides which are *homogeneous* in the 2 dependent variables $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$, and its solutions depend in a simple manner on the time variable t (see (2)), the more general system

$$\dot{\tilde{x}}_1(t) = \mathbf{i}\omega\tilde{x}_1(t) + \frac{\tilde{x}_1(t) + \alpha_1\tilde{x}_2(t)}{\beta_1[\tilde{x}_1(t)]^2 + (\alpha_1\beta_1 + \alpha_2\beta_2)\tilde{x}_1(t)\tilde{x}_2(t) + \beta_2[\tilde{x}_2(t)]^2}, \tag{4a}$$

$$\dot{\tilde{x}}_2(t) = \mathbf{i}\omega\tilde{x}_2(t) - \frac{\tilde{x}_2(t) + \alpha_2\tilde{x}_1(t)}{\beta_1[\tilde{x}_1(t)]^2 + (\alpha_1\beta_1 + \alpha_2\beta_2)\tilde{x}_1(t)\tilde{x}_2(t) + \beta_2[\tilde{x}_2(t)]^2}, \tag{4b}$$

where $\mathbf{i} = \sqrt{-1}$ is the *imaginary unit* and ω is an *arbitrary nonvanishing real* parameter, is *isochronous: all* its nonsingular solutions are *completely-periodic* in the (*real*) time variables t, with a period which is 4 *times* (or for a *subset* of solutions only 2 *times*) the basic period $T = \pi/|\omega|$: see [2]. But of course, due to the presence of the *imaginary* unit $\mathbf{i}$ (see (4)), this system (4) of ODEs lives in the *complex* world, namely its 2 dependent variables $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ must, and its 4 parameters α_n and β_n $(n = 1, 2)$ may, be considered to be *complex* numbers; so, in terms of *real* variables and *real* parameters, it amounts to a system of 4 *nonlinear* ODEs in 4 *real* dependent variables featuring 9 *arbitrary real* parameters. The fact that such a system—that the interested reader might like to write out *explicitly*—is *isochronous* seems a *remarkable* finding; possibly relevant in *applicative* contexts. ■

Final remark. The results reported above imply that the 2 variables $x_n(t)$ or $\tilde{x}_n(t)$ evolve in time as a *linear* superposition with constant coefficients of 2 simple functions of time: see (1). This is likely to motivate the *experts* on systems of nonlinear evolution equations to consider the finding reported in this short communication to be rather *trivial*. On the other hand *practitioners* might find that the systems of 2 nonlinear systems of ODEs (1) or (4) describe interesting phenomenologies. So it might be useful that the scientific community become aware of this finding; which might possibly deserve to be recorded in the website **EqWorld**. ■

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